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Solvability of Fractional Boundary Value Problems for a Combined Caputo Derivative

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Abstract

This paper deals with a class of boundary value problems of fractional differential equations involving the combined Caputo derivative, which can reflect both the past and the future nonlocal memory effects. This fractional derivative is a convex combination of the left and right Caputo fractional derivative of different order. By using the fractional Gronwall-type inequalities and some fixed point theorems, we have established some sufficient conditions for the existence and uniqueness of solutions for fractional boundary value problems with the combined Caputo fractional derivative. Various examples are given to illustrate the applications of the results.

Keywords: *Combined Caputo derivative; Existence; Fixed point.*
2010 Mathematics Subject Classification: *34A12.*

1. Introduction

Fractional integration and differentiation have been widely used in variety of the branch of science including physics, engineering, chemistry, biology and finance [12, 15, 16, 22, 23]. Some recent books cover the fractional calculus [10, 17, 13]. Fractional calculus can better describe the memory properties of the physical process than the standard integer order calculus. Geometric and physical interpretation of fractional differentiation and integration can be found in [18]. Recently, the existence and uniqueness of solutions for fractional differential equations have attracted a good deal of attention and have been developed by many authors; (see, e.g., [10, 17, 13, 1, 2, 6, 5, 9, 38, 25, 27, 24, 28, 26] and the references therein). On the other hand, analytical and numerical solutions for fractional differential equations using different definitions for fractional derivatives and integrals have been proposed and studied in the literature [20, 29, 30, 31, 32, 33, 34, 35].
In this paper, we consider the combined Caputo operator ${}_{a}^{C}D_{b}^{\nu,\mu}$ which is a convex combination of the left Caputo fractional derivative of order ν and the right Caputo fractional derivative of order μ on $[a,b]$. The idea of combining the left and right fractional derivatives is not new. The Riesz space fractional derivative is the combined fractional Riemann–Liouville derivative [10, 17] of the same order. This symmetric fractional derivative involves the fractional Riemann–Liouville derivative instead of the Caputo derivative. Some nonconservative models such as stationarity–conservation laws with variable coefficients can be better described using the Riesz space derivative [11]. However, this symmetric fractional derivative is not suitable for some variational problems which have continuous solution with symmetric fractional derivatives [19]. Thus, we can better describe a more general class of fractional boundary/initial value problems by using of Caputo derivative.
A considerable number of papers has been devoted to the solvability of fractional initial/boundary value problems involving the fractional Riemann–Liouville and/or the Caputo fractional derivatives in the literature . However, both of these fractional operators are one sided operator and thus, they hold either past or future memory effects. In contrast, the main feature of the combined Caputo fractional operator is that it is a two sided operator which holds both the history and future non-local memory effects. This property of the combined Caputo fractional operator plays a decisive role in the mathematical modelling of physical processes on a finite domain because the present states of many processes depend both on the past and future concentrations. Some applications of the combined Caputo derivative to anomalous diffusion problems have been studied in [3, 21].
Numerical solutions of anomalous diffusion problems with two sided operator have been proposed and studied in [7, 37, 21, 4, 3, 36]. Recently, the authors obtained some existence and uniqueness results of fractional differential equations in [4] and investigated the existence of positive solutions for the fractional boundary value problems with the Riesz-Caputo derivative in [8]. The Riesz-Caputo derivative is a combined Caputo operator with particularly chosen parameters.
To the best of our knowledge, there is no work on the existence of the fractional boundary value problems (FBVPs) of the combined Caputo fractional differential equations with the Dirichlet boundary condition. In this paper, we investigate the existence and uniqueness of solutions for the following fractional boundary value problems:

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$$
\begin{aligned}
&{}^{C}_{a}D^{\nu,\mu}_{b} u(\eta) = F(\eta, u(\eta)) \quad \nu \in (1,2], \quad a \le \eta \le b, \\
&\qquad u(a) = u_a, \quad u(b) = u_b,
\end{aligned} \tag{1.1}
$$

where ${}^{C}_{a}D^{\nu,\mu}_{b}$ is the combined Caputo derivative defined below and $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $u_a, u_b \in \mathbb{R}$.
The remainder of paper is organized as follows. Section 2 introduces some preliminaries, definitions and lemmas which are useful in proving main results. Section 3 provides some sufficient conditions for the existence and uniqueness of solutions for the problem (1.1). We establish these results by using two fixed point theorems, namely, Schauder and Leray–Schauder. Finally, some numerical examples are given to illustrate the applications of the main results.

