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ASSOCIATED CURVES OF A FRENET CURVE IN THE DUAL LORENTZIAN SPACE

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Abstract. In this work, we firstly introduce notions of principal directed curves and principal donor curves which are associated curves of a Frenet curve in the dual Lorentzian space $\mathbb{D}^3_1$. We give some relations between the curvature and the torsion of a dual principal directed curve and the curvature and the torsion of a dual principal donor curve. We show that the dual principal directed curve of a dual general helix is a plane curve and obtain the equation of dual general helix by using position vector of plane curve. Then we show that the principal donor curve of a circle in $\mathbb{D}^2$ or a hyperbola in $\mathbb{D}^2_1$ and the principal directed curve of a slant helix in $\mathbb{D}^3_1$ are a helix and general helix, respectively. We explain with an example for the second case. Finally, according to causal character of the principal donor curve of principal directed rectifying curve in $\mathbb{D}^3_1$, we show this curve to correspond to any timelike or spacelike ruled surface in Minkowski 3−space $\mathbb{R}^3_1$.

1. Introduction

It is very interesting to study curves in both dual space $\mathbb{D}^3$ and dual Lorentzian space $\mathbb{D}^3_1$. Because a differentiable curve on dual unit sphere in $\mathbb{D}^3$ represents a ruled surface in Euclidean 3−space $\mathbb{R}^3$ with the aid of the E. Study mapping. Similarly, a differentiable curve on dual pseudo hyperbolic space $\mathbb{H}^2_0$ in $\mathbb{D}^3_1$ corresponds to a timelike ruled surface in Minkowski 3−space $\mathbb{R}^3_1$ and the timelike (resp. spacelike) curve on dual pseudo sphere $\mathbb{S}^2_1$ in $\mathbb{D}^3_1$ corresponds to any spacelike (resp. timelike) ruled surface in $\mathbb{R}^3_1$. Therefore, we can say something about ruled surfaces in $\mathbb{R}^3$ or $\mathbb{R}^3_1$ when examining curves in $\mathbb{D}^3$ or $\mathbb{D}^3_1$, respectively [9, 16–18].

Keywords. Dual Lorentzian space, associated curves, dual general helix, dual slant helix, principal directed rectifying curve, ruled surface.

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In this paper, we examine associated curves of a Frenet curve in $\mathbb{D}_1^3$ and show these curves to correspond to any timelike or spacelike ruled surfaces in Minkowski 3−space $\mathbb{R}_1^3$. For this purpose, we recall the fundamental properties of $\mathbb{R}_1^3$ and $\mathbb{D}_1^3$.

$\mathbb{R}_1^3$ is the 3−dimensional Lorentzian space (or Minkowski 3−space) with symmetric, bilinear and non-degenerate metric given by

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3$$

for vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in Euclidean 3−space $\mathbb{R}^3$. In $\mathbb{R}_1^3$, the Lorentzian vector product of u and v is defined by

$$u \times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

We know that a vector and a curve have three different categories, namely, spacelike, timelike and null, depending on their causal characters. Then a vector u is said to be spacelike, timelike or null (lightlike) if $\langle u, u\rangle > 0$ (or $u = 0$), $\langle u, u\rangle < 0$, $\langle u, u\rangle = 0$ (and $u \neq 0$), respectively. Similarly, a curve γ is called spacelike, timelike or null (lightlike) if its velocity vector is spacelike, timelike or null vector, respectively. We also state that Frenet curves are timelike curves and spacelike curves with spacelike or timelike principal normal vector. Lastly, a surface is named non-degenerate (or degenerate) if induced metric on its tangent plane is non-degenerate (or degenerate). The pseudo sphere of radius $r > 0$ in $\mathbb{R}_1^3$ denoted by

$$S_1^2 = \{p \in \mathbb{R}_1^3 : \langle p, p\rangle = r^2,\ r > 0\}$$

and the pseudo hyperbolic space of radius $r > 0$ in $\mathbb{R}_1^3$ denoted by

$$H_0^2 = \{p \in \mathbb{R}_1^3 : \langle p, p\rangle = -r^2,\ r > 0\}$$

are non-degenerate surfaces [2, 12, 13].

A number expressed as

$$\widehat{a} = a + \xi a^* \text{ or } \widehat{a} = (a, a^*)$$

is called a dual number for $\forall a, a^* \in \mathbb{R}$ and the set of all dual numbers is indicated by $\mathbb{D}$, where ξ is called as dual unit with properties

$$\xi \neq 0,\ 0\xi = \xi 0 = 0,\ 1\xi = \xi 1 = \xi,\ \xi^2 = 0.$$

Equality and some operations on $\mathbb{D}$ are defined as follows:

i) Equality: $\widehat{a} = \widehat{b}$ for $\widehat{a} = a + \xi a^*$, $\widehat{b} = b + \xi b^*$ iff $a = b$ and $a^* = b^*$.

ii) Addition: $\widehat{a} + \widehat{b} = (a + \xi a^*) + (b + \xi b^*) = (a + b) + \xi(a^* + b^*)$.

iii) Multiplication: $\widehat{ab} = (a + \xi a^*)(b + \xi b^*) = ab + \xi(ab^* + a^*b)$.

iv) Division: $\frac{\widehat{a}}{\widehat{b}} = \frac{a}{b} + \xi(\frac{a^*b - ab^*}{b^2})$, $b \neq 0$.

We note that $\mathbb{D}$ is a commutative ring according to the above addition and multiplication operations. Also f on $\mathbb{D}$ is defined by

$$f(\widehat{a}) = f(a + \xi a^*) = f(a) + \xi a^* f'(a),$$

where $f^{'}$ represents the derivative of f . For example,

$$\sin(\widehat{a}) = \sin(a + \xi a^*) = \sin a + \xi a^* \cos a$$

(see [9, 17, 19] for more details).

A dual vector $\widehat{x}$ is an ordered triple of dual numbers $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ and also a dual vector $\widehat{x}$ has the form $\widehat{x} = x + \xi x^*$ for $\forall x = (x_1, x_2, x_3)$, $x^* = (x_1^*, x_2^*, x_3^*) \in \mathbb{R}^3$, where x and x^* are the real and dual parts of $\widehat{x}$, respectively. The set of all dual vectors which is denoted as $\mathbb{D}^3$ is a module on the ring $\mathbb{D}$. The Lorentzian inner product of dual vectors $\widehat{x}$ and $\widehat{y}$ is defined by

$$\langle \widehat{x}, \widehat{y} \rangle = \langle x, y \rangle + \xi(\langle x, y^* \rangle + \langle x^*, y \rangle).$$

The dual space $\mathbb{D}^3$ together with this Lorentzian inner product is called dual Lorentzian space and it is represented by $\mathbb{D}^3_1$. The causal characterization of a dual vector $\widehat{x} = x + \xi x^*$ depends on the causal characterization of x, that is a dual vector $\widehat{x}$ is called to be spacelike, timelike, null (lightlike) if the vector x is spacelike, timelike, null (lightlike), respectively. The Lorentzian vector product of dual vectors $\widehat{x} = (\widehat{x_1}, \widehat{x}_2, \widehat{x}_3)$ and $\widehat{y} = (\widehat{y}_1, \widehat{y}_2, \widehat{y}_3)$ in $\mathbb{D}^3_1$ is defined by

$$\widehat{x} \times \widehat{y} = (\widehat{x}_3\widehat{y}_2 - \widehat{x}_2\widehat{y}_3, \widehat{x}_3\widehat{y}_1 - \widehat{x}_1\widehat{y}_3, \widehat{x}_1\widehat{y}_2 - \widehat{x}_2\widehat{y}_1).$$

If $x \neq 0$, then the norm of $\widehat{x}$ is given by

$$\|\widehat{x}\| = \sqrt{|< \widehat{x}, \widehat{x} >|} = \|x\| + \xi \frac{< x, x^* >}{\|x\|^2}.$$

A dual vector $\widehat{x}$ with norm $1+\xi 0 = (1, 0) \in \mathbb{D}$ is called a dual unit vector. Therefore, dual pseudo sphere and dual pseudo hyperbolic space are defined by

$$\mathbb{S}^2_1 = \left\{ \widehat{x} = x + \xi x^* | \ \ \|\widehat{x}\| = (1, 0) \ ; \ x, \ x^* \in \mathbb{R}^3_1 \text{ and the vector } \widehat{x} \text{ is spacelike} \right\}$$

and

$$\mathbb{H}^2_0 = \left\{ \widehat{x} = x + \xi x^* | \ \ \|\widehat{x}\| = (1, 0) \ ; \ x, \ x^* \in \mathbb{R}^3_1 \text{ and the vector } \widehat{x} \text{ is timelike} \right\},$$

respectively.

