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Research Article

Some Properties of r-Small Submodules

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Abstract

In this work, some properties of r-small submodules are investigated. It is proved that the finite sum of r-small submodules is r-small. It is also proved that every homomorphic image of an r-small submodule is r-small. Let M be an R-module and $N,K\leq M$. If $K\ll_r M$ and $(N+K)/K\ll_r M/K$, then $N\ll_r M$. Let $f:M\to N$ be an R-module epimorphism and $Kerf\ll_r M$. If $T\ll_r N$, then $f^{-1}(T)\ll_r M$. Let M be an R-module. Then $Rad(RadM)=\sum_{L\ll_r M} L$.

Keywords: Small Submodules, Maximal Submodules, Radical, Supplemented Modules.

r-Küçük Alt Modüllerinin Bazı Özellikleri

Öz

Bu çalışmada r-küçük alt modüllerle ilgili birtakım özellikler incelendi. r-küçük alt modüllerin sonlu toplamlarının r-küçük olduğu gösterildi. Ayrıca r-küçük alt modüllerin homomorfik görüntülerinin de r-küçük olduğu gösterildi. M bir R-modül ve $N,K\leq M$ olsun. Eğer $K\ll_r M$ ve $(N+K)/K\ll_r M/K$ ise $N\ll_r M$ olur. $f:M\to N$ bir R-modül epimorfizması ve $Kerf\ll_r M$ olsun. Eğer $T\ll_r N$ ise bu durumda $f^{-1}(T)\ll_r M$ olur. M bir R-modül olsun. Bu durumda $Rad(RadM)=\sum_{L\ll_r M} L$ olur.

Anahtar Kelimeler: Küçük Alt Modüller, Maksimal Alt Modüller, Radikal, Tümlenmiş Modüller.

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1. Introduction

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. Let M be an R-module and $U,V \leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. Let M be an R-module and $U,V \leq M$. If $M=U+V$ and $U \cap V \leq RadV$, then V is called a *generalized* (*Radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*Radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. Let M be an R-module and $K \leq V \leq M$. We say V *lies above* K in M if $V/K \ll M/K$.

More details about supplemented modules are in Clark et al. (2006), Nebiyev and Pancar (2013), Wisbauer (1991) and (Zöschinger, 1974). More details about generalized (Radical) supplemented modules are in Xue (1996) and (Wang and Ding, 2006).

2. Preliminaries

Lemma 2.1. Let M be an R-module. The following assertions are hold.

(1) If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.

(2) Let N be an R-module and $f: M \rightarrow N$ be an R-module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if f is an R-module epimorphism and $Kerf \ll M$.

(3) If $L \leq M$ and $K \ll L$, then $K \ll M$.

(4) If $K_1,K_2,\ldots,K_n \ll M$, then $K_1+K_2+\ldots+K_n \ll M$.

(5) Let $K_1,K_2,\ldots,K_n,L_1,L_2,\ldots,L_n \leq M$. If $K_i \ll L_i$ for every $i=1,2,\ldots,n$, then $K_1+K_2+\ldots+K_n \ll L_1+L_2+\ldots+L_n$.

Proof. See Clark et al. (2006), 2.2 and (Wisbauer, 1991, 19.3).

Lemma 2.2. Let M be an R-module. The following assertions are hold.

(1) $RadM = \sum_{L \ll M} L$.

(2) Let N be an R-module and $f: M \rightarrow N$ be an R-module homomorphism. Then $f(RadM) \leq RadN$. If $Kerf \leq RadM$, then $f(RadM)=Radf(M)$.

(3) If $N \leq M$, then $RadN \leq RadM$.

(4) For $K,L \leq M$, $RadK+RadL \leq Rad(K+L)$.

(5) $Rx \ll M$ for every $x \in RadM$.

Proof. See (Wisbauer, 1991, 21.5 and 21.6).

3- Theorems and Proofs

r-Small Submodules

Definition 3.1. Let M be an R-module and $N\leq M$. If $N\ll RadM$, then N is called a *radical small* (or briefly *r-small*) submodule of M and denoted by $N\ll_r M$ (See (Nebiyev and Ökten, 2020)).

Clearly we can see that if M is a radical module ($RadM=M$) and $N\leq M$, then $N\ll_r M$ if and only if $N\ll M$.

Proposition 3.2. Let M be an R-module and $N\leq M$. If $N\ll_r M$, then $N\ll M$.
Proof. Since $N\ll_r M$, $N\ll RadM$. Then by Lemma 2.1(3), $N\ll M$.

Lemma 3.3. Let $N\leq M$. If $N\ll M$ and $RadM$ is a supplement submodule in M, then $N\ll_r M$.
Proof. Since $N\ll M$ and $RadM$ is a supplement submodule in M, by Wisbauer (1991) 41.1(5), $N=N\cap RadM\ll RadM$. Hence $N\ll_r M$, as desired.