2. Preliminaries

In this section, we give some useful definitions and lemmas that will be used in this paper.

Definition 2.1. *[10] Let $\nu > 0$. The left and right Riemann-Liouville fractional integral of a function $f \in L^1([a,b])$ of order ν defined as, respectively*

$$I^{\nu}_{a} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-s)^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

$${}_{b}I^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_x^b (s-x)^{\nu-1} f(s)\, ds, \quad x \in [a,b].$$

Definition 2.2. *(Combined Riemann Fractional Integral) Let $\nu, \mu \in (0,1]$, $\gamma \in [0,1]$. The combined Caputo fractional integral of a function $f \in L^1([a,b])$ of order ν defined as*

$${}^{\gamma}_{b}I^{\nu,\mu}_{a} f(x) = I^{\nu}_{a} f(x) + {}_{b}I^{\mu} f(x), \quad x \in [a,b].$$

Note that the Riesz fractional integral is an example of the combined Riemann fractional integral operator when $\nu = \mu$ and takes the following form

$${}_{b}I^{\nu}_{a} f(x) = \frac{1}{\Gamma(\nu)} \int_a^b |x-s|^{\nu-1} f(s)\, ds, \quad x \in [a,b]. \tag{2.1}$$

Definition 2.3. *[10] Let $\nu \in (0,1]$. The left and right Caputo fractional derivative of a function $f \in AC([a,b])$ of order ν defined as, respectively*

$$
\begin{aligned}
{}^{C}_{a}D^{\nu}_{x} f(x) &= \frac{1}{\Gamma(1-\nu)} \int_a^x (x-s)^{-\nu} f'(s)\, ds = I^{1-\nu}_{a} f'(x), \\
{}^{C}_{x}D^{\nu}_{b} f(x) &= \frac{-1}{\Gamma(1-\nu)} \int_x^b (s-x)^{-\nu} f'(s)\, ds = -{}_{b}I^{n+1-\nu} f'(x),
\end{aligned}
$$

where $AC([a,b])$ is the space of absolutely continuous functions on $[a,b]$.

Definition 2.4. *[15] Let $\nu, \mu \in (0,1], \gamma \in [0,1]$. The combined Caputo fractional derivative ${}^{C}_{a}D^{\nu,\mu}_{b}$ of order (ν,μ) of a function $f \in AC[a,b]$ defined by*

$${}^{C}_{a}D^{\nu,\mu}_{b} f(x) = \gamma {}^{C}_{a}D^{\nu}_{x} f(x) + (1-\gamma) {}^{C}_{x}D^{\mu}_{b} f(x) = \gamma I^{1-\nu}_{a} f'(x) - (1-\gamma) {}_{b}I^{1-\mu} f'(x).$$

Lemma 2.5. *[10] Let $f \in C^n[a,b]$ and $\nu, \mu \in (n, n+1]$. Then we have the following relations*

$$I^{\nu}_{a} {}^{C}_{a}D^{\nu}_{x} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k,$$

$${}_{b}I^{\mu} {}^{C}_{x}D^{\mu}_{b} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^(k) f^{(k)}(b)}{k!} (b-x)^k.$$

Definitions and lemmas above lead to have the following

$$
\begin{aligned}
{}^{\gamma}_{b}I^{\nu,\mu}_{a} {}^{C}_{a}D^{\nu,\mu}_{b} f(x) &= \gamma \left(I^{\nu}_{a} {}^{C}_{a}D^{\nu}_{x} f(x) + {}_{b}I^{\mu} {}^{C}_{a}D^{\nu}_{x} f(x) \right) + (-1)^{n+1}(1-\gamma) \left(I^{\nu}_{a} {}^{C}_{x}D^{\mu}_{b} f(x) + {}_{b}I^{\mu} {}^{C}_{x}D^{\mu}_{b} f(x) \right) \\
&= \gamma I^{\nu}_{a} {}^{C}_{a}D^{\nu}_{x} f(x) + (-1)^{n+1}(1-\gamma) {}_{b}I^{\mu} {}^{C}_{x}D^{\mu}_{b} f(x).
\end{aligned}
$$

If $\nu, \mu \in (0,1]$ then we have the following simplified form

$${}^{\gamma}_{b}I^{\nu,\mu}_{a} {}^{C}_{a}D^{\nu,\mu}_{b} f(x) = f(x) - \gamma f(a) - (1-\gamma) f(b) \tag{2.2}$$

The following fractional Gronwall inequalities will be frequently used in the sequel.

