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Adomian polynomials method for dynamic equations on time scales

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Abstract

A recent study on solving nonlinear differential equations by a Laplace transform method combined with the Adomian polynomial representation, is extended to the more general class of dynamic equations on arbitrary time scales. The derivation of the method on time scales is presented and applied to particular examples of initial value problems associated with nonlinear dynamic equations of first order.

Keywords: Time scale, Adomian polynomials, Laplace transform, Dynamic equation.
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1. Introduction

In a recent paper, a series solution method based on combining the Laplace transform and Adomian polynomial expansion was proposed to find an approximate solution of nonlinear differential equations [8]. It uses the expansion in Adomian polynomials defined in [1, 2]. An important drawback of the Laplace transform method is the fact that it cannot be applied in the case of nonlinear differential equation in general. In order to cope with this problem, the authors of [8] suggested the use of Adomian polynomial expansion of the nonlinear function of the dependent variable involved in the differential equation.

In this work, we propose a counterpart of this method on an arbitrary time scale and derive its general formulation for a dynamic equation of any order. We confirm that when the time scale is the set of real numbers, our method reduces to that in [8].

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Our presentation is organized as follows. First, we recollect some preliminary information on time scales in Secton 2. In Section 3, we derive the method for an n-th order nonlinear dynamic equation. The next section contains the application of the method to specific examples of first order nonlinear dynamic equations. The last section is devoted to conclusion and some further directions for study.

2. Preliminaries

We start this section with a review of some basic concepts on time scales which are used throughout the paper. A detailed information on basic calculus on time scales can be found in [3, 4, 5].

Definition 2.1. *A time scale, usually denoted by $\mathbb{T}$, is an arbitrary nonempty closed subset of the real numbers. On a time scale $\mathbb{T}$,*

1. *the forward jump operator $\sigma : \mathbb{T} \longmapsto \mathbb{T}$ is defined as*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

2. *the backward jump operator $\rho : \mathbb{T} \longmapsto \mathbb{T}$ is defined as*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

3. *the set $\mathbb{T}^\kappa$ is defined as*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if} \quad \sup\mathbb{T} < \infty \\ \\ \mathbb{T} \quad \text{otherwise}, \end{cases}$$

4. *the graininess function $\mu : \mathbb{T} \longmapsto [0, \infty)$ is defined as*

$$\mu(t) = \sigma(t) - t.$$

Clearly, $\sigma(t) \geq t$ for any $t \in \mathbb{T}$ and $\rho(t) \leq t$ for any $t \in \mathbb{T}$. We set

$$\inf\emptyset = \sup\mathbb{T}, \quad \sup\emptyset = \inf\mathbb{T}.$$

Definition 2.2. *A point $t \in \mathbb{T}$ is called*

1. *right (respectively left) dense if $\sigma(t) = t < \sup\mathbb{T}$ (respectively $\rho(t) = t > \inf\mathbb{T}$),*
2. *right (respectively left) scattered if $\sigma(t) > t$ (respectively $\rho(t) < t$),*
3. *isolated if it both right and left scattered.*

Definition 2.3. *Let $f : \mathbb{T} \longmapsto \mathbb{R}$ be a function and let $t \in \mathbb{T}^\kappa$. If for any $\epsilon > 0$ there is a neighborhood B of t, $B = (t-\delta, t+\delta) \cap \mathbb{T}$ with $\delta > 0$, such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for} \quad \text{all} \quad s \in B, \quad s \neq \sigma(t),$$

then $f^\Delta(t)$ is called the delta derivative (Hilger derivative or derivative) of f at t.

If $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$, then f is delta differentiable (Hilger differentiable or differentiable) in T^κ.

Clearly, the delta derivative is well-defined and reduces to the classical derivative when $\mathbb{T}$ is the set of the real numbers. We refer the reader to [3], [4] and [5] for more information on the delta derivative.

Next, we recall the definite integral on time scales.

Definition 2.4. 1. *A function $f : \mathbb{T} \longmapsto \mathbb{R}$ having finite right limits at all right dense points and finite left limits at all left dense points of $\mathbb{T}$ is called regulated.*

2. *A function $f : \mathbb{T} \longmapsto \mathbb{R}$ which is regulated and continuous at right dense points of $\mathbb{T}$ is called rd-continuous. The set of rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.*
3. *A continuous function $f : \mathbb{T} \longmapsto \mathbb{R}$ is called pre-differentiable with region of differentiation D, if*
   (a) $D \subset \mathbb{T}^\kappa$,
   (b) $\mathbb{T}^\kappa \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$,
   (c) f *is differentiable at each* $t \in D$.

Theorem 2.1 ([3],[4],[5]). *Let $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}$, $f : \mathbb{T}^\kappa \longmapsto \mathbb{R}$ be a given regulated function. Then there exists exactly one pre-differentiable function F satisfying*

$$F^\Delta(t) = f(t) \quad \text{for} \quad \text{all} \quad t \in D, \quad F(t_0) = x_0.$$

Definition 2.5. *If $f : \mathbb{T} \longmapsto \mathbb{R}$ is a regulated function, any function F defined in Theorem 2.1 is a pre-antiderivative of f. For a regulated function f the indefinite integral is given as*

$$\int f(t)\Delta t = F(t) + c,$$

with an integration constant c. The Cauchy integral of f is

$$\int_\tau^s f(t)\Delta t = F(s) - F(\tau) \quad \text{for} \quad \text{all} \quad \tau, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \longmapsto \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \longmapsto \mathbb{R}$ whenever we have

$$F^\Delta(t) = f(t) \quad ,$$

for all $t \in \mathbb{T}^\kappa$.

