

THE ROLE OF THE GEOMETRIC MODELS IN THE EXPLANATION OF DETERMINANT AND THE PROPERTIES OF A DETERMINANT

Yasin SOYLU
Atatürk University
Agri Educational Faculty
The Mastery of Science Teaching Department
Agri, TURKEY

INTRODUCTION

Every branch of science has its own special methods teaching within the perspective of its purposes. A teaching method which is appropriate for the structure of mathematics should be according with these stated purposes below (Van de Wella, 1989);

The students;

- **Conceptual knowledge of mathematics**
- **Procedural knowledge of mathematics**
- **Connections between conceptual and procedural knowledge**

These three purposes are called as connectional knowledge. Conceptual knowledge can be defined as knowledge of mathematical structures (concepts and its elements) and giving them with symbols; and benefiting from its utilities; the knowledge of procedural techniques of mathematics and giving them with symbols; formatting the connections and relations among methods, symbols and concepts.

By studying students' knowledge of mathematics in terms of learning psychology that mentions two kinds of mathematical knowledge. First one is an entirely mechanical data consist of some abilities such as recognizing the symbols, doing the operations ; second one is the ability to put symbols into some mathematical concepts, forming some relationships among them and doing operations by using them (Baki, 1998). While in procedural knowledge, it is necessary to know only how to use knowledge without needing to know the meaning of a concept or an operation; in the conceptual knowledge, the act of conception becomes important (Baki, 1997). Conceptual knowledge and procedural knowledge are two-dependent components. Both conceptual and procedural knowledge are very important in mathematics (Hiebert, and Carpenter, 1992).

A permanent and functional learning in mathematics is only possible with balancing conceptual and procedural knowledge (Baki, 1998). It has been more important to have operational knowledge in mathematics, whereas the conceptual knowledge should be predominantly focused on. In other words, the conceptual and operational data are not balanced in teaching mathematics. For the conceptual and operational data are not balanced in teaching mathematics, the subjects are not learned conceptually (İşleyen, and Işık, 2003). For the lessons are not explained conceptually, the subjects are memorized instead of being learned. Most students are not aware of that there are concepts at the basis of the subjects they learn, and they do not know what mathematics means.

They believe that learning mathematics is to operate on meaningless symbols, and they try to learn mathematics by memorizing it (Oaks, 1990).

Generally, it is stated that abstract concepts are difficult to articulate, and this may be the reason why the students have difficulty in understanding it, however, this problem can be eliminated or at least reduced by making the concepts concrete or giving concrete means. That the more concrete but less abstract subjects can be learned more easily is a fact admitted by everybody (Ersoy, 1997 and Baki, A., 2002).

The concretization of mathematics has been frequently used recently as an essential idea in literature for solving problems and constituting the mathematical concepts in the minds of students meaningfully. (di Seassa, 1994; Dubinsky, 1994; Duval, 1995; Eisenberg & Dreyfus, 1991; Glasensferd, 1991; Janvier, 1987; Kaput, 1994; Presmeg, 1986; Steinbring, 1991; Vinner, 1989; Zimmermann & Cunningham, 1990, *eta*) (Hitt, F., 2001).

USE OF GEOMETRY IN TEACHING MATHEMATICS

The abstraction in mathematics affects the students negatively from the point of sensational learning, and as a result of this, they get the opinion that they can not learn mathematical concepts, so they try to learn mathematics by memorization. The use of concretization or concrete materials method in learning and teaching of abstract concepts affect students in a positive way both emotionally and mentally. Because concretization of concepts in mathematics makes it possible to give meaning to abstract mathematical concepts and to better understanding of the mathematical relations (Soylu, Y., 2001). The mathematical concepts are generally abstract and require a high level of mental activity. It can be said that the method of concretization which means to introduce the algebraic and abstract concepts with the help of geometrical models may be helpful to students in showing how a logical theory can be formed through a physical or external model. Because drawing pictures about abstract concepts causes to an interpretation in mind. Students may learn the concepts permanently and properly by classifying, putting into sequence, and outlining the concepts they learned concrete or formally. Many mathematics practitioners admit that geometry is concrete. To make an introduction to linear algebra lesson with geometry may be helpful to students in developing analytic thinking about basic linear algebra concepts. The use of geometry in algebraic lesson is defended by both mathematicians and in mathematics books (Banchoff, and Wermer, 1991).

In a study which purposed to compare the efficiency of concretization through geometry and traditional teaching methods, it was observed that the method of concretization through geometry is more efficient than the traditional teaching methods on the teaching of linear transforms, the concepts about linear transforms, and determinant (Soylu, 2005).