Let $\widehat{\gamma}(\sigma) = \gamma(\sigma) + \xi\gamma^*(\sigma)$ be a dual curve with parameter $\sigma \in \mathbb{R}$ in $\mathbb{D}^3_1$. The real curve $\gamma(\sigma)$ is called the (real) indicatrix of $\widehat{\gamma}(\sigma)$. If every $\gamma(\sigma)$ and $\gamma^*(\sigma)$ are differentiable, then $\widehat{\gamma}(\sigma)$ is differentiable in $\mathbb{D}^3_1$. The dual arc length of the dual curve $\widehat{\gamma}$ is given by

$$\widehat{s} = \int_0^s \|\widehat{\gamma}(\sigma)'\| \, d\sigma = \int_0^s \|\gamma(\sigma)'\| \, d\sigma + \xi \int_0^s < t, \gamma^*(\sigma) > d\sigma = s + \xi s^*,$$

where s and t is arclength and the unit tangent vector of γ, respectively. As in $\mathbb{R}^3_1$ we call timelike dual curves and spacelike dual curves with spacelike or timelike dual principal normal vector as dual Frenet curves (or Frenet curves in $\mathbb{D}^3_1$). Assume that $\widehat{\gamma}$ is a reparametrization curve with the parametrization s of the indicatrix. Hence the dual Frenet formulae for the dual unit speed Frenet curve $\widehat{\gamma}$ are

$$\frac{d}{d\widehat{s}}\begin{bmatrix}\widehat{t}\\ \widehat{n}\\ \widehat{b}\end{bmatrix}=\begin{bmatrix}0 & \widehat{\kappa} & 0\\ -\varepsilon_0\varepsilon_1\widehat{\kappa} & 0 & \widehat{\tau}\\ 0 & -\varepsilon_1\varepsilon_2\widehat{\tau} & 0\end{bmatrix}\begin{bmatrix}\widehat{t}\\ \widehat{n}\\ \widehat{b}\end{bmatrix}, \tag{1}$$

such that $< t,t >= \varepsilon_0 = \pm1$, $< n,n >= \varepsilon_1 = \pm1$ and $< b,b >= \varepsilon_2 = \pm1$, where

$$\begin{array}{rccl}\widehat{\kappa}: & \mathbb{R} & \to & \mathbb{D}\\ & s & \to & \widehat{\kappa}(s)=\kappa(s)+\xi\kappa^*(s)\end{array}$$

is nowhere pure dual curvature and

$$\begin{array}{rccl}\widehat{\tau}: & \mathbb{R} & \to & \mathbb{D}\\ & s & \to & \widehat{\tau}(s)=\tau(s)+\xi\tau^*(s)\end{array}$$

is nowhere pure dual torsion [4,14,16–20].

Let $\widehat{\gamma}$ be a dual unit speed Frenet curve in $\mathbb{D}^3_1$ and $\widehat{W}$ be a dual unit vector field along $\widehat{\gamma}$. The curve $\widehat{\gamma}_0$ in $\mathbb{D}^3_1$ is called the $\widehat{W}-$directional dual curve of $\widehat{\gamma}$ if the dual unit tangent vector $\widehat{t}_0$ of $\widehat{\gamma}_0$ is equal to $\widehat{W}$. Moreover $\widehat{\gamma}$ is called the $\widehat{W}-$donor dual curve of $\widehat{\gamma}_0$. Thus, we can define three different dual curves by special selection of $\widehat{W}$:

i) If $\widehat{W}=\widehat{t}$, then $\widehat{t}_0=\widehat{t}$. In this case $\widehat{\gamma}$ and $\widehat{\gamma}_0$ are the same dual curves.

ii) If $\widehat{W}=\widehat{n}$, then $\widehat{t}_0=\widehat{n}$. In this case $\widehat{\gamma}_0$ is called the dual principal directional curve of $\widehat{\gamma}$ and $\widehat{\gamma}$ is called the dual principal donor curve of $\widehat{\gamma}_0$.

iii) If $\widehat{W}=\widehat{b}$ then $\widehat{t}_0=\widehat{b}$. In this case $\widehat{\gamma}_0$ is called the dual binormal directional curve of $\widehat{\gamma}$ and $\widehat{\gamma}$ is called the dual binormal donor curve of $\widehat{\gamma}_0$ [1,7,8,11].

In this paper, we obtain firstly some relations between the curvature and the torsion of a principal directed curve and the curvature and the torsion of a principal donor curve in $\mathbb{D}^3_1$. We see that the principal directed curve of a dual general helix is a plane curve and give the equation of a dual general helix by using position vector of a plane curve. Then we show that the principal donor curve of a circle in $\mathbb{D}^2$ or a hyperbola in $\mathbb{D}^2_1$ is a dual helix and we also obtain that the principal directed curve of a dual slant helix is a dual general helix. We give an example for simple closed dual slant helix. Finally, according to causal character of the principal donor curve of a principal directed rectifying curve in $\mathbb{D}^3_1$, we show that this curve to correspond to any timelike or spacelike ruled surface in Minkowski 3−space $\mathbb{R}^3_1$.

2. Principal Directional and Principal Donor Curves of a Frenet Curve in $\mathbb{D}^3_1$

In this section, we examine principal directional and principal donor curves of a Frenet curve in the dual Lorentzian space $\mathbb{D}^3_1$. Firstly, we state that the causal characterization of a curve $\widehat{\gamma}$ in $\mathbb{D}^3_1$ depends on the causal characterization of a curve

γ which is the real part of $\widehat{\gamma}$. Then we give the following Lemma from Lemma 3.1 in [8].

Lemma 1. *There is no timelike dual general helix or spacelike dual general helix with spacelike principal normal that provides the condition* $\left|\frac{\widehat{\tau}}{\widehat{\kappa}}\right| = (1,0)$ *in the dual Lorentzian space* $\mathbb{D}_1^3$.

Now we give the following theorem which expresses the relationship between the dual curvature and torsion of $\widehat{\gamma}(s)$ and the dual curvature and torsion of $\widehat{\gamma}_0$ which is the principal direction of $\widehat{\gamma}$.

Theorem 1. *Let* $\widehat{\gamma}$ *be a dual unit speed Frenet curve with the dual curvature* $\widehat{\kappa}$ *and the dual torsion* $\widehat{\tau}$ *and* $\widehat{\gamma}_0$ *be the principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}_1^3$. *Then the dual curvature* $\widehat{\kappa}_0$ *and the dual torsion* $\widehat{\tau}_0$ *of* $\widehat{\gamma}_0$ *is*

$$\widehat{\kappa}_0 = \sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}, \quad \widehat{\tau}_0 = \frac{\widetilde{\varepsilon}_2\varepsilon_1\varepsilon_2\widehat{\kappa}^2}{\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right), \tag{2}$$

where $\varepsilon_0 =< t,t >$, $\varepsilon_1 =< n,n >$, $\varepsilon_2 =< b,b >$, $\widetilde{\varepsilon}_1 =< n_0,n_0 >$ *and* $\widetilde{\varepsilon}_2 =< b_0,b_0 >$ *such that* $\{t,n,b\}$ *and* $\{t_0,n_0,b_0\}$ *Frenet frames of the curves* γ *and* γ_0, *respectively.*

Proof. Since $\widehat{\gamma}_0$ is the principal direction curve of a dual unit speed Frenet curve $\widehat{\gamma}$, the equations $\widehat{t}_0 = \widehat{n}$ and $\frac{d\widehat{t}_0}{d\widehat{s}} = \frac{d\widehat{n}}{d\widehat{s}}$ are provided. Considering the dual Frenet formulae (1) we have

$$\frac{d\widehat{t}_0}{d\widehat{s}} = -\varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{t} + \widehat{\tau}\widehat{b}$$

and

$$\widehat{\kappa}_0^2 < \widehat{n}_0, \widehat{n}_0 >= \varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2.$$

Therefore, we obtain

$$< n_0, n_0 >= \frac{\varepsilon_0\kappa^2 + \varepsilon_2\tau^2}{\kappa_0^2}. \tag{3}$$

The dual curvature of $\widehat{\gamma}$ is also

$$\widehat{\kappa}_0 = \sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}.$$

Thus, the dual Frenet vectors along $\widehat{\gamma}_0$ are

$$\widehat{t}_0 = \widehat{n},\ \ \widehat{n}_0 = \frac{-\varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{t} + \widehat{\tau}\widehat{b}}{\sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}}, \widehat{b}_0 = \widetilde{\varepsilon}_0\widetilde{\varepsilon}_1\frac{\widehat{\kappa}\widehat{b} + \varepsilon_1\varepsilon_2\widehat{\tau}\widehat{t}}{\sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}}. \tag{4}$$