More details on delta integral can be found in [3], [4] and [5].

In the following discussion we need the definition of the generalized exponential function on time scales. Its definition is based on the regressive functions, that is, functions $f : \mathbb{T} \to \mathbb{R}$ satisfying

$$1 + \mu(t)f(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is usually denoted by $\mathcal{R}(\mathbb{T})$ or $\mathcal{R}$. The set $\mathcal{R}$ endowed with the operation $\oplus$ defined as

$$(f \oplus g)(t) = f(t) + g(t) + \mu(t)f(t)g(t),$$

is a group called regressive group $(\mathcal{R}, \oplus)$. For any $f \in \mathcal{R}$, we define

$$(\ominus f)(t) = -\frac{f(t)}{1 + \mu(t)f(t)} \quad \text{for all} \quad t \in \mathbb{T}^\kappa,$$

and the operation $\ominus$ in $\mathcal{R}$ as

$$(f \ominus g)(t) = (f \oplus (\ominus g))(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

Clearly, for $f, g \in \mathcal{R}$, we have

$$f \ominus g = \frac{f - g}{1 + \mu g}.$$

We also need the Hilger complex numbers which are defined by

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : \ z \neq -\frac{1}{h} \right\},$$

for $h > 0$ and $\mathbb{C}_0 = \mathbb{C}$. We also define

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} : \ -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h} \right\},$$

for $h > 0$, and $\mathbb{Z}_0 = \mathbb{C}$. Finally, the cylindrical transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined as

$$\xi_h(z) := \frac{1}{h}\operatorname{Log}(1 + zh),$$

where Log is the principal logarithm function. If $h = 0$, we take $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Definition 2.6. *For $f \in \mathcal{R}$, the generalized exponential function is defined as*

$$e_f(t, s) = e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} = e^{\int_s^t \frac{1}{\mu(\tau)}\operatorname{Log}(1+\mu(\tau)f(\tau))\Delta\tau} \quad \textit{for} \quad s, t \in \mathbb{T}.$$

More infomation on the generalized exponential function can be found in [3, 5].

Below we give the definition of Laplace transform on time scales.

Definition 2.7. *[5, 6] Denote by $\mathbb{T}_0$, a time scale such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = \infty$. For a function $f : \mathbb{T}_0 \to \mathbb{C}$, define the set*

$$\begin{aligned} \mathcal{D}(f) \quad = \quad & \{z \in \mathbb{C} : \ 1 + z\mu(t) \neq 0 \ \textit{for all} \ t \in \mathbb{T}_0 \\ & \textit{and the improper integral} \int_0^\infty f(x)e^{\sigma}_{\ominus z}(x, 0)\Delta x \ \textit{exists} \Big\}, \end{aligned}$$

where $e^{\sigma}_{\ominus z}(x, 0) = (e_{\ominus z} \circ \sigma)(x, 0) = e_{\ominus z}(\sigma(x), 0)$.

For all $z \in \mathcal{D}(f)$, the Laplace transform of the function f is defined as

$$\mathcal{L}(f)(z) = \int_0^\infty f(x)e^{\sigma}_{\ominus z}(x, 0)\Delta x. \tag{1}$$

Definition 2.8. *The monomials $h_k(t, s)$, $k \in \mathbb{N}_0$ on a time scale $\mathbb{T}$ are defined as follows [5].*

$$\begin{aligned} h_0(t, s) \quad &= \quad 1, \\ h_{k+1}(t, s) \quad &= \quad \int_s^t h_k(\tau, s)\Delta\tau, \end{aligned}$$

for $t, s \in \mathbb{T}$ and $k \in \mathbb{N}_0$.

Note that $h_k^{\Delta}(t, s) = h_{k-1}(t, s), \quad t, s \in \mathbb{T}, \quad k \in \mathbb{N}$.

It is shown in [6] that the Laplace transform of a monomial $h_k(t, t_0)$ is

$$\mathcal{L}(h_k(t, t_0))(z) = \frac{1}{z^{k+1}}. \tag{2}$$

The Taylor formula on a general time scale is given as follows.

Theorem 2.2 ([3, 5])**.** *Let $n \in \mathbb{N}$. Suppose f is n times Δ-differentiable on $\mathbb{T}^{\kappa^n}$. Let also, $s \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$. Then*

$$f(t) = \sum_{k=0}^{n-1} h_k(t, s)f^{\Delta^k}(s) + \int_s^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau.$$

3. Adomian polynomials method on time scales

In this section we derive the method and present its application to a dynamic equation of arbitray order with a nonlinear term.

Let $\mathbb{T}$ be a time scale with forward jump operator σ, delta differentiation operator Δ and graininess function μ. In the rest of the paper we assume that μ is delta differentiable on $\mathbb{T}$. Denote the set consisting of all possible strings $\Lambda_{n,k}$ of length n, containing exactly k times σ and $n-k$ times Δ operators by $S_k^{(n)}$. The following theorem is needed in the derivation of the method.

