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Asymptotic behaviours of the solutions of neutral type Volterra integro-differential equations and some numerical solutions via differential transform method

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ABSTRACT

In this manuscript, we consider the first order neutral Volterra integro-differential equation (NVIDE) with delay argument. Firstly, we obtain novel sufficient conditions to establish the asymptotic behaviours of solutions of considered NVIDE using the Lyapunov method and present an example to demonstrate the applicability of proposed method. Secondly, we get some numerical solutions for a particular case of considered NVIDE via the differential transformation method (DTM). The results of this manuscript are novel and they improve some existing ones in the literature.

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1. Introduction

Volterra integral and integro-differential equations were first introduced by Vito Volterra in 1926. Subsequently, these equations have been frequently used in technical fields of scientific and engineering. Volterra integro-differential equations (VIDEs), which are known as a famous mathematical model in the related literature, and delay differential equations/systems have been seen in many practical fields such as electrical circuit, glass forming process, biology, physics, chemistry, control theory, economics (see, [1-13]). Besides, many researchers have done studies on the qualitative behaviors of solutions and some numerical solutions of these equations in recent years. Lyapunov's direct method is an important tool to discuss the some qualitative behaviors of solutions of ordinary and functional differential equations and integro-differential equations. This technique is theoretically very attractive, and there are many applications where it is natural to use it. However, it is rather difficult to construct a meaningful Lyapunov functional for a non-linear ordinary or functional differential equation and a non-linear functional VIDE.

DTM, which is a semi-analytical-numerical method, is based on the expansion of the Taylor series. The method principle was first used by Pukhov [14] who applied to solve linear and non-linear physical process problems, and by Zhou[15] who applied to linear and non-linear initial value problems in electrical circuit analysis. In obtaining numerical, analytical and precise solutions of ordinary and partial differential equations, this approach is beneficial and has been widely studied and applied in recent years (see, [16-20]). DTM is a reliable approach that needs less effort and does not need linearization, according to existing methods in the literature. By using this method, it is possible to solve integral and integro-differential equations [21], differential difference equations [22], partial differential equations [23], fractional-order differential equations [24].

In this manuscript, we consider the first order NVIDE with delay argument:

$$\frac{d}{dt}[u(t)+c(t)u(t-\xi(t))]=-a(t)\mu(u)-b(t)u(t-\xi(t))+\int_{t-\xi(t)}^{t} k(t,s)\,f(u(s))ds \\ +h(t,u(t),u(t-\xi(t)),\qquad t\geq 0, \tag{1}$$

where $a(t),\ b(t), c(t):[t_0,\infty)\to[0,\infty),\ t_0\geq 0,$ are continuous and $c(t)$ is differentiable with $|c(t)|\leq c_0<1\ (c_0-\text{constant})$; $h,\ k$ and f with $f(0)=0,$ are real-valued and continuous functions on their respective domains. The varying delay argument $\xi(t):[0,\infty)\to[0,\xi_0]$ is continuous and differentiable and satisfying

$$0\leq\xi(t)\leq\xi_0,\quad 0\leq\xi'(t)\leq\delta<1, \tag{2}$$

where ξ_0 and δ are positive constants.

For each solution $u(t)$ of equation (1), we assume the following existence initial condition:

$$u(\theta)=\varphi(\theta),\qquad \theta\in[t_0-\xi_0,t_0],$$

where $\varphi\in C([t_0-\xi_0,\ t_0],\ \Re).$

Define

$$\mu_0(u)=\begin{cases}\mu(u)u^{-1}, & u\neq 0\\ \dfrac{d\mu(0)}{dt}, & u=0.\end{cases} \tag{3}$$

Hence, from (1) and (3), we have

$$\begin{aligned}\frac{d}{dt}[u(t)+c(t)u(t-\xi(t))]=-a(t)\mu_0(u)u(t)-b(t)u(t-\xi(t))+\int_{t-\xi(t)}^{t}k(t,s)f(u(s))ds\\ +h(t,u(t),u(t-\xi(t)),\qquad t\geq 0.\end{aligned} \tag{4}$$

2. Main problem

In this section, before proceeding further, we will state some assumptions for main result.