Corollary 3.4. Let $N\leq M$. If $N\ll M$ and $RadM$ is a direct summand of M, then $N\ll_r M$.
Proof. Clear from Lemma 3.3.

Corollary 3.5. Let M be an R-module and $RadM$ be a supplement submodule in M. Then $Rx\ll_r M$ for every $x\in RadM$.
Proof. Let $x\in RadM$. Then by Lemma 2.2(5), $Rx\ll M$. Since $RadM$ is a supplement submodule in M, by Lemma 3.3, $Rx\ll_r M$.

Corollary 3.6. Let M be an R-module and $RadM$ be a direct summand of M. Then $Rx\ll_r M$ for every $x\in RadM$.
Proof. Clear from Corollary 3.5.

Lemma 3.7. Let M be an R-module with $Rad(RadM)=RadM$. If $Rx\ll M$ for $x\in M$, then $Rx\ll_r M$.
Proof. Since $Rx\ll M$, by Lemma 1.2(1), $Rx\leq RadM$ and $x\in RadM=Rad(RadM)$. Then by Lemma 2.2(5), $Rx\ll RadM$ and $Rx\ll_r M$.

Corollary 3.8. Let M be an R-module and $RadM$ be a Rad-supplement submodule in M. If $Rx\ll M$ for $x\in M$, then $Rx\ll_r M$.
Proof. Clear from Lemma 3.7, since $Rad(RadM)=RadM$.

Corollary 3.9. Let M be an R-module with $Rad(RadM)=RadM$. Then $Rx\ll_r M$ for every $x\in RadM$.
Proof. Let $x\in RadM$. By Lemma 2.2(5), $Rx\ll M$. Then by Lemma 3.7, $Rx\ll_r M$.

Corollary 3.10. Let M be an R-module and $RadM$ be a Rad-supplement submodule in M. Then $Rx\ll_r M$ for every $x\in RadM$.
Proof. Clear from Corollary 3.9, since $Rad(RadM)=RadM$.

Proposition 3.11. If $N \ll_r M$, then $N \ll K$ for every maximal submodule K of M.
Proof. Since $N \ll_r M$, $N \ll RadM$ and since $RadM \leq K$ for every maximal submodule K of M, by Lemma 2.1(3), $N \ll K$.

Let M be an R-module. It is defined the relation β^* on the set of submodules of an R-module M by $X\beta^*Y$ if and only if $Y+K=M$ for every $K \leq M$ such that $X+K=M$ and $X+T=M$ for every $T \leq M$ such that $Y+T=M$ (See (Birkenmeier et al., 2010)).

Lemma 3.12. Let $N \ll_r M$ and $L \leq RadM$. If $N\beta^*L$ in $RadM$, then $L \ll_r M$.
Proof. Clear from Birkenmeier et al. (2010) Theorem 2.6(*i*).

Corollary 3.13. Let $N \ll_r M$ and $L \leq RadM$. If L lies above N in $RadM$, then $L \ll_r M$.
Proof. Since L lies above N in $RadM$, $N\beta^*L$ in $RadM$. Then by Lemma 3.12, $L \ll_r M$ holds.

Lemma 3.14. Let M be an R-module and $N \leq K \leq M$. If $N \ll_r K$, then $N \ll_r M$.
Proof. Since $N \ll_r K$, $N \ll RadK$. By Lemma 2.2(3), $RadK \leq RadM$. Then by Lemma 2.1(3), $N \ll RadM$ and $N \ll_r M$.

Lemma 3.15. Let M be an R-module and $N \leq K \leq M$. If $K \ll_r M$, then $N \ll_r M$.
Proof. Since $K \ll_r M$, $K \ll RadM$. Then by Lemma 2.1(1), $N \ll RadM$. Hence $N \ll_r M$, as desired.

Lemma 3.16. Let M be an R-module and $N,K \leq M$. If $N \ll_r M$, then $(N+K)/K \ll_r M/K$.
Proof. Since $N \ll_r M$, $N \ll RadM$. By Lemma 2.1, $(N+K)/K \ll (RadM+K)/K$. By Lemma 2.2, $(RadM+K)/K \leq Rad(M/K)$. Then by Lemma 2.1(3), $(N+K)/K \ll Rad(M/K)$. Hence $(N+K)/K \ll_r M/K$, as desired.

Lemma 3.17. Let M be an R-module and $N,K \leq M$. If $K \ll_r M$ and $(N+K)/K \ll_r M/K$, then $N \ll_r M$.
Proof. Since $K \leq RadM$, $Rad(M/K)=(RadM)/K$. Since $(N+K)/K \leq Rad(M/K)=(RadM)/K$, $N \leq RadM$. Let $N+T=RadM$ for $T \leq RadM$. Then $(N+K)/K+(T+K)/K=(N+T)/K=(RadM)/K$ and since $(N+K)/K \ll Rad(M/K)=(RadM)/K$, $(T+K)/K=(RadM)/K$ and $T+K=RadM$. Since $K \ll RadM$, $T=RadM$. Hence $N \ll RadM$ and $N \ll_r M$, as required.