Theorem 3.1. *[5] For every $m, n \in \mathbb{N}_0$ we have*

$$h_n(t,s)h_m(t,s) = \sum_{l=m}^{m+n} \left(\sum_{\Lambda_{l,m} \in S_m^{(l)}} h_n^{\Lambda_{l,m}}(s,s) \right) h_l(t,s),$$

for every $t, s \in \mathbb{T}$.

For $s \in \mathbb{T}$, $l, m, n \in \mathbb{N}_0$, set

$$A_{l,m,n,s} = \sum_{\Lambda_{l,m} \in S_m^{(l)}} h_n^{\Lambda_{l,m}}(s,s).$$

By Theorem 3.1, for any $m, n \in \mathbb{N}_0$, we have

$$h_n(t,s)h_m(t,s) = \sum_{l=m}^{m+n} A_{l,m,n,s} h_l(t,s). \tag{3}$$

For $n \in \mathbb{N}_0$, $t, s \in \mathbb{T}$, define the polynomials

$$H_n^1(t,s) = (h_1(t,s))^n, \quad t, s \in \mathbb{T}.$$

Note that on any time scale $h_1(t,s) = t - s$ and we have

$$H_n^1(t,s)H_m^1(t,s) = (t-s)^n(t-s)^m = (t-s)^{n+m} = H_{n+m}^1(t,s), \quad t, s \in \mathbb{T}.$$

Note also that

$$H_1^1(t,s) = h_1(t,s), \tag{4}$$

and by (3), we get

$$\begin{aligned} H_2^1(t,s) &= h_1(t,s)h_1(t,s) \\ &= \sum_{l=1}^{2} A_{l,1,1,s} h_l(t,s) \\ &= A_{1,1,1,s} h_1(t,s) + A_{2,1,1,s} h_2(t,s) \\ &= A_{1,1,1,s} H_1^1(t,s) + A_{2,1,1,s} h_2(t,s), \end{aligned}$$

whereupon

$$h_2(t,s) = -\frac{A_{1,1,1,s}}{A_{2,1,1,s}} H_1^1(t,s) + \frac{1}{A_{2,1,1,s}} H_2^1(t,s),$$

and so on. Below we denote by B_i^j, $i, j \in \mathbb{N}$, the constants for which

$$H_n^1(t,s) = B_1^n h_1(t,s) + B_2^n h_2(t,s) + \cdots + B_n^n h_n(t,s), \quad t, s \in \mathbb{T}. \tag{5}$$

Example 3.1. *Let $\alpha \in \mathbb{R}$. Then using the Taylor formula and the fact that*

$$(e_\alpha(t,s))^{\Delta^k} = \alpha^k e_\alpha(t,s),$$

the Taylor series of $e_\alpha(t,s)$ yields

$$\begin{aligned} e_\alpha(t,s) &= 1 + \alpha h_1(t,s) + \alpha^2 h_2(t,s) + \cdots \\ &= 1 + \alpha H_1^1(t,s) \\ &\quad + \alpha^2 \left(-\frac{A_{1,1,1,s}}{A_{2,1,1,s}} H_1^1(t,s) + \frac{1}{A_{2,1,1,s}} H_2^1(t,s) \right) + \cdots \\ &= 1 + \left(\alpha - \alpha^2 \frac{A_{1,1,1,s}}{A_{2,1,1,s}} + \cdots \right) H_1^1(t,s) \\ &\quad + \left(\frac{\alpha^2}{A_{2,1,1,s}} + \cdots \right) H_2^1(t,s) + \cdots . \end{aligned}$$

Now, suppose that $u : \mathbb{T} \to \mathbb{R}$ is a given function which has a convergent series expansion of the form

$$u = \sum_{j=0}^{\infty} u_j. \tag{6}$$

Suppose also that $f : \mathbb{R} \to \mathbb{R}$ is a given analytic function such that

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{7}$$

where A_n, $n \in \mathbb{N}_0$, are given by

$$\begin{aligned} A_0 &= f(u_0) \\ A_n &= \sum_{\nu=1}^{n} c(\nu, n) f^{(\nu)}(u_0), \quad n \in \mathbb{N}. \end{aligned}$$

Here the functions $c(\nu, n)$ denote the sum of products of ν components u_j of u given in (6), whose subscripts sum up to n, divided by the factorial of the number of repeated subscripts, i.e.,

$$\begin{aligned} A_0 &= f(u_0), \\ A_1 &= c(1,1) f'(u_0) \\ &= u_1 f'(u_0), \end{aligned}$$

$$\begin{aligned}
A_2 &= c(1,2)f'(u_0) + c(2,2)f''(u_0) \\
&= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0), \\
A_3 &= c(1,3)f'(u_0) + c(2,3)f''(u_0) + c(3,3)f'''(u_0) \\
&= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0), \\
A_4 &= c(1,4)f'(u_0) + c(2,4)f''(u_0) + c(3,4)f'''(u_0) \\
&\quad + c(4,4)f^{(4)}(u_0) \\
&= u_4 f'(u_0) + \left(u_1 u_3 + \frac{u_2^2}{2}\right) f''(u_0) + \frac{u_1^2 u_2}{2} f'''(u_0) \\
&\quad + \frac{u_1^4}{4!} f^{(4)}(u_0)
\end{aligned}$$