2.1. Assumptions

(A1) There exists a positive constant β, such that

$$|k(t,s)|\leq\beta,\ \text{for all } t\geq 0\ .$$

(A2) There is an $M>0$, such that $|u|,|z|\leq M$ imply that,

$$|f(u)-f(z)|\leq|u-z|\quad\text{and}\quad f(0)=0.$$

(A3) Let $h(t,u,u(t-\xi(t))\in\Re$ be a non-linear uncertainty such that

$$\left|h(t,u,u(t-\xi(t))\right| \le q_1\left|u(t)\right| + q_2\left|u(t-\xi(t))\right|, \quad q_1, q_2 \ge 0 .$$

(A4) $\tilde{\mu}$ be a positive constant such that $1 \le \mu_0(u) \le \tilde{\mu}$ for all $u \in \Re$.

Theorem 1: Assumptions (A1)–(A4) are satisfied. Then, the zero solution of NVIDE (4) is asymptotically stable, if there exists a constant c_0 such that $|c(t)| \le c_0 < 1$ and

$$\Lambda = \left(\Lambda_{jk}\right) < 0, \tag{5}$$

where Λ is a 3×3 symmetric matrix with the elements $\Lambda_{11} = -2a(t) + q_1\left(c_0 + 2\right) + (\beta+1)\xi_0 + q_2 + 1,$ $\Lambda_{12} = -\left(c_0 a(t) + b(t)\right), \Lambda_{13} = 0, \Lambda_{22} = q_2 - (1-\delta) + c_0\left(-2b(t) + \beta\xi_0 + q_1 + 2q_2\right), \Lambda_{23} = 0, \Lambda_{33} = \beta + \beta c_0 - (1-\delta).$

Proof. Consider the legitimate Lyapunov functional as

$$W(t) = [u(t) + c(t)u(t-\xi(t))]^2 + \int_{t-\xi(t)}^{t} u^2(s)ds + \int_{t-\xi(t)}^{t}\int_{s}^{t} f^2(u(v))dvds.$$

From the calculation of the time derivative of the Lyapunov functional W, we have

$$\begin{aligned}
\frac{dW}{dt} &= 2[u(t)+c(t)u(t-\xi(t))][-a(t)\mu_0(u)u(t) - b(t)u(t-\xi(t)) \\
&\quad + \int_{t-\xi(t)}^{t} k(t,s) f(u(s))ds + h(t,u(t),u(t-\xi(t))] + u^2(t) \\
&\quad -(1-\xi'(t))u^2(t-\xi(t)) + \int_{t-\xi(t)}^{t} f^2(u(t))ds - (1-\xi'(t))\int_{t-\xi(t)}^{t} f^2(u(v))dv \\
&= -2a(t)\mu_0(u)u^2(t) - 2b(t)u(t)u(t-\xi(t)) + 2u(t)\int_{t-\xi(t)}^{t} k(t,s) f(u(s))ds \\
&\quad +2u(t)h(t,u(t),u(t-\xi(t)) - 2a(t)c(t)\mu_0(u)u(t)u(t-\xi(t)) \\
&\quad -2b(t)c(t)u^2(t-\xi(t)) + 2c(t)u(t-\xi(t))\int_{t-\xi(t)}^{t} k(t,s) f(u(s))ds \\
&\quad +2c(t)u(t-\xi(t))h(t,u(t),u(t-\xi(t)) + u^2(t) - (1-\xi'(t))u^2(t-\xi(t)) \\
&\quad +\xi(t) f^2(u(t)) - (1-\xi'(t))\int_{t-\xi(t)}^{t} f^2(u(v))dv \\
&\le -2a(t)u^2(t) - 2b(t)u(t)u(t-\xi(t)) + 2\beta|u(t)|\int_{t-\xi(t)}^{t} |f(u(s))|ds \\
&\quad +2q_1u^2(t) + 2q_2|u(t)||u(t-\xi(t))| - 2a(t)c(t)u(t)u(t-\xi(t))
\end{aligned}$$