By taking differentiation of equation (4) with respect to $\widehat{s}$ and this is written in the equation

$$\widehat{\tau}_0 = -\widetilde{\varepsilon}_2 < \frac{d\widehat{b}_0}{d\widehat{s}}, \widehat{n}_0 >,$$

we have

$$\widehat{\tau}_0 = -\widetilde{\varepsilon}_2 < \left(\varepsilon_0\varepsilon_2 \frac{\widehat{\kappa}\left(\widehat{\kappa}\frac{d\widehat{\tau}}{d\widehat{s}} - \widehat{\tau}\frac{d\widehat{\kappa}}{d\widehat{s}}\right)}{\left(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)\right)^{\frac{3}{2}}}\right)\widehat{t} - \left(\varepsilon_1\varepsilon_2 \frac{\widehat{\tau}\left(\widehat{\kappa}\frac{d\widehat{\tau}}{d\widehat{s}} - \widehat{\tau}\frac{d\widehat{\kappa}}{d\widehat{s}}\right)}{\left(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)\right)^{\frac{3}{2}}}\right)\widehat{b}, \frac{-\varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{t} + \widehat{\tau}\widehat{b}}{\sqrt{\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right)}} > .$$

Then we get

$$\widehat{\tau}_0 = \frac{\widetilde{\varepsilon}_2\varepsilon_1\varepsilon_2\widehat{\kappa}^2}{\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right).$$

□

We can write the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ of $\widehat{\gamma}$ in terms of the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ of $\widehat{\gamma}_0$ in the following theorem:

Theorem 2. *Let $\widehat{\gamma}$ be a dual unit speed spacelike Frenet curve having a spacelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ be a spacelike principal direction of $\widehat{\gamma}$ in $\mathbb{D}^3_1$.*

(a) If $\kappa > |\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with spacelike dual principal normal. Then the curvature and the torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\cosh(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = \widehat{\kappa}_0(s)\sinh(\int \widehat{\tau}_0(s)d\widehat{s}) \tag{5}$$

(b) If $\kappa < |\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with timelike dual principal normal. Then the curvature and the torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s) = \widehat{\kappa}_0(s)\sinh(\int \widehat{\tau}_0(s)d\widehat{s}), \qquad \widehat{\tau}(s) = -\widehat{\kappa}_0(s)\cosh(\int \widehat{\tau}_0(s)d\widehat{s}) \tag{6}$$

Proof. (*a*) If $\kappa > |\tau|$, as a result of (3), $\widehat{\gamma}_0$ is a spacelike dual curve with spacelike dual principal normal. Then by using (2) the curvature and the torsion functions of $\widehat{\gamma}_0$ are,

$$\widehat{\kappa}_0^2(s) = \widehat{\kappa}^2(s) - \widehat{\tau}^2(s), \quad \widehat{\tau}_0(s) = \frac{\widehat{\kappa}^2(s)}{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right) \tag{7}$$

respectively. Firstly we replace $\frac{\widehat{\tau}}{\widehat{\kappa}}$ in the second equation of (7) with $\widehat{f}$. Then the second equation of (7) is rewritten as

$$\widehat{\tau}_0(s) = \frac{1}{1 - \left(\frac{\widehat{\tau}(s)}{\widehat{\kappa}(s)}\right)^2}\frac{d\widehat{f}(s)}{d\widehat{s}} = \frac{1}{1 - \widehat{f}^2(s)}\frac{d\widehat{f}(s)}{d\widehat{s}},$$

where

$$\widehat{f}(s) = f(s) + \xi f^*(s) = \frac{\tau(s)}{\kappa(s)} + \xi(\frac{\tau^*(s)}{\kappa(s)} - \frac{\tau(s)\kappa^*(s)}{\kappa^2(s)}).$$

On the other hand, since $\kappa > |\tau|$, $|f(s)|$ is less than 1. Thus, we get that

$$\int \widehat{\tau}_0(s)d\widehat{s} = \int \frac{\frac{d\widehat{f}(s)}{d\widehat{s}}}{1 - \widehat{f}^2(s)}d\widehat{s} = \tanh^{-1}\widehat{f}(s) + \widehat{c},$$

where $\widehat{c}$ is dual constant. If we take $\widehat{c}=0$ without breaking the generality, then we obtain

$$\widehat{f}(s)=\tanh\left(\int\widehat{\tau}_0(s)d\widehat{s}\right).$$

By using $\widehat{f}=\frac{\widehat{\tau}}{\widehat{\kappa}}$ we obtain

$$\widehat{\tau}(s)=\tanh\left(\int\widehat{\tau}_0(s)d\widehat{s}\right)\widehat{\kappa}(s).$$

If this equation is written in place of the first equation of (7) and the necessary arrangements are made, then both equations of (5) are obtained.

(b) The proof is similar to the proof of the statement (a). □

Similarly, we can write Theorem 3 and Theorem 4.

Theorem 3. *Let $\widehat{\gamma}$ be a dual unit speed spacelike Frenet curve having a timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ be a timelike principal direction of $\widehat{\gamma}$ in $\mathbb{D}_1^3$. Then the dual curvature and the dual torsion of principal donor curve of $\widehat{\gamma}_0$ are*

$$\widehat{\kappa}(s)=\widehat{\kappa}_0(s)\cos(\int\widehat{\tau}_0(s)d\widehat{s}),\qquad \widehat{\tau}(s)=-\widehat{\kappa}_0(s)\sin(\int\widehat{\tau}_0(s)d\widehat{s}). \tag{8}$$

Theorem 4. *Let $\widehat{\gamma}$ be a dual unit speed timelike Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ be principal direction of $\widehat{\gamma}$ in $\mathbb{D}_1^3$.*
(a) If $\kappa<|\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with spacelike dual principal normal. Then the dual curvature and the dual torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s)=\widehat{\kappa}_0(s)\sinh(\int\widehat{\tau}_0(s)d\widehat{s}),\qquad \widehat{\tau}(s)=\widehat{\kappa}_0(s)\cosh(\int\widehat{\tau}_0(s)d\widehat{s}). \tag{9}$$

(b) If $\kappa>|\tau|$, then $\widehat{\gamma}_0$ is a spacelike dual curve with timelike dual principal normal. Then the dual curvature and the dual torsion of principal donor curve of $\widehat{\gamma}_0$ are

$$\widehat{\kappa}(s)=\widehat{\kappa}_0(s)\cosh(\int\widehat{\tau}_0(s)d\widehat{s}),\qquad \widehat{\tau}(s)=-\widehat{\kappa}_0(s)\sinh(\int\widehat{\tau}_0(s)d\widehat{s}). \tag{10}$$

3. Principal Directional Curves of General Helices in $\mathbb{D}_1^3$

In this section, we show that principal directional curves of general helices in $\mathbb{D}_1^3$ is plane curves. Then we obtain the position vectors of dual general helices with the aid of this plane curves (see [5, 12] for general helix in $\mathbb{R}_1^3$).

Theorem 5. *A dual unit speed Frenet curve $\widehat{\gamma}$ in $\mathbb{D}_1^3$ is a general helix iff the principal directional curve of $\widehat{\gamma}$ is a plane curve.*

Proof. Since $\widehat{\gamma}$ is a dual unit speed Frenet curve, we will only give the proof for a spacelike dual Frenet curve with timelike principal normal.

($\Rightarrow$) Let $\widehat{\gamma}(s)$ be a dual unit speed Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and $\widehat{\gamma}_0$ be the principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Then it is clear that

$$\frac{\widehat{\tau}}{\widehat{\kappa}} = -\tan\left(\int \widehat{\tau}_0(s)d\widehat{s}\right) \tag{11}$$

from the equation (8). By taking derivative of (11) with respect to $\widehat{s}$ we have

$$\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right) = -\widehat{\tau}_0(s)\sec^2\left(\int \widehat{\tau}_0(s)d\widehat{s}\right) = 0.$$

Since $\sec^2\left(\int \widehat{\tau}_0(s)d\widehat{s}\right) \neq 0$, we say that $\widehat{\tau}_0(s) = 0$. Then $\widehat{\gamma}_0$ is a plane curve in $\mathbb{D}^3_1$.

($\Leftarrow$) Let $\widehat{\gamma}_0$ which is principal directional curve of $\widehat{\gamma}$ be a plane curve in $\mathbb{D}^3_1$. Then $\widehat{\tau}_0 = 0$. As a result of $\widehat{\kappa} \neq 0$, $\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}}{\widehat{\kappa}}\right) = 0$ and $\frac{\widehat{\tau}}{\widehat{\kappa}}$ is a dual constant from (2). Consequently the Frenet curve $\widehat{\gamma}$ is a general helix in $\mathbb{D}^3_1$. □

Similarly, we can also prove in case $\widehat{\gamma}$ is a timelike curve or a spacelike curve with spacelike principal normal $\mathbb{D}^3_1$.