and so on. Suppose now that u is also given by the convergent series

$$u = \sum_{n=0}^{\infty} c_n H_n^1(t, t_0). \tag{8}$$

We wish to find the respected transformed series for $f(u)$. From (6), we have

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} c_n H_n^1(t, t_0),$$

and hence,

$$u_n = c_n H_n^1(t, t_0) \quad n \in \mathbb{N}_0.$$

Thus, we obtain a series representation for f of the form

$$\begin{aligned}
f(u) &= f\left(\sum_{n=0}^{\infty} c_n H_n^1(t, t_0)\right) \\
&= \sum_{n=0}^{\infty} A^n(c_0, c_1, \ldots, c_n) H_n^1(t, t_0).
\end{aligned}$$

which compared with he expansion (7) gives the coefficients $A^n(c_0, c_1, \ldots, c_n)$ as

$$A^n(c_0, c_1, \ldots, c_n) H_n^1(t, t_0) = A_n(u_0, u_1, \ldots, u_n), \quad n = 0, 1, \ldots.$$

For $n = 0$, we have

$$\begin{aligned}
u_0 &= c_0 H_0^1(t, t_0). \\
&= c_0.
\end{aligned}$$

Thus,

$$\begin{aligned} A^0(c_0)H_0^1(t,t_0) &= A^0(c_0) \\ &= A_0(u_0). \end{aligned}$$

For $n=1$, we find

$$\begin{aligned} A^1(c_0,c_1)H_1^1(t,t_0) &= A_1(u_0,u_1) \\ &= u_1 f'(u_0), \end{aligned}$$

or

$$A^1(c_0,c_1)H_1^1(t,t_0) = c_1 H_1^1(t,t_0) f'(u_0),$$

whereupon

$$\begin{aligned} A^1(c_0,c_1) &= c_1 f'(u_0) \\ &= c_1 f'(c_0) \\ &= A_1(c_0,c_1). \end{aligned}$$

For $n=2$, we have

$$A^2(c_0,c_1,c_2)H_2^1(t,t_0) = A_2(u_0,u_1,u_2)$$

or

$$A^2(c_0,c_1,c_2)H_2^1(t,t_0) = u_2 f'(u_0) + \frac{u_1^2}{2} f''(u_0).$$

Then

$$\begin{aligned} A^2(c_0,c_1,c_2)H_2^1(t,t_0) &= c_2 H_2^1(t,t_0) f'(c_0) + \frac{c_1^2 (H_1^1(t,t_0))^2}{2} f''(c_0) \\ &= \left(c_2 f'(c_0) + \frac{c_1^2}{2} f''(c_0) \right) H_2^1(t,t_0), \end{aligned}$$

whereupon

$$\begin{aligned} A^2(c_0,c_1,c_2) &= c_2 f'(c_0) + \frac{c_1^2}{2} f''(c_0) \\ &= A_2(c_0,c_1,c_2). \end{aligned}$$

For $n=3$, we find

$$\begin{aligned} A^3(c_0,c_1,c_2,c_3)H_3^1(t,t_0) &= A_3(u_0,u_1,u_2,u_3) \\ &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0), \end{aligned}$$

or

$$A^3(c_0,c_1,c_2,c_3)H_3^1(t,t_0) = c_3 H_3^1(t,t_0) f'(c_0) + c_1 c_2 H_3^1(t,t_0) f''(c_0) + \frac{c_1^3}{3!} f'''(c_0) H_3^1(t,t_0),$$

whereupon

$$\begin{aligned} A^3(c_0, c_1, c_2, c_3) &= c_3 f'(c_0) + c_1 c_2 f''(c_0) + \frac{c_1^3}{3!} f'''(c_0) \\ &= A_3(c_0, c_1, c_2, c_3), \end{aligned}$$

and continuing in this way we get the following result.

Theorem 3.2. *Let $u : \mathbb{T} \to \mathbb{R}$ be a function with a convergent expansion given in* (8)*. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function having the form* (7)*. Then*

$$f(u) = f\left(\sum_{n=0}^{\infty} c_n H_n^1(t, t_0)\right) = \sum_{n=0}^{\infty} A_n(c_0, c_1, \dots, c_n) H_n^1(t, t_0).$$

Example 3.2. *For $\alpha = 1$, consider $u = e_\alpha(t, t_0)$ and $f(u) = u^2$. Using Example 3.1, we have*

$$e_\alpha(t, t_0) = \sum_{m=0}^{\infty} c_m H_m^1(t, t_0)$$

where

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \alpha - \alpha^2 \frac{A_{1,1,1,s}}{A_{2,1,1,s}} + \cdots +, \\ c_2 &= \frac{\alpha^2}{A_{2,1,1,s}} + \cdots, \\ &\vdots \end{aligned}$$

Note that

$$(e_\alpha(t, t_0))^2 = c_0^2 + 2 c_0 c_1 H_1^1(t, t_0) + \cdots. \tag{9}$$

On the other hand, by Theorem 3.2, we obtain

$$(e_\alpha(t, t_0))^2 = \sum_{m=0}^{\infty} A_m H_m^1(t, t_0)$$

and

$$\begin{aligned} A_0(u_0) &= A_0(c_0) \\ &= 1 \\ &= c_0^2, \\ A_1(u_0, u_1) &= c_1 f'(c_0) \\ &= 2 c_0 c_1 \end{aligned}$$

and so on, i.e., we get (9)*.*

In what follows, we present the Adomian polynomials method for a dynamic equation of arbitrary order on a general time scale $\mathbb{T}$. With $\mathcal{L}$ we will denote the Laplace transform on $\mathbb{T}$ given in (1). Suppose that $t_0 \in \mathbb{T}$. Consider the initial value problem (IVP)