$$
\begin{aligned}
&-2b(t)c(t)u^2(t-\xi(t))+2\beta c(t)\left|u(t-\xi(t))\right|\int_{t-\xi(t)}^{t}\left|f(u(s))\right|ds \\
&+2q_1c(t)\left|u(t)\right|\left|u(t-\xi(t))\right|+2q_2c(t)u^2(t-\xi(t))+u^2(t) \\
&-(1-\xi'(t))u^2(t-\xi(t))+\xi(t)f^2(u(t))-(1-\xi'(t))\int_{t-\xi(t)}^{t}f^2(u(v))dv \\
\leq&\left(-2a(t)+2q_1+1\right)u^2(t)-2\left(c_0a(t)+b(t)\right)u(t)u(t-\xi(t)) \\
&+\beta\int_{t-\xi(t)}^{t}\left(u^2(t)+f^2(u(s))\right)ds+q_2\left(u^2(t)+u^2(t-\xi(t))\right) \\
&+\left(2q_2c_0-2c_0b(t)-(1-\xi'(t))\right)u^2(t-\xi(t)) \\
&+\beta c_0\int_{t-\xi(t)}^{t}\left(u^2(t-\xi(t))+f^2(u(s))\right)ds+q_1c_0\left(u^2(t)+u^2(t-\xi(t))\right) \\
&+\xi(t)f^2(u(t))-(1-\xi'(t))\int_{t-\xi(t)}^{t}f^2(u(v))dv.
\end{aligned}
$$

From (2), we obtain

$$
\begin{aligned}
\frac{dW}{dt}\leq&\left(-2a(t)+q_1\left(c_0+2\right)+(\beta+1)\xi_0+q_2+1\right)u^2(t) \\
&-2\left(c_0a(t)+b(t)\right)u(t)u(t-\xi(t)) \\
&+\left(q_2+c_0\left(-2b(t)+\beta\xi_0+q_1+2q_2\right)-(1-\delta)\right)u^2(t-\xi(t)) \\
&+[\beta+\beta c_0-(1-\delta)]\int_{t-\xi(t)}^{t}f^2(u(s))ds.
\end{aligned}
$$

The last estimate implies that

$$\frac{dW}{dt}\leq\psi^T(t)\Lambda\Psi(t),$$

where $\psi^T(t)=[u(t)\quad u(t-\xi(t)\quad (\int_{t-\xi(t)}^{t}f^2(u(s))ds)^{1/2}]$ and Λ is defined in (5). Therefore if the matrix Λ is negative definite, $\frac{dW}{dt}$ is negative. (5) implied that there exists a constant sufficiently small $\eta>0$ such that $\frac{dW}{dt}\leq-\eta\left\|u_t\right\|$. Thus, equation (4) is asymptotically stable according to [6, Theorem 8.1, pp. 292–293]. This completes the proof of Theorem 1.

Example 1: As a special case of the equation (4), we consider the following the first order NVIDE

$$\frac{d}{dt}[u(t)+0.24u(t-0.3)]=-1.8u(t)-0.5u(t-0.3)-0.6\int_{t-0.3}^{t}u(s)ds,\quad t\geq 0. \tag{6}$$

Here

$$a(t)=1.8, b(t)=0.5, c(t)=0.24, |k(t,s)|\leq\beta=0.6, q_1=q_2=0, \xi(t)=0.3\ . \tag{7}$$

Under the above assumptions, by solving (5) using Matlab, it can be easily determined that all eigenvalues of this matrix are negative. As a result, it is clear that all the conditions of Theorem 1 hold. This discussion implies that the zero solution of equation (6) with (7) is asymptotically stable.