Theorem 6. *Let $\widehat{\gamma}$ be a spacelike plane curve with the dual curvature $\widehat{\kappa}$ in $\mathbb{D}^3_1$.*
(a) If the principal normal vector of $\widehat{\gamma}$ in $\mathbb{D}^3_1$ is a spacelike, then the position vector of $\widehat{\gamma}$ is given by

$$\widehat{\gamma}(s) = \int \left(0, \cos\left(\int \widehat{\kappa}(s)d\widehat{s}\right), \sin\left(\int \widehat{\kappa}(s)d\widehat{s}\right)\right) d\widehat{s}, \tag{12}$$

(b) If the principal normal vector of $\widehat{\gamma}$ in $\mathbb{D}^3_1$ is a timelike, then the position vector of $\widehat{\gamma}$ is given by

$$\widehat{\gamma}(s) = \int \left(\sinh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), \cosh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), 0\right) d\widehat{s}. \tag{13}$$

Proof. Let $\widehat{\gamma}$ be a spacelike plane curve with the dual curvature $\widehat{\kappa}$ in $\mathbb{D}^3_1$. Since $\widehat{\gamma}$ is a spacelike dual curve, $\left\langle \widehat{t}, \widehat{t}\right\rangle = (1, 0)$. On the other hand if we consider the dual Frenet formulae (1) and $\widehat{\theta} = \int \widehat{\kappa}(s)d\widehat{s}$, then the following statements hold:
(a) If the principal normal vector of $\widehat{\gamma}$ is spacelike, then $\widehat{t}(s) = \left(0, \cos\widehat{\theta}, \sin\widehat{\theta}\right)$. Therefore, we have the equation (12).
(b) If the principal normal vector of $\widehat{\gamma}$ is timelike, then $\widehat{t}(s) = \left(\sinh\widehat{\theta}, \cosh\widehat{\theta}, 0\right)$. Therefore, we have the equation (13). □

Theorem 7. *The position vector $\widehat{\gamma}$ of a timelike plane curve with the dual curvature $\widehat{\kappa}$ in $\mathbb{D}^3_1$ is given by*

$$\widehat{\gamma}(s) = \int \left(\cosh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), \sinh\left(\int \widehat{\kappa}(s)d\widehat{s}\right), 0\right) d\widehat{s}. \tag{14}$$

Theorem 8. *Let $\widehat{\gamma}$ be a dual unit speed spacelike general helix having a spacelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau} = \widehat{m}\widehat{\kappa}$ for dual constant $\widehat{m} = m + \xi m^*$ in $\mathbb{D}_1^3$.*
(a) If $\frac{|\tau|}{\kappa} = |m| < 1$, then the position vector $\widehat{\gamma}$ is given by

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{1-\widehat{m}^2}} \int \left(\widehat{m}, \sin\left(\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right), -\cos\left(\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right) \right) d\widehat{s}, \tag{15}$$

and the principal directional curve of $\widehat{\gamma}$ is a spacelike plane curve with a spacelike principal normal in $\mathbb{D}^2$,
(b) If $\frac{|\tau|}{\kappa} = |m| > 1$ then the position vector $\widehat{\gamma}$ is given by

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{\widehat{m}^2-1}} \int \left(\cosh\left(\sqrt{\widehat{m}^2-1} \int \widehat{\kappa}(s) d\widehat{s} \right), \sinh\left(\sqrt{\widehat{m}^2-1} \int \widehat{\kappa}(s) d\widehat{s} \right), \widehat{m} \right) d\widehat{s}. \tag{16}$$

and the principal directional curve of $\widehat{\gamma}$ is a spacelike plane curve with a timelike principal normal in $\mathbb{D}_1^2$.

Proof. Let $\widehat{\gamma}_0$ be principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}_1^3$. $\widehat{\gamma}_0$ is a spacelike dual curve because $\widehat{\gamma}$ has a spacelike principal normal.
In case (a) we can say that $\widehat{\gamma}_0$ has the dual Frenet vectors,

$$\begin{cases} \widehat{t_0}(s) = \left(0, \cos\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right], \sin\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right]\right) \\ \widehat{n}_0(s) = \left(0, -\sin\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right], \cos\left[\int \widehat{\kappa}_0(s) d\widehat{s}\right]\right) \\ \widehat{b}_0(s) = (1, 0, 0) \end{cases}$$

by using (12). If we consider the equation (11) and $0 < |m| < 1$, then the equations

$$\widehat{\kappa}(s) = \frac{\widehat{\kappa}_0(s)}{\sqrt{1-\widehat{m}^2}} \quad \text{and} \quad \widehat{\tau}(s) = \widehat{m}\widehat{\kappa}(s)$$

are hold. From (4) and $\widehat{\kappa}_0(s) = \widehat{\kappa}(s)\sqrt{1-\widehat{m}^2}$, the dual unit tangent vector $\widehat{t}$ is obtained as

$$\widehat{t} = \frac{1}{\sqrt{1-\widehat{m}^2}} \left(\widehat{m}, \sin\left[\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right], -\cos\left[\sqrt{1-\widehat{m}^2} \int \widehat{\kappa}(s) d\widehat{s} \right] \right).$$

Hence, if $\frac{|\tau|}{\kappa} = |m| < 1$, then a spacelike general helix with a spacelike principal normal in $\mathbb{D}_1^3$ is given by the equation (15).
(b) The proof is similar to the proof of the statement (a). □

Similarly, we have Theorem 9 and Theorem 10.

Theorem 9. *Let $\widehat{\gamma}$ be a dual unit speed spacelike general helix having timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau} = \widehat{m}\widehat{\kappa}$ for dual* *constant* $\widehat{m} = m + \xi m^*$ *in* $\mathbb{D}^3_1$. *The position vector of* $\widehat{\gamma}$ *is given by*

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{1+\widehat{m}^2}} \int \left(\sinh\left[\sqrt{1+\widehat{m}^2}\int \widehat{\kappa}(s) d\widehat{s}\right], \cosh\left[\sqrt{1+\widehat{m}^2}\int \widehat{\kappa}(s) d\widehat{s}\right], -\widehat{m} \right) d\widehat{s} \tag{17}$$

and the principal directional curve of $\widehat{\gamma}$ *is a timelike plane curve in* $\mathbb{D}^2_{1.}$.

Theorem 10. *Let* $\widehat{\gamma}$ *be a dual unit speed timelike general helix with the dual curvature* $\widehat{\kappa}$ *and the dual torsion* $\widehat{\tau} = \widehat{m}\widehat{\kappa}$ *for dual constant* $\widehat{m} = m + \xi m^*$ *in* $\mathbb{D}^3_1$.
(*a*) *If* $\frac{|\tau|}{\kappa} = |m| > 1$, *then the position vector of* $\widehat{\gamma}$ *is given by*

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{\widehat{m}^2-1}} \int \left(\widehat{m}, \sin\left[\sqrt{\widehat{m}^2-1}\int \widehat{\kappa}(s) d\widehat{s}\right], -\cos\left[\sqrt{\widehat{m}^2-1}\int \widehat{\kappa}(s) d\widehat{s}\right] \right) d\widehat{s} \tag{18}$$

and the principal directional curve of $\widehat{\gamma}$ *is a spacelike plane curve with spacelike principal normal in* $\mathbb{D}^2$,
(*b*) *If* $\frac{|\tau|}{\kappa} = |m| < 1$, *then the position vector of* $\widehat{\gamma}$ *is given by*

$$\widehat{\gamma}(s) = \frac{1}{\sqrt{1-\widehat{m}^2}} \int \left(\cosh\left[\sqrt{1-\widehat{m}^2}\int \widehat{\kappa}(s) d\widehat{s}\right], \sinh\left[\sqrt{1-\widehat{m}^2}\int \widehat{\kappa}(s) d\widehat{s}\right], \widehat{m} \right) d\widehat{s} \tag{19}$$

and the principal directional curve of $\widehat{\gamma}$ *is a spacelike plane curve with timelike principal normal in* $\mathbb{D}^2_1$.

Taking into consideration the above three theorems, the following three results are obtained:

Corollary 1. *Let* $\widehat{\gamma}$ *be a dual unit speed spacelike Frenet curve with a spacelike principal normal and* $\widehat{\gamma}_0$ *be a spacelike principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}^3_1$. *Then* $\widehat{\gamma}_0$ *is a plane curve in* $\mathbb{D}^2$ *or* $\mathbb{D}^2_1$ *iff* $\widehat{\gamma}$ *is a general helix in* $\mathbb{D}^3_1$ *with inequalities* $\kappa > |\tau|$ *or* $\kappa < |\tau|$, *respectively. Furthermore* $\widehat{\gamma}_0$ *is a circle in* $\mathbb{D}^2$ *or spacelike hyperbola in* $\mathbb{D}^2_1$ *if and only if* $\widehat{\gamma}$ *is a helix in* $\mathbb{D}^3_1$ *with* $\kappa > |\tau|$ *or a helix* $\kappa < |\tau|$,*respectively.*

Corollary 2. *Let* $\widehat{\gamma}$ *be a dual unit speed spacelike Frenet curve with a timelike principal normal and* $\widehat{\gamma}_0$ *be a timelike principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}^3_1$. *Then* $\widehat{\gamma}_0$ *is a plane curve iff* $\widehat{\gamma}$ *is a general helix in* $\mathbb{D}^3_1$. *Furthermore* $\widehat{\gamma}_0$ *is a timelike hyperbola if and only if* $\widehat{\gamma}$ *is a helix in* $\mathbb{D}^3_1$.