$$\begin{cases} y^{\Delta^n} + a_1 y^{\Delta^{n-1}} + \cdots + a_n y = f(y), \quad t > t_0, \\ y(t_0) = y_0, \quad y^{\Delta}(t_0) = y_1, \quad \ldots, \quad y^{\Delta^{n-1}}(t_0) = y_{n-1}, \end{cases} \tag{10}$$

where $a_i \in \mathbb{R}$, $i \in \{1, \ldots, n\}$, $y_i \in \mathbb{R}$, $i \in \{0, \ldots, n-1\}$, are given constants, $f : \mathbb{R} \to \mathbb{R}$ is an analytic function. We will search a solution of the IVP (10), in the form

$$y(t) = \sum_{j=0}^{\infty} c_j H_j^1(t, t_0), \quad t \geq t_0.$$

Assume that

$$f(y) = \sum_{j=0}^{\infty} A_j(c_0, \ldots, c_j) H_j^1(t, t_0), \quad t \geq t_0.$$

Using the formula (2) given in [5], that is,

$$\mathcal{L}\left(h_k(t, t_0)\right)(z) = \frac{1}{z^{k+1}}, \quad k \in \mathbb{N}_0,$$

we get

$$\begin{aligned} \mathcal{L}\left(H_0^1(t, t_0)\right)(z) &= \frac{1}{z}, \\ \mathcal{L}\left(H_j^1(t, t_0)\right)(z) &= \sum_{k=1}^{j} B_k^j \mathcal{L}(h_k(t, t_0))(z) \\ &= \sum_{k=1}^{j} B_k^j \frac{1}{z^{k+1}}, \quad j \in \mathbb{N}. \end{aligned}$$

Let $Y(z) = \mathcal{L}(y(t))(z)$. We take the Laplace transform of both sides of the dynamic equation in (10) and using the initial conditions we obtain

$$\begin{aligned} &z^n Y(z) - \sum_{l=0}^{n-1} z^l y_{n-1-l} + a_1 z^{n-1} Y(z) - a_1 \sum_{l=0}^{n-2} z^l y_{n-2-l} \\ &\quad + \cdots + a_n Y(z) = \sum_{j=0}^{\infty} \left(A_j(c_0, \ldots, c_j) \sum_{k=1}^{j} B_k^j \frac{1}{z^{k+1}} \right) \end{aligned}$$

or

$$\begin{aligned} \left(z^n + a_1 z^{n-1} + \cdots + a_n\right) Y(z) &= \sum_{l=0}^{n-1} z^l y_{n-1-l} + a_1 \sum_{l=0}^{n-2} z^l y_{n-2-l} \\ &\quad + \cdots + a_{n-1} y_0 + A_0(c_0) \frac{1}{z} \\ &\quad + \sum_{j=1}^{\infty} \left(A_j(c_0, \ldots, c_j) \sum_{k=1}^{j} B_k^j \frac{1}{z^{k+1}} \right). \end{aligned}$$

From this equation we get

$$\begin{aligned} Y(z) &= \frac{1}{z^n + a_1 z^{n-1} + \cdots + a_n} \Bigg(\sum_{l=0}^{n-1} z^l y_{n-1-l} + a_1 \sum_{l=0}^{n-2} z^l y_{n-2-l} \\ &+ \cdots + a_{n-1} y_0 + A_0(c_0) \frac{1}{z} \\ &+ \sum_{j=1}^{\infty} \left(A_j(c_0, \ldots, c_j) \sum_{k=1}^{j} B_k^j \frac{1}{z^{k+1}} \right) \Bigg). \end{aligned}$$

Consequently,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \Bigg(\frac{1}{z^n + a_1 z^{n-1} + \cdots + a_n} \Bigg(\sum_{l=0}^{n-1} z^l y_{n-1-l} + a_1 \sum_{l=0}^{n-2} z^l y_{n-2-l} \\ &+ \cdots + a_{n-1} y_0 + A_0(c_0) \frac{1}{z} \\ &+ \sum_{j=1}^{\infty} \left(A_j(c_0, \ldots, c_j) \sum_{k=1}^{j} B_k^j \frac{1}{z^{k+1}} \right) \Bigg) \Bigg)(t) \end{aligned}$$

or by the linearity of the inverse Laplace transform,

$$\begin{aligned} y(t) &= \sum_{l=0}^{n-1} y_{n-1-l} \mathcal{L}^{-1} \left(\frac{z^l}{z^n + a_1 z^{n-1} + \cdots + a_n} \right)(t) \\ &+ a_1 \sum_{l=0}^{n-2} y_{n-2-l} \mathcal{L}^{-1} \left(\frac{z^l}{z^n + a_1 z^{n-1} + \cdots + a_n} \right)(t) \\ &+ \cdots \\ &+ a_{n-1} y_0 \mathcal{L}^{-1} \left(\frac{1}{z^n + a_1 z^{n-1} + \cdots + a_n} \right)(t) \\ &+ A_0(c_0) \mathcal{L}^{-1} \left(\frac{1}{z^{n+1} + a_1 z^n + \cdots + a_n z} \right)(t) \\ &+ \sum_{j=1}^{\infty} \left(A_j(c_0, \ldots, c_j) \sum_{k=1}^{j} B_k^j \mathcal{L}^{-1} \left(\frac{1}{z^{n+k+1} + a_1 z^{n+k} + \cdots + a_n z^{k+1}} \right)(t) \right), \end{aligned}$$

$t \geq t_0$.