Lemma 2.6. *[4] Assume that $\nu,\mu > 0$, $\beta_i(x), i = 1,2$ are non-negative nondecreasing and locally integrable functions, $\alpha_i(x), i = 1,2$ are non-negative nondecreasing continuous real valued functions and assume also that $v(x)$ is nonnegative and locally integrable on $[a,b]$ satisfying*

$$v(x) \leq \beta_1(x) + \alpha_1(x)\int_a^x (x-s)^{\nu-1}v(s)\,ds.$$

Then we have the following

$$v(x) \leq \beta_1(x)\mathrm{E}_{\nu,1}(\alpha_1\Gamma(\nu)(x-a)^{\nu}),$$

where $\mathrm{E}_{\nu,1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\nu+1)}$ *is the generalized Mittag-Leffler function.*
If $v(x)$ satisfies the following inequality

$$v(x) \leq \beta_2(x) + \alpha_2(x)\int_x^b (s-x)^{\mu-1}v(s)\,ds,$$

then we have

$$v(x) \leq \beta_2(x)\mathrm{E}_{\mu,1}(\alpha_2\Gamma(\mu)(b-x)^{\mu}).$$

Lemma 2.7. *Assume that the conditions of Lemma 2.6 hold true. If $v(x)$ is nonnegative and locally integrable on $[a,b]$ satisfying*

$$v(x) \leq \beta_1(x) + \alpha_1(x)\int_a^x (x-s)^{\nu-1}v(s)\,ds + \beta_2(x) + \alpha_2(x)\int_x^b (s-x)^{\mu-1}v(s)\,ds,$$

then we have the following

$$v(x) \leq (\beta_1(x)+\beta_2(x))\mathrm{E}_{\mu,1}(\alpha_2\Gamma(\mu)(b-x)^{\mu})\mathrm{E}_{\nu,1}(\alpha_1\Gamma(\nu)(x-a)^{\nu}).$$

Proof. We infer from Lemma 2.6 that

$$\begin{aligned} v(x) &\leq \Big(\beta_1(x)+\beta_2(x)+\alpha_2\int_x^b (s-x)^{\mu-1}v(s)\,ds\Big)\mathrm{E}_{\nu,1}(\alpha_1\Gamma(\nu)(x-a)^{\nu}) \\ &\leq (\beta_1(x)+\beta_2(x))\mathrm{E}_{\mu,1}(\alpha_2\Gamma(\mu)(b-x)^{\mu})\mathrm{E}_{\nu,1}(\alpha_1\Gamma(\nu)(x-a)^{\nu}). \end{aligned}$$

□

Lemma 2.8. *[24] Let X be a Banach space and B be a closed and convex subset of X. If C is a open subset of B and $T : C \to C$ is a continuous and compact operator, then one of the following hols:*

1. *The operator has a fixed point in C,*
2. *There is a point $c \in \partial C$ with $0 < \mu < 1$ such that $c = \mu T(c)$.*

Lemma 2.9. *[4] Let X be a Banach space with $Y \subset X$ closed, bounded and convex, and $T : Y \to Y$ is completely continuous. Then T has a fixed point in Y.*

3. Existence Results

Let $E = C[a,b]$ denote the Banach space with the norm defined as $\|u\| = \sup\{|u(t)| : t \in J = [a,b]\}$.
We say that $u \in C^2(J)$ with ${}_a^CD_b^{\nu,\mu}u$ exists on J is a solution of the problem (1.1) if u solves the equation ${}_a^CD_b^{\nu,\mu}u(x) = F(x,u(x))$ for each $t \in J$ and the conditions $u(a) = u_0$ and $u(b) = u_b$ are fulfilled. We denote the set of positive real numbers by $\mathbb{R}_{>0}$ and the set of non-negative real numbers by $\mathbb{R}_{\geq 0}$.
In order to prove the existence results for the problem (1.1), the following lemmas are useful.