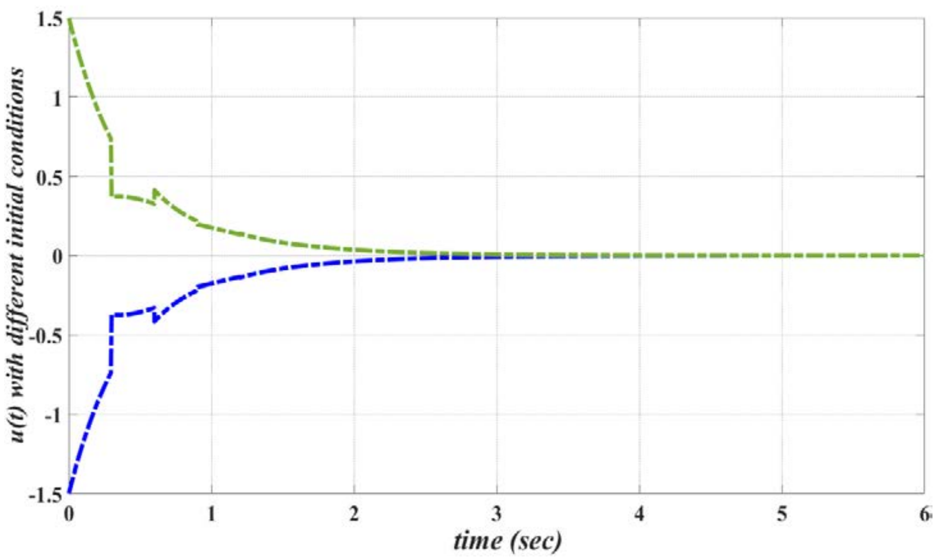

Figure 1 *The simulation of the Example 1.*

3. DTM and numerical experiment

The theory of differential transformation can be found in [14]-[15]. In this manuscript we will explain briefly. The differential transformation of function $u(t)$ is defined as

$$U(k)=\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}, \tag{8}$$

where $u(t)$ is the original function and $U(k)$ is the transformed function.

Differential inverse transformation of $U(k)$ is defined as

$$u(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}. \tag{9}$$

From (8) and (9), if the function $u(t)$ can be expressed in a finite series as follows

$$u(t)=\sum_{k=0}^{\infty}U(k)t^{k}=U(0)+U(1)t+U(2)t^{2}+\ldots, \tag{10}$$

then it is called series solution of the DTM.
The following fundamental theorems can be easily deduced from equations (8) and (9) (also, see [20], [21]).

Theorem 2: If $u(t)=\dfrac{du(t)}{dt}$, then $U(k)=\dfrac{(k+1)!}{k!}U(k+1)=(k+1)U(k+1)$.

Theorem 3: If $u(t)=\delta u(t),$ then $U(k)=\delta U(k),$ where δ is a constant.

Theorem 4: If $u(t)=u(t-a),\ a>0$ and reel constant, then

$$U(k)=\sum_{i=k}^{N}(-1)^{i-k}\binom{i}{k}a^{i-k}U(i),\ N\to\infty.$$

Theorem 5: If $u(t)=\dfrac{d}{dt}u(t-a)$, then

$$U(k)=(k+1)\sum_{i=k+1}^{N}(-1)^{i-k-1}\binom{i}{k+1}a^{i-k-1}U(i),\ N\to\infty.$$

Theorem 6: If $u(t)=\int_{t_0}^{t}u(s)ds$, then $U(k)=\dfrac{U(k-1)}{k},\ k\geq 1.$

Now, we demonstrate potentiality, advantages and effectiveness of our method on an example.

Example 2:

$$\frac{d}{dt}[u(t)+0.25u(t-0.2)]=-2u(t)-0.2u(t-0.4)-0.3\int_{t-0.3}^{t}u(s)ds,\quad t\geq 0, \tag{11}$$

$$u(0)=2.5\ . \tag{12}$$

As a special case of the equation (4), under initial condition $u(0)=2.5,$ we consider the first order NVIDE (11) with delay argument. Taking into account Theorems 2-6, applying DTM on both sides of equation (11) and condition (12), we obtain the following recurrence relation

$$U(0)=2.5,$$

$$(k+1)U(k+1)=[-0.25(k+1)\sum_{i=k+1}^{N}(-1)^{i-k-1}\binom{i}{k+1}0.2^{i-k-1}U(i)-2U(k)$$

$$-0.2\sum_{i=k}^{N}(-1)^{i-k}\binom{i}{k}(0.4)^{i-k}U(i)-0.3\frac{U(k-1)}{k}],\qquad k=0,\ 1,\ 2.$$

Using this recurrence relation, the following series coefficients $U(k)$ can be obtained.