After computing the inverse Laplace transform of the right-hand-side, we equate the coefficients of the functions $h_k(t, t_0)$ on both sides. In general, this results in a nonlinear system for the constants c_k, $k \in \mathbb{N}_0$.

4. Examples of IVPs for first order nonlinear dynamic equations

As a particular case, we consider an IVP associated with a first order dynamic equation of the form

$$y^{\Delta} = f(y), \quad t > t_0, \quad y(t_0) = 0, \tag{11}$$

where $f:\mathbb{R}\to\mathbb{R}$ is an analytic function. We propose a solution of the IVP (11), in the form

$$y(t)=\sum_{j=0}^{\infty}c_jH_j^1(t,t_0),\quad t\geq t_0.$$

Like in the general case, we suppose that

$$f(y)=\sum_{j=0}^{\infty}A_j(c_0,\ldots,c_j)H_j^1(t,t_0),\quad t\geq t_0.$$

On the other hand, by (5) we have

$$y(t)=c_0+\sum_{j=1}^{\infty}\sum_{k=1}^{j}c_jB_k^jh_k(t,t_0),\quad t\geq t_0,\tag{12}$$

and

$$f(y)=A_0(c_0)+\sum_{j=1}^{\infty}\sum_{k=1}^{j}A_j(c_0,\ldots,c_j)B_k^jh_k(t,t_0),\quad t\geq t_0.\tag{13}$$

Let

$$\mathcal{L}\left(y(t)\right)(z)=Y(z).$$

Then we have

$$\mathcal{L}\left(y^{\Delta}(t)\right)(z)=zY(z)-y(t_0)=zY(z).$$

Taking the Laplace transform of both sides of the dynamic equation (11) we obtain

$$\begin{aligned}zY(z)&=\mathcal{L}\left(A_0(c_0)+\sum_{j=1}^{\infty}\sum_{k=1}^{j}A_j(c_0,\ldots,c_j)B_k^jh_k(t,t_0)\right)(z)\\&=A_0(c_0)\frac{1}{z}+\sum_{j=1}^{\infty}\sum_{k=1}^{j}A_j(c_0,\ldots,c_j)B_k^j\frac{1}{z^{k+1}}.\end{aligned}$$

Then we arrive at

$$Y(z)=A_0(c_0)\frac{1}{z^2}+\sum_{j=1}^{\infty}\sum_{k=1}^{j}A_j(c_0,\ldots,c_j)B_k^j\frac{1}{z^{k+2}}.$$

Now, by taking the inverse Laplace transform of both sides, we get

$$y(t)=A_0(c_0)h_1(t,t_0)+\sum_{j=1}^{\infty}\sum_{k=1}^{j}A_j(c_0,\ldots,c_j)B_k^jh_{k+1}(t,t_0).$$

Employing (12), we have

$$c_0+\sum_{j=1}^{\infty}\sum_{k=1}^{j}c_jB_k^jh_k(t,t_0)=A_0(c_0)h_1(t,t_0)+\sum_{j=1}^{\infty}\sum_{k=1}^{j}A_j(c_0,\ldots,c_j)B_k^jh_{k+1}(t,t_0).$$

In order to equate the coefficients of the time scale monomials $h_k(t,t_0)$ on both sides, we reorder the sums as follows.

$$\begin{aligned}&c_0+\left(\sum_{j=1}^{\infty}c_jB_1^j\right)h_1(t,t_0)+\sum_{k=2}^{\infty}\left(\sum_{j=k}^{\infty}c_jB_k^j\right)h_k(t,t_0)\\&=A_0(c_0)h_1(t,t_0)+\sum_{k=2}^{\infty}\sum_{j=k-1}^{\infty}A_j(c_0,\ldots,c_j)B_{k-1}^jh_k(t,t_0).\end{aligned}$$

This results in the following nonlinear system for determining the constants c_j, $j = 0, 1, \ldots$.

$$\begin{aligned} c_0 &= 0, \\ \sum_{j=1}^{\infty} c_j B_1^j &= A_0(c_0) = f(0), \\ \sum_{j=k}^{\infty} c_j B_k^j &= \sum_{j=k-1}^{\infty} A_j(c_0, \ldots, c_j) B_{k-1}^j, \quad k \geq 2. \end{aligned} \tag{14}$$

Notice that the system is infinite and nonlinear in its unknowns. However, the nonlinearity is of polynomial type. This is a result of the nonlinear structure of the function f.

Remark 4.1. *If $\mathbb{T} = \mathbb{R}$, we have $H_1^k(t, t_0) = h_k(t, t_0) = \dfrac{(t-t_0)^k}{k!}$ for $k \in \mathbb{N}$ and hence, $B_k^j = k!\delta_{k,j}$ for $k \in \mathbb{N}$ and $j = 1, \ldots k$. In this case, the system (14) becomes*

$$\begin{aligned} c_0 &= 0, \\ k!c_k &= (k-1)!A_{k-1}(c_0, \ldots, c_{k-1}), \quad k = 1, 2, 3, \ldots, \\ \textit{or simply } c_k &= \tfrac{1}{k} A_{k-1}(c_0, \ldots, c_{k-1}) \quad k = 1, 2, 3, \ldots, \end{aligned} \tag{15}$$

which is consistent with the study given in [8].