Lemma 3.1. *The solution $u(x)$ to the fractional boundary value problem (FBVP) of the combined Caputo differential equation (1.1) is given by the following integral equations*

$$u(x) = \gamma u_a + (1-\gamma)u_b + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}F(s,u(s))\,ds + \frac{1}{\Gamma(\nu)}\int_x^b (s-x)^{\mu-1}F(s,u(s))\,ds. \tag{3.1}$$

Proof. Applying the combined Riemann integral operator to both side of the problem (1.1) and using the relation (2.2) reveal that for $\nu,\mu \in (0,1]$

$$u(x) = \gamma u_a + (1-\gamma)u_b + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}F(s,u(s))\,ds + \frac{1}{\Gamma(\nu)}\int_x^b (s-x)^{\mu-1}F(s,u(s))\,ds. \tag{3.2}$$

This is the desired result and we complete the proof. □

By the help of Lemma 3.1, we define the integral operator $K : C(J) \to C(J)$ as follows

$$Ku(x) = \gamma u_a + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}F(s,u(s))\,ds + (1-\gamma)u_b + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}F(s,u(s))\,ds. \tag{3.3}$$

Theorem 3.2. *Assume that $\phi \in C(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$, $\psi \in C(J, \mathbb{R}_{>0})$ and $R \in L^1(J, \mathbb{R}_{\geq 0})$. Assume further that ϕ is nondecreasing and the following hold*

A1. $|F(x,u(x))| \leq \phi(|u|)\psi(x) + R(x)$,

A2. $\phi(t) \leq \dfrac{t}{||\psi|| \max\{(b-a)^{\mu},(b-a)^{\nu}\}}$.

Then the FBVP (1.1) has a solution $u(x) \in C(J)$.

Proof. Define a closed, bounded and convex set $S_r := \{u \in C(J) : ||u|| < r\}$, where

$$r = \mathrm{E}_{\mu,1}(1)\mathrm{E}_{\nu,1}(1)\Big(\max\{|u_a|,|u_b|\} + 2M\Big) + 1,$$

where $M = \max\left\{\dfrac{1}{\Gamma(\nu)}\sup_{x\in J}\displaystyle\int_a^x (x-s)^{\nu-1}R(s)\,ds, \dfrac{1}{\Gamma(\mu)}\sup_{x\in J}\displaystyle\int_x^b (s-x)^{\mu-1}R(s)\,ds\right\}$.

Note that the existence of M follows from the fact that $R \in L^1(J, \mathbb{R}_{\geq 0})$. We prove that the operator K has a fixed point by making use of Lemma 2.8.
Let $\lambda \in (0,1)$ and $u \in S_r$ with

$$u = \lambda K u. \tag{3.4}$$

We show that $||u|| \neq r$. The equation (3.4) implies that

$$\begin{aligned}
|u(x)| \leq& \gamma|u_a| + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}F(s,u(s))\,ds + (1-\gamma)|u_b| + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}F(s,u(s))\,ds \\
\leq& \gamma|u_a| + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}(\phi(|u(s)|)\psi(s)+R(s))\,ds + (1-\gamma)|u_b| + \frac{1}{\Gamma(\nu)}\int_x^b (s-x)^{\mu-1}(\phi(|u(s)|)\psi(s)+R(s))\,ds \\
\leq& \left(\gamma|u_a| + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}R(s)\,ds\right) + \frac{1}{\Gamma(\nu)||\psi||\max\{(b-a)^{\nu},(b-a)^{\mu}\}}\int_a^x (x-s)^{\nu-1}\psi(s)|u(s)|\,ds \\
&+ \left((1-\gamma)|u_b| + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1} + R(s)\,ds\right) + \frac{1}{\Gamma(\mu)||\psi||\max\{(b-a)^{\nu},(b-a)^{\mu}\}}\int_x^b (s-x)^{\mu-1}\psi(s)|u(s)|\,ds \\
\leq& (\gamma|u_a| + M) + ((1-\gamma)|u_b| + M) + \frac{1}{\max\{(b-a)^{\nu},(b-a)^{\mu}\}}\left(\frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}|u(s)|\,ds + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}|u(s)|\,ds\right).
\end{aligned}$$