In mathematics, writing, telling, and drawing (feel, sound, and look) provides a stronger learning through three sensory organs. Because; learning is an act of perception. As swimming can not be learned by only watching, mathematics can not be learned by only signing. Learning a more efficient mathematics may be obtained with listening, writing and drawing. The use of geometry in mathematics provides a better understanding of mathematics (Hacısalıhođlu, 1998).

Mathematical concepts are abstract concepts. These concepts can not be understood exactly without presenting some concrete things. If these concepts stay only in our minds, not touched and felt, they become unimportant.

Because of this, it must be considered important to explain mathematical concepts (graphic, drawings, and tables etc.) with their visual presentations. (Chiappini, and Bottino, 2001). Harel points out that if geometrical presentations enliven the given subjects in the minds of the students, it may prove a helpful to model for geometry to comprise linear algebra concepts (Gueudet-Chartier, 2003b).

For seeking answer to the questions of "Whether teaching mathematics may be possible by concretization through geometry?" and "If geometrical shapes have an important place in making conceptual things meaningful?", Harel, give a lecture based on concretization principle to a group of students in which he explained vector spaces subject with using a geometrical presentation. To another group, he explained with traditional teaching methods. At the results of these explanations, he saw that the group that taught with using geometry became more successful than the others. But he found some difficulties about the use of geometrical models. At the result of that study, Harel, pointed out that geometry must be used carefully, (Gueudet-Chartier, 2003a).

GEOMETRICAL REPRESENTATIONS IN R^n

It may be supposed that the use of geometry in linear algebra teaching will be limited the use of only R, R^2 and R^3 , and with this method, it will cause problems for subjects not presented through represented geometrically. That is, it may be considered that the use of concretization through geometry will be useful in only R, R^2 and R^3 spaces, and will be invalid for other space subjects. But, if R^n be thought as being a simple generalization of R, R^2 and R^3 spaces, we can teach the concepts about larger dimensional spaces more efficiently with benefiting from spaces which we can show geometrically. However, the fact of forming isomorphism between spaces can eliminate this problem. For example, isomorphism can be established between R^3 and $P_2(R)$, and R^n and $P_{n-1}(R)$.

R, R^2 and R^3 spaces are isomorphs to R^n . Because of this, a problem given in R^n can be generalized in R^n after solved in R, R^2 and R^3 and stated clearly (Gueudet-Chartier, 2003b). For example; if linear isomorphism can be set up between $L: R^3 \rightarrow P_2(R)$?

As being $\forall \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \in R^3$, the transformation of L which is defined as

$$L \left(\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \right) = a_2 x^2 + a_1 x + a_0 \text{ is a linear transformation.}$$

As being $\forall \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}, \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} \in R^3$ and $\lambda \in R$,

$$L \left(\lambda \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} \right) = L \left(\begin{bmatrix} \lambda a_2 + b_2 \\ \lambda a_1 + b_1 \\ \lambda a_0 + b_0 \end{bmatrix} \right) = (\lambda a_2 + b_2)x^2 + (\lambda a_1 + b_1)x + (\lambda a_0 + b_0)$$

is obtained. Let's look if L linear transformation is 1:1 and surjective; (if it is $\dim \text{Ker}(L)=0$, L linear transformation is 1:1.

$$\forall \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \in R^3 \quad L \left(\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} \right) = a_2x^2 + a_1x + a_0 = 0 = 0x^2 + 0x + 0x$$

becomes $a_2 = 0, a_1 = 0$ ve $a_0 = 0$.

From this, it becomes $\text{Ker}(L)=0$ and $\dim \text{Ker}(L)=0$. So L is 1:1. If V is a finite dimensional vector space and $T:V \rightarrow W$ is a linear transformation from the theorem of $\dim(\text{Ker}T) + \dim(\text{Im}T) = \dim(V)$, it becomes $\dim(\text{Im}L) = \dim(R^3)$. Here L linear transformation is the surjective. As L linear transformation is 1:1 and the surjectiv, $L:R^3 \rightarrow P_2(R)$ transformation is a linear isomorphism.

One of the most important purposes of this study is to explain determinants and its properties meaningfully to the students. By giving meanings to basic concepts, algebraic structures can be applied to advanced subjects. If the students can make the subjects he learnt abstract and do the practices about these subjects, he will be articulated the subject (meaningfully). So the model used in concretization principle should contain a meaningful relation and comprise a logical regulation. This logical relationship should be in a comprehensible position and suitable for mental and sensational situation of the student. The geometrical presentations should be appropriate for the level of students. Improprity of the model may reduce the expected mental and sensational effect an even it could cause to a conceptual confusion. (Touger, 1986) pointed out that when given an improper model, students become disappointed and some students find to use that model as a harder job than their own efforts.