Next, we give some particular examples.

Example 4.1. *As a first example we consider an IVP associated with a linear dynamic equation of first order of the form*

$$y^{\Delta}(t) = ay(t) + b, \quad y(0) = 0, \tag{16}$$

where a, b are real constants. Assume that

$$y(t) = \sum_{j=0}^{\infty} c_j H_j^1(t, 0), \quad t \geq 0,$$

where c_j, $j = 0, 1, \ldots$, are the coefficients to be determined. By Theorem 3.2, we have

$$f(y) = ay(t) + b = \sum_{j=0}^{\infty} A_j(c_0, \ldots, c_j) H_j^1(t, 0), \quad t \geq 0,$$

where

$$\begin{aligned} A_0 &= f(c_0) \\ &= ac_0 + b \\ A_1 &= c_1 f'(c_0) \\ &= ac_1 \\ A_2 &= c_2 f'(c_0) + \tfrac{c_1^2}{2!} f''(c_0) \\ &= ac_2 \\ A_3 &= c_3 f'(c_0) + c_1 c_2 f''(c_0) + \tfrac{c_1^3}{3!} f'''(c_0) \\ &= ac_3 \\ A_4 &= c_4 f'(c_0) + \left(c_1 c_3 + \tfrac{c_2^2}{2}\right) f''(c_0) + \tfrac{c_1^2 c_2}{2} f'''(c_0) + \tfrac{c_1^4}{4!} f^{(4)}(c_0) \\ &= ac_4 \\ &\ldots \\ A_n &= ac_n \end{aligned}$$

since $f'(c_0) = a$ *and* $f^{(k)}(c_0) = 0$ *for* $k \geq 2$. *Therefore, the system* (14) *for this example takes the form*

$$\begin{aligned} c_0 &= 0, \\ \sum_{j=1}^{\infty} c_j B_1^j &= b, \\ \sum_{j=k}^{\infty} c_j B_k^j &= \sum_{j=k-1}^{\infty} a c_{j-1} B_{k-1}^j, \quad k \in \mathbb{N}, k \geq 2. \end{aligned} \tag{17}$$

This is an infinite linear system having the following triangular form

$$\begin{aligned} c_0 &= 0 \\ c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 + \cdots &= b \\ c_2 B_2^2 + c_3 B_2^3 + c_4 B_2^4 + \cdots &= a(c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 + \cdots) = ab \\ c_3 B_3^3 + c_4 B_3^4 + c_5 B_3^5 + \cdots &= a(c_2 B_2^2 + c_3 B_2^3 + c_4 B_2^4 + \cdots) = a^2 b \\ &\cdots \\ c_n B_n^n + c_{n+1} B_n^{n+1} + \cdots &= a(c_{n-1} B_{n-1}^{n-1} + c_n B_{n-1}^n + \cdots) = a^{n-1} b. \\ &\cdots \end{aligned}$$

In the next two examples we take f to be a nonlinear function.

Example 4.2. *Consider the initial value problem associated with the first order nonlinear dynamic equation of the form*

$$y^{\Delta}(t) = e^{y(t)}, \quad t \geq 0, \quad y(0) = 0, \tag{18}$$

where $e^{y(t)}$ *is the exponential function on the set of real numbers. Assume that the solution has the series representation*

$$y(t) = \sum_{j=0}^{\infty} c_j H_j^1(t, 0), \quad t \geq 0,$$

where c_j, $j \in \mathbb{N}_0$, *are the coefficients to be determined. By Theorem 3.2 we have*

$$f(y) = e^{y(t)} = \sum_{j=0}^{\infty} A_j(c_0, \ldots, c_j) H_j^1(t, 0), \quad t \geq 0,$$

where

$$\begin{aligned} A_0 &= f(c_0) \\ &= e^{c_0} \\ A_1 &= c_1 f'(c_0) \\ &= c_1 e^{c_0} \\ A_2 &= c_2 f'(c_0) + \frac{c_1^2}{2!} f''(c_0) \\ &= \left(c_2 + \frac{c_1^2}{2!}\right) e^{c_0} \\ A_3 &= c_3 f'(c_0) + c_1 c_2 f''(c_0) + \frac{c_1^3}{3!} f'''(c_0) \\ &= \left(c_3 + c_1 c_2 + \frac{c_1^3}{3!}\right) e^{c_0} \\ A_4 &= c_4 f'(c_0) + \left(c_1 c_3 + \frac{c_2^2}{2}\right) f''(c_0) + \frac{c_1^2 c_2}{2} f'''(c_0) + \frac{c_1^4}{4!} f^{(4)}(c_0) \\ &= \left(c_4 + c_1 c_3 + \frac{c_2^2}{2} + \frac{c_1^2 c_2}{2} + \frac{c_1^4}{4!}\right) e^{c_0} \\ &\cdots \end{aligned} \tag{19}$$