Corollary 3. *Let* $\widehat{\gamma}$ *be a dual unit speed timelike Frenet curve and* $\widehat{\gamma}_0$ *be a spacelike principal directional curve of* $\widehat{\gamma}$ *in* $\mathbb{D}^3_1$. *Then* $\widehat{\gamma}_0$ *is a plane curve in* $\mathbb{D}^2$ *or* $\mathbb{D}^2_1$ *iff* $\widehat{\gamma}$ *is a general helix in* $\mathbb{D}^3_1$ *with inequalities* $\kappa < |\tau|$ *or* $\kappa > |\tau|$, *respectively. Furthermore* $\widehat{\gamma}_0$ *is a circle in* $\mathbb{D}^2$ *or a spacelike hyperbola in* $\mathbb{D}^2_1$ *iff* $\widehat{\gamma}$ *is a helix in* $\mathbb{D}^3_1$ *with* $\kappa < |\tau|$ *or* $\kappa > |\tau|$, *respectively.*

Consequently, the general helices are characterized in $\mathbb{D}^3_1$ according to the associated curve as follows:

Theorem 11. *A general helix in* $\mathbb{D}^3_1$ *is the principal donor curve of some planar curves.*

4. Principal Directional Curves of Slant Helices in $\mathbb{D}^3_1$

In this section, we examine the causal characters of general helices which are principal directional curves of slant helices according to causal characters of slant helices in $\mathbb{D}^3_1$. We are state that the connections between general helices and slant helices in $\mathbb{D}^3_1$ as follows:

Let $\widehat{\gamma}$ be a Frenet curve in $\mathbb{D}^3_1$ and $\widehat{W}$ be a dual unit vector along the dual Frenet curve $\widehat{\gamma}$. If $\widehat{W}$ has a constant dual angle with a constant dual vector $\widehat{V}$ along $\widehat{\gamma}$, then the tangent vector of $\widehat{\gamma}_0$, which is the $\widehat{W}-$directional curve of $\widehat{\gamma}$, also has a constant dual angle with $\widehat{V}$ along $\widehat{\gamma}$. Conversely, if the dual unit tangent vector of the Frenet curve $\widehat{\gamma}_0$ in $\mathbb{D}^3_1$ makes a constant dual angle with the constant vector $\widehat{V}$ in $\mathbb{D}^3_1$ then $\widehat{\gamma}$ is the $\widehat{W}-$donor curve of $\widehat{\gamma}_0$.

In the expression given above, we take principal normal vector instead of $\widehat{W}$ along $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Then $\widehat{\gamma}$ is a dual slant helix (slant helix in $\mathbb{D}^3_1$) that is the principal normal vector of $\widehat{\gamma}$ makes a constant dual angle with a constant vector $\widehat{V}$ in $\mathbb{D}^3_1$ iff the principal directional curve of $\widehat{\gamma}$ is a general helix in $\mathbb{D}^3_1$ that is the dual unit tangent vector of $\widehat{\gamma}_0$ makes a constant dual angle with a constant vector $\widehat{V}$ in $\mathbb{D}^3_1$. On the other hand, a slant helix is the principal donor curve of a general helix and a general helix is the principal directional curve of a slant helix in $\mathbb{D}^3_1$ (see [3, 15] for slant helices)

Now let $\widehat{\gamma}_0$ be a spacelike general helix having a spacelike dual principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$ in $\mathbb{D}^3_1$. Then the spacelike principal donor curve $\widehat{\gamma}_1$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_1 = \widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_1 = \widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. A timelike principal donor curve $\widehat{\gamma}_2$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa_2} = \widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_2 = \widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The dual Frenet curves $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ hold the equations of slant helices:

$$\frac{\widehat{\kappa}_1^2}{\left(\widehat{\kappa}_1^2 - \widehat{\tau}_1^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_1}{\widehat{\kappa}_1}\right) = \frac{\cosh^2\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]}{\widehat{\kappa}_0}\frac{d}{d\widehat{s}}\left(\tanh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right) = \widehat{c} \tag{20}$$

and

$$\frac{-\widehat{\kappa}_2^2}{\left(\widehat{\tau}_2^2 - \widehat{\kappa}_2^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_2}{\widehat{\kappa}_2}\right) = -\frac{\sinh^2\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]}{\widehat{\kappa}_0}\frac{d}{d\widehat{s}}\left(\coth\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right) = \widehat{c}, \tag{21}$$

respectively.

Let $\widehat{\gamma}_0$ be a spacelike general helix having a timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$ in $\mathbb{D}^3_1$. Then the spacelike principal donor curve $\widehat{\gamma}_3$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_3 = \widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_3 = -\widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The timelike principal donor curve $\widehat{\gamma}_4$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_4 = \widehat{\kappa}_0(s)\cosh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_4 = -\widehat{\kappa}_0(s)\sinh\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The dual Frenet curves $\widehat{\gamma}_3$ and $\widehat{\gamma}_4$ hold the equations of slant helices:

$$\frac{\widehat{\kappa}_3^2}{\left(\widehat{\tau}_3^2-\widehat{\kappa}_3^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_3}{\widehat{\kappa}_3}\right) = \frac{\sinh^2\left[\widehat{c}\int\widehat{\kappa}_0 d\widehat{s}\right]}{\widehat{\kappa}_0(s)}\frac{d}{d\widehat{s}}\left(-\coth\left[\widehat{c}\int\widehat{\kappa}_0 d\widehat{s}\right]\right) = \widehat{c} \tag{22}$$

and

$$\frac{-\widehat{\kappa}_4^2}{\left(\widehat{\kappa}_4^2-\widehat{\tau}_4^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_4}{\widehat{\kappa}_4}\right) = -\frac{\cosh^2\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]}{\widehat{\kappa}_0(s)}\frac{d}{d\widehat{s}}\left(-\tanh\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]\right) = \widehat{c}, \tag{23}$$

respectively.

Finally, let $\widehat{\gamma}_0$ be a timelike general helix with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}_0 = \widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$ in $\mathbb{D}^3_1$. Then the principal donor curve $\widehat{\gamma}_5$ of $\widehat{\gamma}_0$ has the dual curvature $\widehat{\kappa}_5 = \widehat{\kappa}_0(s)\cos\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$ and the dual torsion $\widehat{\tau}_5 = -\widehat{\kappa}_0(s)\sin\left[\widehat{c}\int\widehat{\kappa}_0(s)d\widehat{s}\right]$. The dual Frenet curve $\widehat{\gamma}_5$ hold the equation of slant helix:

$$\frac{-\widehat{\kappa}_5^2}{\left(\widehat{\tau}_5^2+\widehat{\kappa}_5^2\right)^{3/2}}\frac{d}{d\widehat{s}}\left(\frac{\widehat{\tau}_5}{\widehat{\kappa}_5}\right) = \frac{-\cos^2\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]}{\widehat{\kappa}_0(s)}\frac{d}{d\widehat{s}}\left(-\tan\left[\widehat{c}\int\widehat{\kappa}_0(s) d\widehat{s}\right]\right) = \widehat{c}. \tag{24}$$

The value of a dual slant helix equation is called the dual slant helix constant. Then we can write following proposition:

Proposition 1. *Let $\widehat{\gamma}_0(s)$ be a general helix with the dual curvature $\widehat{\kappa}_0$ and the dual torsion $\widehat{\tau}_0$ and $\widehat{\gamma}$ be the principal donor curve of $\widehat{\gamma}_0$ in $\mathbb{D}^3_1$. Then $\widehat{\gamma}$ is a dual slant helix with the dual slant helix constant $\frac{\widehat{\tau}_0}{\widehat{\kappa}_0}$.*

In the previous section, general helices were constructed in $\mathbb{D}^3_1$ with the help of plane curves. The above methods gave idea to construct slant helix with the help of general helices in $\mathbb{D}^3_1$. Now, by using the method in the third chapter the slant helices will be constructed from the general helices in $\mathbb{D}^3_1$.