Corollary 3.18. Let M be an R-module, $N \leq M$ and $K \ll_r M$. Then $N \ll_r M$ if and only if $(N+K)/K \ll_r M/K$.
Proof. Clear from Lemma 3.16 and Lemma 3.17.

Lemma 3.19. Let $f: M \rightarrow N$ be an R-module homomorphism. If $K \ll_r M$, then $f(K) \ll_r f(M)$.
Proof. Since $K \ll_r M$, $K \ll RadM$. By Lemma 2.1(2) and Lemma 2.2(2), $f(K) \ll f(RadM) \leq Radf(M)$. Hence $f(K) \ll_r f(M)$, as desired.

Corollary 3.20. Let $f: M \rightarrow N$ be an R-module homomorphism. If $K \ll_r M$, then $f(K) \ll_r N$.
Proof. Clear from Lemma 3.19 and Lemma 3.14.

Lemma 3.21. Let $f: M \to N$ be an R-module epimorphism and $Kerf \ll_r M$. If $T \ll_r N$, then $f^{-1}(T) \ll_r M$.

Proof. By Lemma 1.2(2), $f(RadM)=Radf(M)=RadN$. Let $x \in f^{-1}(T)$. Then $f(x) \in T \leq RadN= f(RadM)$. Since $f(x) \in f(RadM)$, there exists $a \in RadM$ with $f(x)=f(a)$. Since $f(x)=f(a)$, $f(x-a)=0$ and $x-a \in Kerf \leq RadM$. Then $x=x-a+a \in RadM$ and $f^{-1}(T) \leq RadM$. Let $f^{-1}(T)+K=RadM$ with $K \leq RadM$. Then $T+f(K)=f(RadM)=RadN$ and since $T \ll RadN$, $f(K)=RadN$. Since $f(K)=RadN$, we can see that $K+Kerf=RadM$ and since $Kerf \ll RadM$, $K=RadM$. Hence $f^{-1}(T) \ll_r M$, as required.

Corollary 3.22. Let $f: M \to N$ be an R-module homomorphism and $Kerf \ll_r M$. If $T \ll_r f(M)$, then $f^{-1}(T) \ll_r M$.

Proof. Clear from Lemma 3.21.

Corollary 3.23. Let $f: M \to N$ be an R-module homomorphism and $K \leq RadM$. If $Kerf \ll_r M$ and $f(K) \ll_r f(M)$, then $K \ll_r M$.

Proof. By Corollary 3.22, $K+Kerf=f^{-1}(f(K)) \ll_r M$. Then by Lemma 3.15, $K \ll_r M$, as desired.

Corollary 3.24. Let $f: M \to N$ be an R-module epimorphism and $K \leq RadM$. If $Kerf \ll_r M$ and $f(K) \ll_r N$, then $K \ll_r M$.

Proof. Clear from Corollary 3.23.

Corollary 3.25. Let $f: M \to N$ be an R-module epimorphism and $K \leq RadM$. If $Kerf \ll_r M$, then $K \ll_r M$ if and only if $f(K) \ll_r N$.

Proof. Clear from Lemma 3.19 and Corollary 3.24.

Lemma 3.26. Let M be an R-module and $K,L \leq M$. If $N \ll_r K$ and $T \ll_r L$, then $N+T \ll_r K+L$.

Proof. Since $N \ll_r K$ and $T \ll_r L$, $N \ll RadK$ and $T \ll RadL$. By Lemma 2.1(5) and Lemma 2.2(4), $T+N \ll RadK+RadL \leq Rad(K+L)$. Hence $N+T \ll_r K+L$, as desired.

Corollary 3.27. Let $M_1, M_2, ..., M_k \leq M$. If $N_1 \ll_r M_1$, $N_2 \ll_r M_2$, ..., $N_k \ll_r M_k$, then $N_1+N_2+\ldots+N_k \ll_r M_1+M_2+\ldots+M_k$.

Proof. Clear from Lemma 3.26.

Corollary 3.28. Let M be an R-module and $K_i \ll_r M$ for $i=1,2,\ldots,n$. Then $K_1+K_2+\ldots+K_n \ll_r M$.

Proof. Clear from Corollary 3.27.

Corollary 3.29. Let M be an R-module. Then $Rad(RadM)= \sum_{L \ll_r M} L$.

Proof. Clear from Lemma 2.2(1).

Remark 3.30. The converse of the Proposition 3.2 is not true in general. Consider an R-module M with $0 \neq RadM \ll M$. Since $RadM$ is not small in $RadM$, $RadM$ is not r-small in M.

Example 3.31. Consider the Z-module Z_8. Here $RadZ_8=2Z_8 \ll Z_8$. But $2Z_8$ is not r-small in Z_8. Here also $Rad(RadZ_8)=4Z_8 \neq RadZ_8$.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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