Using Lemma 2.7, we have

$$\begin{aligned}
|u(x)| \leq& \Big(\gamma|u_a| + (1-\gamma)|u_b| + 2M\Big)\mathrm{E}_{\mu,1}\left(\frac{\Gamma(\mu)(b-x)^{\mu}}{\Gamma(\mu)\max\{(b-a)^{\nu},(b-a)^{\mu}\}}\right)\mathrm{E}_{\nu,1}\left(\frac{\Gamma(\nu)(x-a)^{\nu}}{\Gamma(\nu)\max\{(b-a)^{\nu},(b-a)^{\mu}\}}\right) \\
\leq& \Big(\gamma|u_a| + (1-\gamma)|u_b| + 2M\Big)\mathrm{E}_{\mu,1}(1)\mathrm{E}_{\nu,1}(1) \\
<& r,
\end{aligned}$$

which concludes that $||u|| \neq r$.
The continuity of the operator K follows easily from the continuity of the function F.
We next show that the operator K is completely continuous. To this end, take $x_1, x_2 \in J, x_1 < x_2$ and $u \in S_r$. Then we have

$$\begin{aligned}
|Ku(x_1) - Ku(x_2)| \leq& \left|\frac{1}{\Gamma(\nu)}\int_a^{x_1}(x_1-s)^{\nu-1}F(s,u(s))\,ds - \frac{1}{\Gamma(\nu)}\int_a^{x_2}(x_2-s)^{\nu-1}F(s,u(s))\,ds\right| \\
&+ \left|\frac{1}{\Gamma(\mu)}\int_{x_1}^b (s-x_1)^{\mu-1}F(s,u(s))\,ds - \frac{1}{\Gamma(\mu)}\int_{x_2}^b (s-x_2)^{\mu-1}F(s,u(s))\,ds\right| \\
=& \left|\frac{1}{\Gamma(\nu)}\int_a^{x_1}\Big((x_1-s)^{\nu-1} - (x_2-s)^{\nu-1}\Big)F(s,u(s))\,ds - \frac{1}{\Gamma(\nu)}\int_{x_1}^{x_2}(x_2-s)^{\nu-1}F(s,u(s))\,ds\right| \\
&+ \left|\frac{1}{\Gamma(\mu)}\int_{x_2}^b \Big((s-x_1)^{\mu-1} - (s-x_2)^{\mu-1}\Big)F(s,u(s))\,ds + \frac{1}{\Gamma(\mu)}\int_{x_1}^{x_2}(s-x_1)^{\mu-1}F(s,u(s))\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\nu)}\int_a^{x_1}\Big((x_1-s)^{\nu-1}-(x_2-s)^{\nu-1}\Big)(\phi(|u(s)|)\psi(s)+R(s))\,ds+\frac{1}{\Gamma(\nu)}\int_{x_1}^{x_2}(x_2-s)^{\nu-1}(\phi(|u(s)|)\psi(s)+R(s))\,ds\\
&+\frac{1}{\Gamma(\mu)}\int_{x_2}^{b}\Big((s-x_1)^{\mu-1}-(s-x_2)^{\mu-1}\Big)(\phi(|u(s)|)\psi(s)+R(s))\,ds+\frac{1}{\Gamma(\mu)}\int_{x_1}^{x_2}(s-x_1)^{\mu-1}(\phi(|u(s)|)\psi(s)+R(s))\,ds\\
&\le N(\nu)\Big(\int_a^{x_1}\Big((x_1-s)^{\nu-1}-(x_2-s)^{\nu-1}\Big)ds+\int_{x_1}^{x_2}(x_2-s)^{\nu-1}ds\Big)\\
&+\frac{1}{\Gamma(\nu)}\Big(\int_a^{x_1}\Big((x_1-s)^{\nu-1}-(x_2-s)^{\nu-1}\Big)R(s)\,ds+\int_{x_1}^{x_2}(x_2-s)^{\nu-1}R(s)\,ds\\
&+N(\mu)\Big(\int_{x_2}^{b}\Big((s-x_1)^{\mu-1}-(s-x_2)^{\mu-1}\Big)ds+\int_{x_1}^{x_2}(s-x_1)^{\mu-1}ds\Big)\\
&+\frac{1}{\Gamma(\mu)}\Big(\int_{x_2}^{b}\Big((s-x_1)^{\mu-1}-(s-x_2)^{\mu-1}\Big)R(s),ds+\int_{x_1}^{x_2}(s-x_1)^{\mu-1}R(s)\,ds\Big)\\
&\le N(\nu+1)\Big((x_2^\nu-x_1^\nu)+\Big((b-x_1)^\nu-(b-x_2)^\nu\Big)\Big)+\frac{1}{\Gamma(\nu)}\Big(\int_a^{x_1}\Big((x_1-s)^{\nu-1}-(x_2-s)^{\nu-1}\Big)R(s)\,ds+\int_{x_1}^{x_2}(x_2-s)^{\nu-1}R(s)\,ds\Big)\\
&+\frac{1}{\Gamma(\mu)}\Big(\int_{x_2}^{b}\Big((s-x_1)^{\mu-1}-(s-x_2)^{\mu-1}\Big)R(s),ds+\int_{x_1}^{x_2}(s-x_1)^{\mu-1}R(s)\,ds\Big),
\end{aligned}
\tag{3.5}
$$