Theorem 12. *Let $\widehat{\gamma}$ be a dual unit speed spacelike slant helix having a spacelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ in $\mathbb{D}^3_1$ and $\widehat{c} = c+\xi c^*$ be a dual slant helix constant.*

(a) If $\kappa > |\tau|$ and $|c| < 1$, then the position vector $\widehat{\gamma}$ is given by

$$\begin{aligned}\widehat{\gamma}(s) &= -\int\Big(-\tfrac{\sinh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}},\\ &\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]-\tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}},\\ &\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]\\ &+\tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}\Big)d\widehat{s},\end{aligned} \tag{25}$$

where $\widehat{K}_1(s)=\int\sqrt{\widehat{\kappa}^2(s)-\widehat{\tau}^2(s)}d\widehat{s}$.

(b) If $\kappa > |\tau|$ and $|c| > 1$ then the position vector $\widehat{\gamma}$ is given by

$$\begin{aligned}\widehat{\gamma}(s) &= -\int\Big(\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right]\\ &-\tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right]}{\sqrt{\widehat{c}^2-1}},\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right]\\ &-\tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_1(s)\right]}{\sqrt{\widehat{c}^2-1}},-\tfrac{\cosh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{\widehat{c}^2-1}}\Big)d\widehat{s},\end{aligned} \tag{26}$$

where $\widehat{K}_1(s)=\int\sqrt{\widehat{\kappa}^2(s)-\widehat{\tau}^2(s)}d\widehat{s}$.

(c) If $\kappa < |\tau|$ then the position vector $\widehat{\gamma}$ is given by

$$\begin{aligned}\widehat{\gamma}(s) &= -\int\Big(\sinh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right]\\ &-\tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right]}{\sqrt{1+\widehat{c}^2}},\ \sinh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right]\\ &-\tfrac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_2(s)\right]}{\sqrt{1+\widehat{c}^2}},\ \tfrac{\cosh\left[\widehat{c}\widehat{K}_2(s)\right]}{\sqrt{1+\widehat{c}^2}})d\widehat{s}\Big),\end{aligned} \tag{27}$$

where $\widehat{K}_2(s)=\int\sqrt{\widehat{\tau}^2(s)-\widehat{\kappa}^2(s)}d\widehat{s}$.

Proof. Let $\widehat{\gamma}_0$ the principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Since $\widehat{\gamma}_0$ is a general helix with the dual torsion $\widehat{\tau}_0=\widehat{c}\widehat{\kappa}_0$ for dual constant $\widehat{c}$.
(a) From the equation (15) we obtain

$$\begin{cases}\widehat{t}_0(s) &= \tfrac{1}{\sqrt{1-\widehat{c}^2}}\left(\widehat{c},\ \sin\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right],\right.\\ &\left.-\cos\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right),\\ \widehat{n}_0(s) &= \left(0,\ \cos\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right],\ \sin\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right),\\ \widehat{b}_0(s) &= \tfrac{1}{\sqrt{1-\widehat{c}^2}}\left(1,\ -\widehat{c}\sin\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right],\right.\\ &\left.\widehat{c}\cos\left[\sqrt{1-\widehat{c}^2}\int\widehat{\kappa}_0(s)d\widehat{s}\right]\right).\end{cases} \tag{28}$$

On the other hand from (4) it is clear that

$$\widehat{t}=-\cosh\left[\int\widehat{\tau}_0(s)d\widehat{s}\right]\widehat{n}_0+\sinh\left[\int\widehat{\tau}_0(s)d\widehat{s}\right]\widehat{b}_0.$$

If we take into consideration the equation (28) then the dual unit tangent vector $\widehat{t}$ of $\widehat{\gamma}$ can be written as

$$\begin{aligned}\widehat{t} \;=\; & \Big(\tfrac{\sinh[\int \widehat{\tau}_0(s)d\widehat{s}]}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\int \widehat{\tau}_0(s)d\widehat{s}\right]\cos\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right] - \tfrac{\widehat{c}\sinh[\int \widehat{\tau}_0(s)d\widehat{s}]\sin\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right]}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\int \widehat{\tau}_0(s)d\widehat{s}\right]\sin\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right] + \tfrac{\widehat{c}\sinh[\int \widehat{\tau}_0(s)d\widehat{s}]\cos\left[\sqrt{1-\widehat{c}^2}\int \widehat{\kappa}_0(s)d\widehat{s}\right]}{\sqrt{1-\widehat{c}^2}}\Big).\end{aligned}$$

By using the equations $\widehat{K}_1(s) = \int \widehat{\kappa}_0(s)d\widehat{s} = \int \sqrt{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)}d\widehat{s}$ and $\widehat{c}\widehat{K}_1(s) = \int \widehat{\tau}_0(s)d\widehat{s}$ we have

$$\begin{aligned}\widehat{t} \;=\; & \Big(\tfrac{\sinh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left(\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right) - \tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left(\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right)}{\sqrt{1-\widehat{c}^2}}, \\ & -\cosh\left[\widehat{c}\widehat{K}_1(s)\right]\sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right] + \tfrac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1-\widehat{c}^2}}\Big).\end{aligned} \tag{29}$$

If we take into consideration $\widehat{t} = \frac{d\widehat{\gamma}(s)}{d\widehat{s}}$ and integrate both sides of the equation (29) with respect to $\widehat{s}$, then we get (25).
The proofs of (b) and (c) are similar to the proof of the statement (a). □

Similarly, we have Theorem 13 and Theorem 14.

Theorem 13. *Let $\widehat{\gamma}$ be a dual unit speed spacelike slant helix having a timelike principal normal with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ in $\mathbb{D}_1^3$ and $\widehat{c} = c + \xi c^*$ be a dual slant helix constant.*
(a) *If $|c| > 1$ then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}\widehat{\gamma}(s) \;=\; & \int\Big(\tfrac{\sin\left[\widehat{c}\widehat{K}_3(s)\right]}{\sqrt{\widehat{c}^2-1}}, \\ & \tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\cos\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right]}{\sqrt{\widehat{c}^2-1}} - \cos\left[\widehat{c}\widehat{K}_3(s)\right]\sin\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right], \\ & \tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\sin\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right]}{\sqrt{\widehat{c}^2-1}} + \cos\left[\widehat{c}\widehat{K}_3(s)\right]\cos\left[\sqrt{\widehat{c}^2-1}\widehat{K}_3(s)\right]\Big)d\widehat{s},\end{aligned} \tag{30}$$

where $\widehat{K}_3(s) = \int \sqrt{\widehat{\kappa}^2(s) + \widehat{\tau}^2(s)}d\widehat{s}$.

(b) *If $|c| < 1$ then the position vector $\widehat{\gamma}$ is given by*

$$\begin{aligned}\widehat{\gamma}(s) \;=\; & \int\Big(\cos\left[\widehat{c}\widehat{K}_3(s)\right]\sinh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right] \\ & +\tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\cosh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right]}{\sqrt{1-\widehat{c}^2}}, \cos\left[\widehat{c}\widehat{K}_3(s)\right]\cosh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right] \\ & +\tfrac{\widehat{c}\sin\left[\widehat{c}\widehat{K}_3(s)\right]\sinh\left[\sqrt{1-\widehat{c}^2}\widehat{K}_3(s)\right]}{\sqrt{1-\widehat{c}^2}}, \tfrac{\sin\left[\widehat{c}\widehat{K}_3(s)\right]}{\sqrt{1-\widehat{c}^2}}\Big)d\widehat{s}\end{aligned} \tag{31}$$

where $\widehat{K}_3(s) = \int \sqrt{\widehat{\kappa}^2(s) + \widehat{\tau}^2(s)} d\widehat{s}$.

Theorem 14. *Let $\widehat{\gamma}(s)$ be a dual unit speed timelike slant helix with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ in $\mathbb{D}_1^3$ and $\widehat{c} = c + \xi c^*$ be a dual slant helix constant.*
(a) If $\kappa < |\tau|$ and $|c| < 1$ then the position vector $\widehat{\gamma}$ is given by

$$
\begin{aligned}
\widehat{\gamma}(s) \quad = \quad & \int \left(\frac{\sinh\left[\widehat{c}\widehat{K}_2(s)\right]}{\sqrt{1-\widehat{c}^2}}, \cos\left(\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right) \cosh\left[\widehat{c}\widehat{K}_2(s)\right] \right. \\
& - \frac{\widehat{c}}{\sqrt{1-\widehat{c}^2}} \sin\left(\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right) \sinh\left[\widehat{c}\widehat{K}_2(s)\right], \\
& \sin\left[\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right] \cosh\left[\widehat{c}\widehat{K}_2(s)\right] \\
& \left. + \frac{\widehat{c}}{\sqrt{1-\widehat{c}^2}} \cos\left[\sqrt{1-\widehat{c}^2}\widehat{K}_2(s)\right] \sinh\left[\widehat{c}\widehat{K}_2(s)\right] \right) d\widehat{s}
\end{aligned}
\tag{32}
$$

where $\widehat{K}_2(s) = \int \sqrt{\widehat{\tau}^2(s) - \widehat{\kappa}^2(s)} d\widehat{s}$.