THE GEOMETRICAL MEANINGS OF DETERMINANTS

On a K sield, a special scalar equivalents to each A square matrix. This scalar is named as the determinant of A matrix. What means the scalar equivalent to the matrix mentioned in that determinant definition? How can we explain this abstract concept to students? Can we give a lesson with the method of concretization through geometry in explaining determinant subject? If we think that determinant is a function from $M_{n \times n}(R)$ to R ; for $n \leq 3$, we can say yes to the answers of questions above. For $n > 3$, as mentioned above, a generalization can be made by establishing isomorphism.

It must be stressed that in a one-dimensional space; the absolute value of determinants of matrixes in the type of 1×1 indicates length; the absolute value of determinants of matrixes in the type of 2×2 are the area of the rectangle which comprised of line or column vectors of these matrixes, the absolute value of the determinants of matrixes in the type of 3×3 are the volume of solid substance which comprised of line or column vectors of these matrixes.

When determinants are explained in linear algebraic lesson; algebraic definition of determinants should be made and also what means this algebraic definition should be taught to students by using geometry in a meaningful way. So the determinant of a matrix;

A σ permutation's sign represented as $\text{sgn } \sigma$, as being sgn

$$\sigma \begin{cases} 1 & (\sigma \rightarrow \text{asdouble}) \\ -1 & (\sigma \text{ asmono}) \end{cases}$$

and after $\det A = \sum_{\sigma \in P_n} (\text{sgn } \sigma) a_{1j_1} a_{2j_2} \dots a_{nj_n} = \sum_{k=1}^{n!} (\text{sgn } \sigma) a_{1\sigma_k(1)} a_{2\sigma_k(2)} \dots a_{n\sigma_k(n)}$

formule is obtained; what it means should be explained to students.

For example; determinant of $A = \begin{bmatrix} 3 \end{bmatrix}$ matrix is $\det A = 3$. It is presented as in Figure: 1



Figure: 1

From $M_{2 \times 2}(R)$ a A matrix be $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. From algebraic definition, it is $\det A = 2 \cdot 1 - 1 \cdot 1 = 1$.

Here, what means the 1 being the value of determinant? The absolute value of determinant of matrix in the type of 2×2 is the area of rectangle comprised from vectors which composed of lines and columns of A matrix. This area can be presented as in Figure: 2

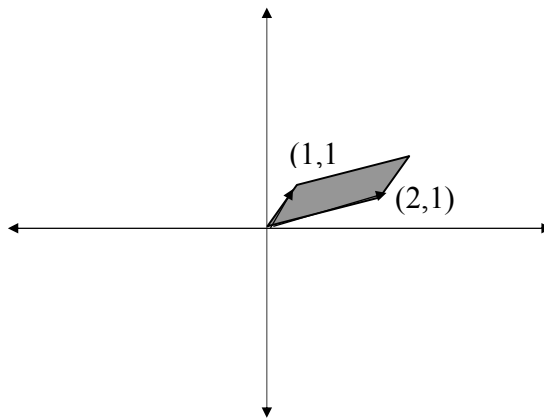


Figure: 2

As a special; let's take $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ unit matrix. From algebraic definition $\det A = 1$ is

obtained. Unit matrix' $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ line vectors' volume composed in

three-dimensional space is presented in Figure: 3.

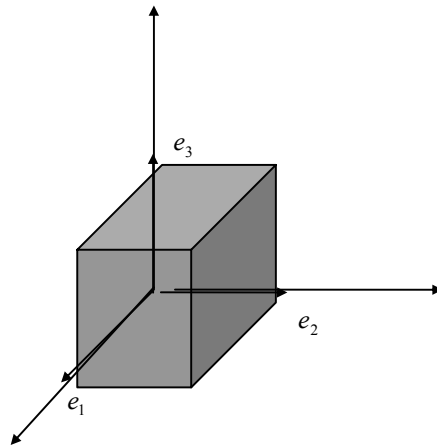


Figure 3

The Picture obtained in Figure: 3 is a 1 unit edged cube. A cube's volume, one edge's length is a unit, is a^3 . So the volume of cube in the figure is 1 br^3 . So it is $\det I = 1$. When doing confirmation of determinants' features; by benefiting from geometric presentations of determinant function it is possible to make a more meaningful explanation of it. For example; from determinants' features, If a A matrix' lines or columns are zero, it is $\det A = 0$.