where $N(\alpha):=\dfrac{r}{\Gamma(\alpha)\max\{(b-a)^\mu,(b-a)^\nu\}}$ for either $\alpha=\nu$ or $\alpha=\mu$. Since the functions in the above integral (3.5) belong to the space of integrable function, namely, L^1- space, because $R\in L^1(J,\mathbb{R}_{\ge 0})$, the right-hand side of the above inequality (3.5) goes to zero when $x_2\to x_1$. This shows that the set KS_r is equicontinuous set in $C(J)$. Uniform boundedness of the set KS_r easily follows from the conditions A1 and A2. Thus we have proved that KS_r is relatively compact and the operator K is completely continuous. Then Lemma 2.8 concludes that the operator K must have a fixed point which is a solution $u(x)$ to the problem (1.1). □

Corollary 3.3. *If $\psi\in C(J,\mathbb{R}_{>0})$ and $R\in L^1(J,\mathbb{R}_{\ge 0})$, and $|F(x,u)|\le\psi(x)|u|+R(s)$ with $||\psi||\le\dfrac{1}{\max\{(b-a)^\mu,(b-a)^\nu\}}$, then there exists a solution $u(x)$ to the problem (1.1).*

Proof. Let $\phi(t)=t$, then ϕ is nondecreasing function with $\phi(|u|)=|u|$. Moreover, the condition $||\psi||\le\dfrac{1}{\max\{(b-a)^\mu,(b-a)^\nu\}}$ is equal to $\dfrac{1}{||\psi||\max\{(b-a)^\mu,(b-a)^\nu\}}\ge 1$ which in turn implies that $\phi(t)=t\le\dfrac{t}{||\psi||\max\{(b-a)^\mu,(b-a)^\nu\}}$. So, all the conditions of Theorem 3.2 are satisfied, thus there exists a solution $u(x)$ of the problem by Theorem 3.2. □

Theorem 3.4. *Assume that R is as in the previous theorem, $T\in C(\mathbb{R},\mathbb{R}_{\ge 0})$ and the following assumptions hold true*

A3. $|F(x,u)|\le T(u)+R(x)$,
A4. $\lim_{t\to\pm\infty}\dfrac{T(t)}{|t|}:=T^*\le\Gamma_\nu^\mu:=\min\{\dfrac{\Gamma(\nu+1)}{(b-a)^\nu},\dfrac{\Gamma(\mu+1)}{(b-a)^\mu}\}$.

Then there exists a solution $u(x)$ to the problem (1.1).

Proof. Set $\delta=\frac{1}{2}\Big(\Gamma_\nu^\mu-T^*\Big)$. Then there is a $r_1>0$ such that

$$T(t)\le\Big(\Gamma_\nu^\mu-\delta\Big)\frac{|r|}{2},\quad for\quad |r|\ge r_1.$$