(b) If $\kappa < |\tau|$ and $|c| > 1$ then $\widehat{\gamma}$ can denoted by

$$
\begin{aligned}
\widehat{\gamma}(s) \quad = \quad & \int \left(\cosh\left[\widehat{c}\widehat{K}_2(s)\right] \sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right] - \frac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right]}{\sqrt{\widehat{c}^2-1}}, \right. \\
& \sinh\left[\widehat{c}\widehat{K}_2(s)\right] \cosh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right] - \frac{\widehat{c}\cosh\left[\widehat{c}\widehat{K}_2(s)\right]\sinh\left[\sqrt{\widehat{c}^2-1}\widehat{K}_2(s)\right]}{\sqrt{\widehat{c}^2-1}}, \\
& \left. - \frac{\cosh\left[\widehat{c}\widehat{K}_2(s)\right]}{\sqrt{\widehat{c}^2-1}}) \right) d\widehat{s},
\end{aligned}
\tag{33}
$$

where $\widehat{K}_2(s) = \int \sqrt{\widehat{\tau}^2(s) - \widehat{\kappa}^2(s)} d\widehat{s}$.

(c) If $\kappa > |\tau|$ then the position vector $\widehat{\gamma}$ is given by

$$
\begin{aligned}
\widehat{\gamma}(s) \quad = \quad & \int (\cosh\left[\widehat{c}\widehat{K}_1(s)\right] \cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right] \\
& - \frac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1+m^2}}, \ \cosh\left[\widehat{c}\widehat{K}_1(s)\right] \sinh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right] \\
& - \frac{\widehat{c}\sinh\left[\widehat{c}\widehat{K}_1(s)\right]\cosh\left[\sqrt{1+\widehat{c}^2}\widehat{K}_1(s)\right]}{\sqrt{1+\widehat{c}^2}}, \ \frac{\sinh\left[\widehat{c}\widehat{K}_1(s)\right]}{\sqrt{1+\widehat{c}^2}}) d\widehat{s},
\end{aligned}
\tag{34}
$$

where $\widehat{K}_1(s) = \int \sqrt{\widehat{\kappa}^2(s) - \widehat{\tau}^2(s)} d\widehat{s}$.

In Theorem 11, general helices in $\mathbb{D}_1^3$ were characterized according to the associated curve. Similarly, the characterization of slant helices in $\mathbb{D}_1^3$ is given as follows:

Theorem 15. *A slant helix in $\mathbb{D}_1^3$ is the second principal donor curve of some plane curves.*

A Frenet curve $\widehat{\gamma}$ in $\mathbb{D}_1^3$ is called a circular slant helix or hyperbolic slant helix if the second principal directional curve of $\widehat{\gamma}$ a circle in $\mathbb{D}^2$ or a hyperbola in $\mathbb{D}_1^2$, respectively. These curves are called simple dual curves.

Now we will deal with simple closed slant helices in $\mathbb{D}_1^3$. Taking into consideration the equations (25)-(27) and (30)-(34), we can state that there are no closed simple dual slant helices given by (25)-(27) and (31)−(34). Therefore we only interest a closed simple dual slant helix given by (30).

Remark 1. *Let $\widehat{\gamma}$ be a spacelike circular slant helix providing the equation (30) and its first principal directional curve of $\widehat{\gamma}_0$ and its second principal directional curve of $\widehat{\gamma}_1$ be a helix with $\frac{|\tau_0|}{\kappa_0} = |c| > 1$ and a circle with radius $\widehat{r}$ in $\mathbb{D}_1^3$, respectively. Since the dual curvature of $\widehat{\gamma}_1$ is $\widehat{\kappa}_1 = \frac{1}{\widehat{r}}$, the dual curvature $\widehat{\kappa}_0$ is expressed by $\widehat{\kappa}_0 = \frac{1}{\widehat{r}\sqrt{\widehat{c}^2-1}}$. Thus the dual function $\widehat{K}_3$ in (30) is given by*

$$\widehat{K}_3(s) = \int \sqrt{\widehat{\kappa}^2 + \widehat{\tau}^2} d\widehat{s} = \int \widehat{\kappa}_0 d\widehat{s} = \frac{\widehat{s}}{\widehat{r}\sqrt{\widehat{c}^2 - 1}}.$$

Therefore, by the a simple integration we can give that $\widehat{\gamma}$ is closed iff $\frac{c}{\sqrt{c^2-1}}$ is rational. Similarly, it appears that other simple dual slant helices are not closed.

Example 1. *A spacelike circular dual slant helix*

$$\widehat{\gamma}(s) = \gamma(s) + \xi\gamma^*(s) \tag{35}$$

of (30) can be denoted by

$$\begin{aligned}\gamma(s) \quad = \quad & -r\left(\frac{1}{c}\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right],\right.\\ & \left(2c^2-1\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right] + 2c\sqrt{c^2-1}\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right],\\ & \left.\left(2c^2-1\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right] - 2c\sqrt{c^2-1}\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\right)\end{aligned} \tag{36}$$

and

$$\begin{aligned}\gamma^*(s) \quad = \quad & \left(\frac{rc^*-r^*c}{c^2}\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right] + \frac{cr^*-c^*r-c^3r^*}{rc(c^2-1)^{\frac{3}{2}}}s\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right],\right.\\ & \left(csr^*\left(1-c^2\right) + c^*sr\left(1-2c^2\right)\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\\ & + \left(\frac{r^*s}{r} + \frac{2cc^*s}{c^2-1}\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]\\ & + \left(r^* - 2r^*c^2 - 4cc^*r\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\\ & -2c\left(\frac{cc^*r+c^2r^*-r^*}{\sqrt{c^2-1}}\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right],\\ & \left(csr^*\left(1-c^2\right) + c^*sr\left(1-2c^2\right)\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]\\ & - \left(\frac{r^*s}{r} + \frac{2cc^*s}{c^2-1}\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\\ & + \left(r^* - 2r^*c^2 - 4cc^*r\right)\cos\left[\frac{cs}{r\sqrt{c^2-1}}\right]\sin\left[\frac{s}{r}\right]\\ & \left.+2c\left(\frac{cc^*r+c^2r^*-r^*}{\sqrt{c^2-1}}\right)\sin\left[\frac{cs}{r\sqrt{c^2-1}}\right]\cos\left[\frac{s}{r}\right]\right).\end{aligned} \tag{37}$$

If we put $c=\frac{3}{2\sqrt{2}}$ *for dual constant* $\widehat{c}=c+\xi c^{*}$ *and* $\widehat{r}=(1,0)$, *then the closed condition* $\frac{c}{\sqrt{c^{2}-1}}=3$ *is provided and an example of a spacelike closed circular dual slant helix with timelike principal normal is given by*

$$\begin{aligned}\widehat{\gamma}_1(s) \quad &= -\left(\tfrac{2\sqrt{2}\cos[3s]}{3},\ \tfrac{5\cos[3s]\cos[s]+3\sin[3s]\sin[s]}{4},\ \tfrac{5\cos[3s]\sin[s]-3\sin[3s]\cos[s]}{4}\right)\\ &+\xi c^{*}\left(\tfrac{8\cos[3s]}{9}-\tfrac{64s\sin[3s]}{3},\ \tfrac{-5s\sin[3s]\cos[s]}{4}+12\sqrt{2}s\cos[3s]\sin[s]\right.\\ &-3\sqrt{2}\cos[3s]\cos[s]-\tfrac{9\sqrt{2}\sin[3s]\sin[s]}{2},\tfrac{-5s\sin[3s]\sin[s]}{4}-12\sqrt{2}s\cos[3s]\cos[s]\\ &\left.-3\sqrt{2}\cos[3s]\sin[s]+\tfrac{9\sqrt{2}\sin[3s]\cos[s]}{2}\right).\end{aligned}$$

If we put $c=2$ *for dual constant* $\widehat{c}=c+\xi c^{*}$ *and* $\widehat{r}=(1,0)$, *then the closed condition* $\frac{c}{\sqrt{c^{2}-1}}=\frac{2}{\sqrt{3}}$ *is not provided and the an example of a spacelike non-closed circular dual slant helix with timelike principal normal is given by*

$$\begin{aligned}\widehat{\gamma}_2(s) \quad = \quad & -\left(\frac{\cos\left[\frac{2s}{\sqrt{3}}\right]}{2},\ 7\cos[s]\cos\left[\frac{2s}{\sqrt{3}}\right]+4\sqrt{3}\sin\left[\frac{2s}{\sqrt{3}}\right]\sin[s],\right.\\ & \left.7\cos\left[\frac{2s}{\sqrt{3}}\right]\sin[s]-4\sqrt{3}\sin\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\right)+\xi c^{*}\left(\frac{\cos\left[\frac{2s}{\sqrt{3}}\right]}{4}-s\frac{\sin\left[\frac{2s}{\sqrt{3}}\right]}{6\sqrt{3}},\right.\\ & -7s\sin\left[\frac{2s}{\sqrt{3}}\right]\cos[s]+\frac{4s}{3}\cos\left[\frac{2s}{\sqrt{3}}\right]\sin[s]-8\cos\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\\ & -\frac{8}{\sqrt{3}}\sin\left[\frac{2s}{\sqrt{3}}\right]\sin[s],\ -7s\sin\left[\frac{2s}{\sqrt{3}}\right]\sin[s]-\frac{4s}{3}\cos\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\\ & \left.-8\cos\left[\frac{2s}{\sqrt{3}}\right]\sin[s]+\frac{8}{\sqrt{3}}\sin\left[\frac{2s}{\sqrt{3}}\right]\cos[s]\right).\end{aligned}$$