Take $A = \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 0 & e & f \end{bmatrix}$ matrix, any of its columns is zero; A matrix' line vectors' shape constituted in a three-dimensional space is as below;

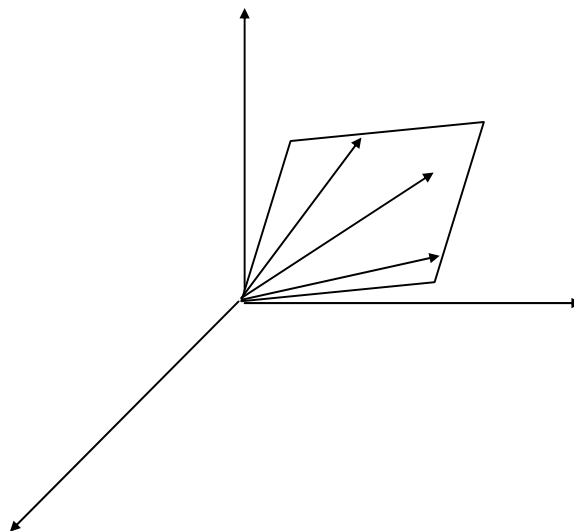


Figure: 4

As seen in Figure 4, A matrix' lines vectors do not point out a volume in three-dimensional space. So it becomes $\det A = 0$.

If A matrix' any of its lines is common of another line, and any of its columns is a common of another column, it is $\det A = 0$.

$A \in M_{2 \times 2}(R)$ matrix be $A = \begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$. As being $x = (a, b)$ and $y = (2a, 2b)$, A matrix' lines vectors' area is as in Figure 5:

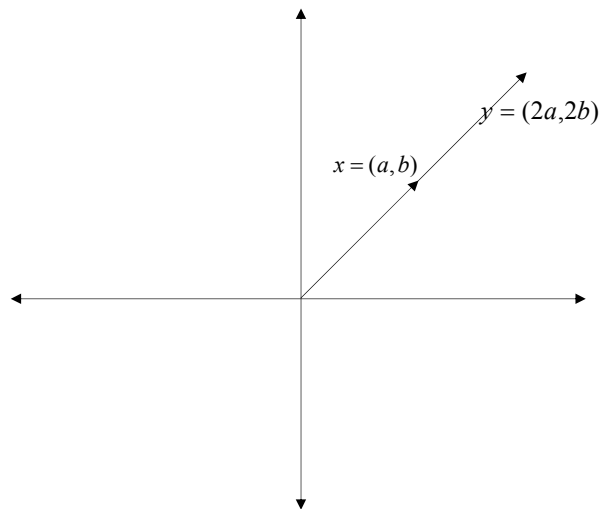


Figure.5

As seen in Figure: 5, A matrix' above line vectors composed a straight line. The straight line does not state area in a two-dimensional space. That is, its area is zero. So A matrix's determinant equals to zero. So it is $\det A = 0$.

METHOD

The Exemplification of the Work

The subject of this study consist of total 82 sophomore students who are in homogeny two different classes (2-A and 2-B) and are taught by the same lecturer, in primary mathematics education, in Kazım Karabekir Education Faculty, in Atatürk University.

The treatment was carried out in second semester of 2005-2006 teaching season. One of the classes was randomly selected as experimental group in which concretization method through geometry used, the other as control group in which traditional teaching methods were used.

Collection of Data

Data has been collected with two ways in this study.

- By searching over the literature concerning the subject; a theoretical basis is established for interpreting the findings and introducing suggestions.
- Information has been gathered by applying Determinant Knowledge Test, Mathematics Attitude Test and Scientific Process Skill Test as pre-test,

- Determinant Knowledge Test as post-test to experimental and control groups.
- Reliability Coefficient of Determinant Knowledge Test been found as 0,75.
- Reliability Coefficient of Mathematics Attitude Test has been found Cronbach Alfa as 0,96 (Duatepe, and Çilesiz, 1999).
- Reliability Coefficient of Scientific Process Skill Test has been found as 0,81 (Doğruöz, 1998).

Analysis of Data

At the testing of hypothesis, "Paired-Samples t test" and percentage-frequency is used. In this study; statistical analysis SPSS/PC (Statistical Package for Social Sciences for Personal Computers) are used as pocket programming. T-test analysis was made in the importance level of 0.05.