Let $M_T=\max_{t\in[-r_1,r_1]}T(t)$ and $r_2:=\max\{r_1,\dfrac{2M_T}{\Gamma_\nu^\mu-\delta}\}$. Let

$$r=\max\Big\{(\Gamma_\nu^\mu/\delta)(2M+\max\{|u_a|,|u_b|\}),r_2\Big\}.$$

If $u\in S_r$, then we have

$$T(t)\le\frac{1}{2}\Big(^\mu-\delta\Big)L$$

and

$$
\begin{aligned}
|Ku(x)|\le&\gamma|u_a|+\frac{1}{\Gamma(\nu)}\int_a^x(x-s)^{\nu-1}|F(s,u(s))|\,ds+(1-\gamma)|u_b|+\frac{1}{\Gamma(\mu)}\int_x^b(s-x)^{\mu-1}|F(s,u(s))|\,ds\\
\le&\gamma|u_a|+\frac{1}{\Gamma(\nu)}\int_a^x(x-s)^{\nu-1}(T(u(s))+R(s))\,ds+(1-\gamma)|u_b|+\frac{1}{\Gamma(\mu)}\int_x^b(s-x)^{\mu-1}(T(u(s))+R(s))\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \gamma|u_a| + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}R(s)\,d + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}T(u(s))\,ds + (1-\gamma)|u_b| + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}R(s)\,ds + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}T(u(s))\,ds\\
&\le (\gamma|u_a|+M) + (\Gamma_\nu^\mu - \delta)\frac{(x-a)^\nu L}{2\Gamma(\nu+1)} + ((1-\gamma)|u_b|+M) + (\Gamma_\nu^\mu - \delta)\frac{(b-x)^\mu L}{2\Gamma(\mu+1)}\frac{\delta L}{\Gamma_\nu^\mu} + (\Gamma_\nu^\mu - \delta)\frac{L}{\Gamma_\nu^\mu}\\
&= r.
\end{aligned}
$$

We have shown that $K : S_r \to S_r$. The complete continuity of K can be proved using similar idea of the proof of Theorem 3.2. Lemma 2.9 implies that K has a fixed point in S_r which is a solution of the problem (1.1). □

Corollary 3.5. *Assume that $F(x,0) \not\equiv 0$ and the assumption*

$$
\lim_{u\to\pm\infty} \max_{x\in[a,b]} \left|\frac{F(x,u)}{u}\right| \le \Gamma_\nu^\mu
$$

holds true. Then there exists a solution $u(x) \in C(J)$ of the FBVP (1.1).

Proof. Set $T(u) = \max_{x\in[a,b]} |F(x,u(x))|$ and $R(x) \equiv 0$. Then $T \in C(\mathbb{R},\mathbb{R}_{\ge 0})$ and the assumptions of Theorem 3.4 are satisfied. Thus, there is a solution $u(x)$ of the problem (1.1) by Theorem 3.4. □

Theorem 3.6. *Assume that $T \in C(\mathbb{R},\mathbb{R}_{\ge 0}), H \in L^1(J,\mathbb{R}_{>0})$ and $R \in L^1(J,\mathbb{R}_{\ge 0})$ and the following assumptions hold true:*

A5. $|F(x,u)| \le T(u)H(x) + R(x)$,
A6. $\lim_{t\to\pm\infty} T(t) < \infty$.

Then there exists a solution $u(x) \in C(J)$ of (1.1).

Proof. The condition A6 implies that $T(t) \le b_1$ when $|r| \ge r_3$ for some positive constants b_1, r_3. Let $M_T = \max\{b_1, \max_{t\in[-r_3,r_3]} T(t)\}$. Then it follows that $T(t) \le M_T \quad \forall t \in \mathbb{R}$. Set

$$
M_H := \max\left\{\frac{1}{\Gamma(\nu)}\sup_{x\in J}\int_a^x (x-s)^{\nu-1}H(s)\,ds, \frac{1}{\Gamma(\mu)}\sup_{x\in J}\int_x^b (s-x)^{\mu-1}H(s)\,ds\right\}.
$$

Let $r = \max\{\max\{|u_a|,|u_b|\} + 2M + 2M_H M_T, r_3\}$.
If $u \in S_r$, we have