For $N = 4,$

$$U(1) = -4.393919443, \quad U(2) = 4.302113487, \quad U(3) = -3.45438152, \quad k = 0,\ 1,\ 2.$$

For $N = 6,$

$$U(1) = -4.391032965, \quad U(2) = 4.310228598, \quad U(3) = -3.580749520, \quad k = 0,\ 1,\ 2.$$

For $N = 8,$

$$U(1) = -4.395762327, \quad U(2) = 4.300262145, \quad U(3) = -3.364722005, \quad k = 0,\ 1,\ 2.$$

Finally, using above mentioned relations, taking $N = 4,\ 6,\ 8$ and using formula (10), we reach approximate solutions of the NVIDE (11) with three iterations as follows:

$N = 4,$

$$u_{DTM}(t) = 2.5 - 4.393919443t + 4.302113487t^2 - 3.45438152t^3,$$

$N = 6,$

$$u_{DTM}(t) = 2.5 - 4.391032965t + 4.310228598t^2 - 3.580749520t^3,$$

$N = 8,$

$$u_{DTM}(t) = 2.5 - 4.395762327t + 4.300262145t^2 - 3.364722005t^3.$$

As a result, it is seen that in the cases of $N = 4,\ N = 6$ and $N = 8,$ our numerical results are almost the same.

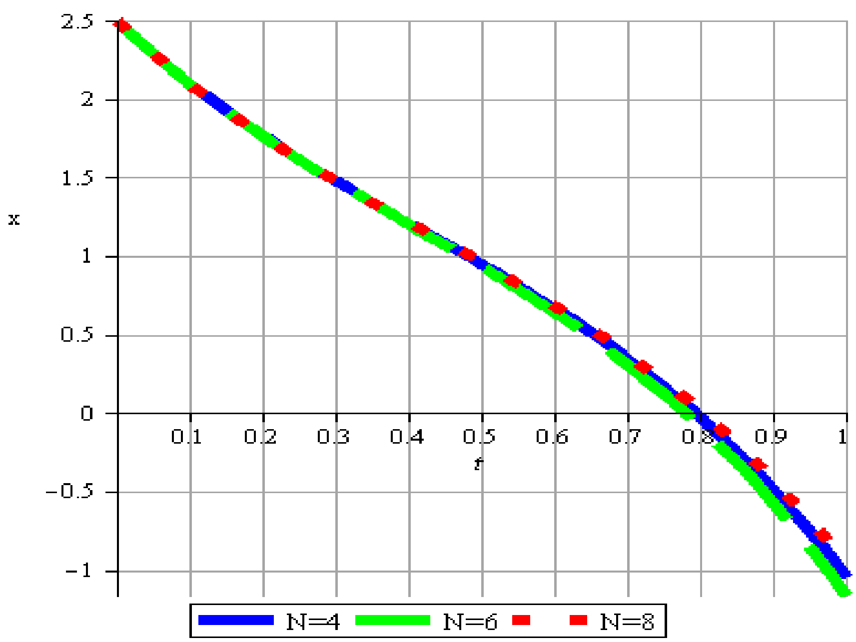

Figure 2 *Comparison between approximate solutions with DTM for* N *and* $0 \le t \le 1.$

Table 1 *Comparison of numerical results with DTM obtained.*

t	*N*=4	*N*=6	*N*=8
0.0	2.5	2.5	2.5
0.1	2.100174808	2.100418240	2.100061666
0.2	1.765665585	1.765556555	1.765940245
0.3	1.475746036	1.473930447	1.477447401
0.4	1.209689859	1.204055421	1.214394804
0.5	0.9467707560	0.9344469780	0.9565941214
0.6	0.6662624282	0.6436206197	0.6838570229
0.7	0.347438578	0.310091852	0.375995174
0.8	-0.030427096	-0.087623823	0.012820244
0.9	-0.488060893	-0.571010904	-0.425856099
1.0	-1.046189108	-1.161553887	-0.960222187

5. Conclusion

In this manuscript, we first derived some novel sufficient conditions to prove the asymptotic behaviours of solutions of the first order NVIDE with delay argument. We provided an example to show the effectiveness of proposed method by Matlab. Thereafter, for a special case of considered NVIDE using DTM, we obtained some numerical approximations as for different N ve t by a suitable computer program. We constructed the Table 1 to make a comparison between the numerical results for $N=4$, $N=6$ and $N=8$. Finally, the simulations (Figure 1 and Figure 2) in Example 1 and Example 2 show that the proposed methods are useful and feasible in terms of the obtained results.

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