Corollary 4. *The closed simple slant helix* $\widehat{\gamma}$ *given by (35) whose real part (36) and dual part (37) in* $\mathbb{D}_1^3$ *is a spacelike circular slant helix with timelike principal normal having slant helix constant* $\widehat{c}=c+\xi c^{*}$ *providing the condition* $\frac{c}{\sqrt{c^{2}-1}}$ *is rational.*

5. Principal Directed Rectifying Curve in $\mathbb{D}_1^3$

In this section, we examine the principal directed rectifying curve whose the position vector always lie in rectifying plane of its principal donor curve in $\mathbb{D}_1^3$ (see [6, 10, 14, 19] for rectifying curve). We show that a principal directional rectifying curve in $\mathbb{D}_1^3$ corresponds to a spacelike or a timelike ruled surface in $\mathbb{R}_1^3$ depending on causal characters of its principal donor curves.

Theorem 16. *Let* $\widehat{\gamma}_0$ *be a pseudo spherical Frenet curve (a Frenet curve lies on* $\mathbb{S}_1^2$ *or* $\mathbb{H}_0^2$*) and* $\widehat{\gamma}$ *be a principal donor curve of* $\widehat{\gamma}_0$ *in* $\mathbb{D}_1^3$. *Then* $\widehat{\gamma}_0$ *is a principal directed rectifying curve.*

Proof. Let $\widehat{\gamma}_0$ be a pseudo spherical Frenet curve and $\widehat{\gamma}$ be a principal donor curve of $\widehat{\gamma}_0$ in $\mathbb{D}_1^3$. According to the dual Frenet frame of $\widehat{\gamma}$, the position vector of $\widehat{\gamma}_0$ is written as

$$\widehat{\gamma}_0(s)=\widehat{\lambda}(s)\widehat{t}(s)+\widehat{\mu}(s)\widehat{n}(s)+\widehat{\beta}(s)\widehat{b}(s), \tag{38}$$

for some dual functions $\widehat{\lambda}$, $\widehat{\mu}$ and $\widehat{\beta}$. Since $\frac{d\widehat{\gamma}_0}{d\widehat{s}} = \widehat{t}_0$, we have

$$\widehat{t}_0 = \widehat{n} = \left(\frac{d\widehat{\lambda}}{d\widehat{s}} - \varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{\mu}\right)\widehat{t} + \left(\widehat{\kappa}\widehat{\lambda} + \frac{d\widehat{\mu}}{d\widehat{s}} - \varepsilon_1\varepsilon_2\widehat{\tau}\widehat{\beta}\right)\widehat{n} + \left(\widehat{\mu}\widehat{\tau} + \frac{d\widehat{\beta}}{d\widehat{s}}\right)\widehat{b}.$$

Thus the system of equations

$$\begin{cases} \frac{d\widehat{\lambda}}{d\widehat{s}} - \varepsilon_0\varepsilon_1\widehat{\kappa}\widehat{\mu} = 0 \\ \widehat{\kappa}\widehat{\lambda} + \frac{d\widehat{\mu}}{d\widehat{s}} - \varepsilon_1\varepsilon_2\widehat{\tau}\widehat{\beta} = 1 \\ \widehat{\mu}\widehat{\tau} + \frac{d\widehat{\beta}}{d\widehat{s}} = 0 \end{cases} \tag{39}$$

is formed. Since $\widehat{\gamma}_0$ is a pseudo spherical Frenet curve, taking into consideration the equation (38) we obtain

$$\varepsilon_0\widehat{\lambda}(s)^2 + \varepsilon_1\widehat{\mu}(s)^2 + \varepsilon_2\widehat{\beta}(s)^2 = \mp\widehat{r}^2.$$

If we take derivative of this last equation with respect to $\widehat{s}$, then we get

$$\varepsilon_0\widehat{\lambda}\frac{d\widehat{\lambda}}{d\widehat{s}} + \varepsilon_1\widehat{\mu}\frac{d\widehat{\mu}}{d\widehat{s}} + \varepsilon_2\widehat{\beta}\frac{d\widehat{\beta}}{d\widehat{s}} = 0 \tag{40}$$

is denoted. By using the equations (39) and (40) it is clear that $\widehat{\mu}(s) = 0$. Hence we can rewrite the equation (38) as

$$\widehat{\gamma}_0(s) = \widehat{\lambda}(s)\widehat{t}(s) + \widehat{\beta}(s)\widehat{b}(s).$$

So the position vector of $\widehat{\gamma}_0(s)$ lies in the rectifying plane of $\widehat{\gamma}$ which is the principal donor curve of $\widehat{\gamma}_0$. Therefore, $\widehat{\gamma}_0$ is principal directed rectifying curve. □

Theorem 17. *Let $\widehat{\gamma}$ be a Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and a pseudo spherical Frenet curve $\widehat{\gamma}_0$ be principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. Then the position vector of $\widehat{\gamma}_0$ lies in the normal plane $S_p\left\{\widehat{n}_0, \widehat{b}_0\right\}$ and the position vector of $\widehat{\gamma}_0$ is given by*

$$\widehat{\gamma}_0(s) = -\frac{\widetilde{\varepsilon}_1(\varepsilon_1\widehat{c}_1\widehat{\kappa} + \varepsilon_2\widehat{c}_2\widehat{\tau})}{(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right))^{3/2}}\widehat{n}_0(s) + \frac{\widehat{c}_1\widehat{\tau} + \varepsilon_0\varepsilon_1\widehat{c}_2\widehat{\kappa}}{(\widetilde{\varepsilon}_1\left(\varepsilon_0\widehat{\kappa}^2 + \varepsilon_2\widehat{\tau}^2\right))^{3/2}}\widehat{b}_0(s) \tag{41}$$

for dual constants $\widehat{c}_1$ and $\widehat{c}_2$.

Proof. Let $\widehat{\gamma}$ is a Frenet curve with the dual curvature $\widehat{\kappa}$ and the dual torsion $\widehat{\tau}$ and a pseudo spherical Frenet curve $\widehat{\gamma}_0$ be principal directional curve of $\widehat{\gamma}$ in $\mathbb{D}^3_1$. We know that the dual curve $\widehat{\gamma}_0$ lies on the rectifying plane of $\widehat{\gamma}$. Then the position vector of $\widehat{\gamma}_0$ can be written by

$$\widehat{\gamma}_0(s) = \widehat{\lambda}\widehat{t}(s) + \widehat{\beta}\widehat{b}(s) \tag{42}$$

for dual functions $\widehat{\lambda}$ and $\widehat{\beta}$. If we take derivative of the equation (42) with respect to $\widehat{s}$, then we have

$$\widehat{n} = \frac{d\widehat{\lambda}}{d\widehat{s}}\widehat{t}(s) + (\widehat{\lambda}\widehat{\kappa} - \varepsilon_1\varepsilon_2\widehat{\beta}\widehat{\tau})\widehat{n}(s) + \frac{d\widehat{\beta}}{d\widehat{s}}\widehat{b}(s).$$

From the last equation it is clear that $\widehat{\lambda} = \widehat{c}_1$ and $\widehat{\beta} = \widehat{c}_2$ are dual constants. Therefore, we obtain (41) by using (4). □

Corollary 5. *Let $\widehat{\gamma}$ be a spacelike Frenet curve with a spacelike principal normal in $\mathbb{D}_1^3$. Then the principal directed rectifying curve of $\widehat{\gamma}$ corresponds to a timelike ruled surface in $\mathbb{R}_1^3$.*

Corollary 6. *Let $\widehat{\gamma}$ be a spacelike Frenet curve with a timelike principal normal in $\mathbb{D}_1^3$. Then the principal directed rectifying curve of $\widehat{\gamma}$ corresponds to a spacelike ruled surface in $\mathbb{R}_1^3$.*

Corollary 7. *Let $\widehat{\gamma}$ be a timelike Frenet curve in $\mathbb{D}_1^3$. Then the principal directed rectifying curve of $\widehat{\gamma}$ corresponds to a timelike ruled surface in $\mathbb{R}_1^3$.*

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