FINDINGS

The analysis results of the findings obtained from Determinant Knowledge Test that has been applied as pre-test, showed that there is not a significant difference statistically between two groups in the rate of achievement concerning determinants and their properties. ($\bar{x}_D = 0,697, \bar{x}_K = 0,930, t = 0,44, p = 0,66$). Mathematics Attitude Test has been applied as pre-test; and according to the analysis results obtained from this test, it is seen that there is not an important statistical difference between experimental and control groups at their attitudes toward mathematics. ($t = 1,149, p = 0,181$). Scientific Process Skill Test has been applied as pre-test, and according to results obtained from this test, it is seen that there is not an important difference between experimental and control groups at the aspect of scientific process skills ($\bar{x}_D = 21,650, \bar{x}_K = 21,765, p = 0,865$). From these results, we can say that experimental and control groups we composed during this study constitutes a homogeneous structure. The data obtained from Determinant Knowledge Test, applied as post-test to experimental and control groups at the end of the study, is as in the Table: 1.

Table: 1.

The findings concerning The Responds of Students to Determinant Knowledge Test

	N	\bar{x}	S	T	p
Deney Grubu	41	69,06	29,83	2,239	0,030
Kontrol Grubu	41	52,55	22,631		

The rate of The Responds of Students to follows question is presented in table: 2.

- If 2×2 matrix' lines or columns are common of each other, show that determinant equals to zero.
- If 3×3 matrix' any one of lines or columns is zero, show that determinant equals to zero.

Table: 2

The Findings concerning the responds students have given to questions.1 and 2.

Sorular	Doğru, %		Kismen Doğru, %		Yanlış, %		Cevapsız, %	
	DG	KG	DG	KG	DG	KG	DG	KG
1	74,5	51,2	11,6	18,6	13,90	20,9	0	9,3
2	41,8	39,5	23,4	27,9	23,2	23,2	11,6	9,4

THE RESULTS AND DISCUSSION

The analysis results obtained from the responds of students to linear algebra knowledge test, applied as post-test, showed that there is a statistically meaningful difference between each of two group's rate of success concerning determinant subject. It is seen that this difference is in the favor of students in experimental group which concretization method through geometry has been taken as principle.

That is to say, it is seen that the rate of success of students from experimental group in which concretization method through geometry is used is higher than that of students from control group in which traditional teaching methods have been applied. ($\bar{x}_D = 69,06$, $\bar{x}_K = 52,55$, $t = 2,239$ and $p = 0,030$).

Depending on the data above, we can say that concretization method through geometry is more efficient than traditional teaching method in the comprehension of determinant subject by students. But, when making concretization method through geometry, not only the geometrical information of students must be taken to consideration, but also a geometrical presentation in a suitable plan must be introduced.

In each of the question 1 and 2. at findings part, it is aimed to find whether determinants equaling to zero by looking the areas and volumes of figures of given matrix' line vectors comprised.

But, while the question 1 requires the use of geometric knowledge at R^2 , the question 2 requires the use of knowledge in R^3 . At these questions, while the rate of response to question 1. of students from experimental group which used geometry much more is %74,5; the rate of response to question 2. has fallen to %41,8.

The most important cause of such a big difference is because of that students get difficulty in showing vectors at R^3 and finding volumes obtained from these vectors. This shows us that for giving an efficient lesson by using concretization method through geometry, it is necessary for students to have adequate geometrical knowledge.

This result shows a parallelism with the results of studies such as "In the end of a research, Harel stated that geometry should be used carefully and appropriately (Harel, 1999). (Touger, 1986), stated that students get trouble when lessons are given with improper geometrical models, even some students find it harder to use such models than their own experiences".

As a result, in this study which was carried out with the aim of comparing the efficiency of concretization method through geometry and traditional teaching method in the teaching of determinants and their properties, according to the results obtained, we can say that concretization method through geometry may be beneficial in linear algebra lesson and in studies toward the aim of explaining subjects of the conceptual level.

The concretization of concepts and definitions in mathematics can be made through different ways. In this study, we have searched the efficiency of concretization method through geometry.

In the teaching of linear algebra lesson, the efficiency of concretization method through geometry can be investigated by the use of materials such as animation and computer programmes like Mathematica.

BIODATA and CONTACT ADDRESSES of AUTHOR



Yasin SOYLU was born at Oltu/Erzurum Turkey, in 1973. He graduated of Mathematics, Faculty of Education, at 19 May University, at 1995. After graduation, He worked as a mathematics teacher for a while.

He has Masters Degree in Educational Mathematics from Ataturk University Graduate School of Natural and Applied Science at 2001. Even so, He has Doctors Degree in Educational Mathematics from Ataturk University Graduate School of Natural and Applied

Science at 2005. He is working at the Department of Science, Faculty of Education Agri, at Ataturk University, since 1999.

Yasin SOYLU
Ataturk University
Agri Faculty of Education
Department of Science Education,
Agri, TURKEY.
Telephone: +905058408167
Emails: yasinsoylu@atauni.edu.tr or
yasinsoylu@gmail.com

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