$$
\begin{aligned}
|Ku(x)| &\le \gamma|u_a| + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}|F(s,u(s))|\,ds + (1-\gamma)|u_b| + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}|F(s,u(s))|\,ds\\
&\le \gamma|u_a| + \frac{1}{\Gamma(\nu)}\int_a^x (x-s)^{\nu-1}(T(u(s))H(s)+R(s))\,ds + (1-\gamma)|u_b| + \frac{1}{\Gamma(\mu)}\int_x^b (s-x)^{\mu-1}(T(u(s))H(s)+R(s))\,ds\\
&\le \gamma|u_a| + \frac{1}{\Gamma(\nu)}\sup_{x\in[a,b]}\int_a^x (x-s)^{\nu-1}R(s)\,ds + \frac{M_T}{\Gamma(\nu)}\sup_{x\in[a,b]}\int_a^x (x-s)^{\nu-1}H(s)\,ds\\
&\quad + (1-\gamma)|u_b| + \frac{1}{\Gamma(\mu)}\sup_{x\in[a,b]}\int_x^b (s-x)^{\mu-1}R(s)\,ds + \frac{M_T}{\Gamma(\mu)}\sup_{x\in[a,b]}\int_x^b (s-x)^{\mu-1}H(s)\,ds\\
&\le (\gamma|u_a|+M) + ((1-\gamma)|u_b|+M) + 2M + 2M_H M_T\\
&\le r.
\end{aligned}
$$

This shows that $K : S_r \to S_r$. By similar argument in the proof of Theorem 3.2, one can show that K is completely continuous. Lemma 2.9 concludes that K must have a fixed point which is a solution $u(x) \in C(J)$ of the problem (1.1). □

Corollary 3.7. *Assume that $H \in L^1(J,\mathbb{R}_{>0})$ and the assumption $|F(x,u(x))| \le H(x)$ holds. Then there exists a solution $u(x) \in C(J)$ of the problem (1.1).*

Proof. Set $T \equiv 1$ and $R \equiv 0$. Then $T \in C(\mathbb{R},\mathbb{R}_{\ge 0})$ and all the conditions of Theorem 3.6 are fulfilled. The proof of the corollary follows from Theorem 3.6. □

4. Numerical Examples

We give several examples to illustrate the application of the main findings in this paper.

Example 4.1. *Consider the following FBVP of the combined Caputo derivative*

$$
\begin{aligned}
&{}^C_aD_b^{0.5,0.4}u(x) = \frac{1}{2}u(x), \quad x \in [0,1],\\
&\quad u(0) = 0, \quad u(1) = 0.
\end{aligned} \tag{4.1}
$$

Here, $F(x,u(x)) = \frac{1}{2}u(x), \nu = 0.5, \mu = 0.4$ and $a = 0, b = 1$.
Let $\phi(t) = t, \quad \psi = \frac{1}{2}, \quad R(x) = 0$, then the assumptions A1 and A2 of Theorem 3.2 are satisfied. Thus there exists a solution $u(x)$ of the problem (4.1).
Moreover, let $T(t) = \frac{1}{2}|t|, \quad R(x) = 0$, then A3 and A4 of Theorem 3.4 are fulfilled. Therefore, there exists a solution $u(x)$ of the problem (4.1) by Theorem 3.4.

Example 4.2. *Consider the following FBVP*

$$\begin{aligned} &{}_{a}^{C}D_{b}^{0.5,0.4}u(x) = \exp(-u^2(x)), \quad x \in [0,1], \\ &\quad u(0) = 0, \quad u(1) = 2. \end{aligned} \tag{4.2}$$

Here $F(x,u(x)) = \exp(-u^2(x)), \quad \nu = 0.5, \mu = 0.4$ and $a = 0, b = 1$.
let $T(u) = \exp(-u^2(x)), \quad H(x) = 1$ and $R(x) = 0$, then $|F(x,u)| \leq T(u)H(x) + R(x)$ and

$$\lim_{t\to\pm\infty} T(t) = \lim_{t\to\pm\infty} \exp(-t^2) = 0 < \infty.$$

Therefore, the conditions A5 and A6 of Theorem 3.6 hold. Therefore, there exists a solution $u(x)$ of the problem (4.2) by Theorem 3.6.

5. Conclusion

In this paper, we present existence results for nonlinear FBVP of the combined Caputo derivative which is an extension of the fractional Caputo derivative. Using this extended fractional derivative, we can better describe a more general class of variational problem. First we establish the existence of solutions of nonlinear FBVP involving the combined Caputo derivative by using of techniques of nonlinear analysis, namely the Leray-Schauder fixed point theorem. Moreover, the existence results based on the Schauder fixed point theorem under some weaker conditions on the nonlinear function are given. Finally, we give some numerical examples to show the existence of the FVBP.

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