The infinite nonlinear system (14) *for this example has the form*

$$\begin{aligned} c_0 &= 0, \\ \sum_{j=1}^{\infty} c_j B_1^j &= A_0(c_0), \\ \sum_{j=k}^{\infty} c_j B_k^j &= \sum_{j=k-1}^{\infty} A_j B_{k-1}^j, \quad k \geq 2, \end{aligned} \tag{20}$$

or, more explicitly,

$$\begin{aligned} c_0 &= 0, \\ c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 + \cdots &= 1, \\ c_2 B_2^2 + c_3 B_2^3 + c_4 B_2^4 + \cdots &= c_1 B_1^1 + \left(c_2 + \tfrac{c_1^2}{2!}\right) B_1^2 + \cdots, \\ c_3 B_3^3 + c_4 B_3^4 + c_5 B_3^5 + \cdots &= \left(c_2 + \tfrac{c_1^2}{2!}\right) B_2^2 + \cdots. \\ \cdots & \end{aligned}$$

Solving this nonlinear system one can approximately obtain c_i, $i \in \mathbb{N}$, *and hence, the approximate solution of the initial value problem which is*

$$y(t) = c_1 H_1^1(t,0) + c_2 H_2^1(t,0) + c_3 H_3^1(t,0) + \cdots \tag{21}$$

Example 4.3. *In the last example we consider the initial value problem associated with the first order nonlinear dynamic equation of the form*

$$y^{\Delta}(t) = y^2 + 1, \quad y(0) = 0. \tag{22}$$

Assume that

$$y(t) = \sum_{j=0}^{\infty} c_j H_j^1(t,0), \quad t \geq 0,$$

where the coefficients c_j, $j \in \mathbb{N}$ *will be determined from the nonlinear system* (14). *Let*

$$f(y) = y^2 + 1 = \sum_{j=0}^{\infty} A_j(c_0, \ldots, c_j) H_j^1(t,0), \quad t \geq 0,$$

where

$$\begin{aligned} A_0 &= f(c_0) \\ A_1 &= c_1 f'(c_0) \\ A_2 &= c_2 f'(c_0) + \tfrac{c_1^2}{2!} f''(c_0) \\ A_3 &= c_3 f'(c_0) + c_1 c_2 f''(c_0) + \tfrac{c_1^3}{3!} f'''(c_0) \\ A_4 &= c_4 f'(c_0) + \left(c_1 c_3 + \tfrac{c_2^2}{2}\right) f''(c_0) + \tfrac{c_1^2 c_2}{2} f'''(c_0) + \tfrac{c_1^4}{4!} f^{(4)}(c_0). \\ \cdots & \end{aligned} \tag{23}$$

Since $f'(c_0) = 2c_0$, $f''(c_0) = 2$ *and* $f^{(m)}(c_0) = 0$ *for* $m \geq 3$, *then we obtain*

$$\begin{aligned} A_0 &= c_0^2 + 1 \\ A_1 &= 2c_0 c_1 \\ A_2 &= 2c_0 c_2 + c_1^2 \\ A_3 &= 2c_0 c_3 + 2c_1 c_2 \\ A_4 &= 2c_0 c_4 + 2c_1 c_3 + c_2^2 \\ \cdots & \end{aligned} \tag{24}$$

The nonlinear infinite system (14) *becomes*

$$\begin{aligned} c_0 &= 0, \\ \sum_{j=1}^{\infty} c_j B_1^j &= A_0(c_0), \\ \sum_{j=k}^{\infty} c_j B_k^j &= \sum_{j=k-1}^{\infty} A_j(c_0, \ldots, c_j) B_{k-1}^j, \quad k \geq 2. \end{aligned} \tag{25}$$

If, in particular, the time scale under consideration is $\mathbb{T} = \mathbb{Z}$, *then*

$$h_0(t,0) = 1, \quad h_1(t,0) = t, \quad h_k(t,0) = \frac{t(t-1)\ldots(t-k+1)}{k!}, k = 2, 3, \cdots,$$

and hence, we compute

$$\begin{aligned} H_1^1(t,0) &= t = h_1(t,0) \\ H_2^1(t,0) &= t^2 = 2h_2(t,0) + h_1(t,0) \\ H_3^1(t,0) &= t^3 = 6h_3(t,0) + 6h_2(t,0) + h_1(t,0) \\ H_4^1(t,0) &= t^4 = 24h_4(t,0) + 36h_3(t,0) + 14h_2(t,0) + h_1(t,0). \\ &\cdots \end{aligned}$$

Then, the system (25) *turns into*

$$\begin{aligned} c_0 &= 0, \\ c_1 + c_2 + c_3 + c_4 + \cdots &= 1, \\ 2c_2 + 6c_3 + 14c_4 + \cdots &= c_1^2 + 2c_1c_2 + (2c_1c_3 + c_2^2) + \cdots, \\ 6c_3 + 36c_4 + \cdots &= 2c_1^2 + 12c_1c_2 + 14(2c_1c_3 + c_2^2) + \cdots, \\ 24c_4 + \cdots &= 12c_1c_2 + 36(2c_1c_3 + c_2^2) + \cdots. \\ &\cdots \end{aligned}$$

5. Conclussion

The method developed in this study makes it possible to use the Laplace transform technique in the case of nonlinear dynamic equations. It is easy to see that the method can be efficiently applied when dealing with initial value problems having homogenous initial conditions. The weakness shows itself in the fact that finding the approximate solution requires solving an infinite nonlinear algebraic system. For computational purposes, one needs to truncate this system. As a future study, the Adomian polynomials method developed in this paper can be also applied to both linear and nonlinear integral equations on time scales which have been recently presented in the books [5, 7].

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