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Research Article

A kernel-based method for Volterra delay integro-differential equations

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Abstract

Volterra integro-differential equations with constant delay $\tau > 0$ are presented in this paper. We used a numerical method based on reproducing kernels to investigate well-known equations. The convergence analysis of the utilized approach is taken into account, which also provides the theoretical structure of the method. In addition, we derive some effective error estimates for the proposed method when applied to Volterra delay integro differential equations. Numerical experiments are carried out to illustrate the efficiency and applicability of the proposed method.

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1. Introduction

Finding approximate analytic solutions of the DIDEs is extremely significant in engineering and physical sciences. In recent years, there have been some research activities concerning the numerical solution of the DIDEs. For example, Bellour and Bousselsal in [2], provided a numerical approach based on using continuous collocation for the numerical solution of the DIDEs, Horvat [12] constructed a polynomial collocation solution of the Volterra DIDEs, Makroglou in [13], considered Volterra DIDEs and applied a block-by-block method based on interpolatory quadrature rules.
In the present paper, we study a numerical method for solving the Volterra DIDEs of the following form

$$\begin{cases} y^{'}(t) - \lambda y(t) - \mu y(t-\tau) = g(t) + \int_0^t H_1(t,s)y(s)ds + \int_{t-\tau}^t \overline{H}_1(t,s)y(s)ds, \ t \in [0,T], \\ y(t) = \Phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $\lambda, \mu \in \Re$, the delay term τ is a positive constant; the initial function $\Phi(t)$ is continuous on $[-\tau, 0]$; $H_1 : S \to \Re$ ($S := \{(t,s) | 0 \leq s \leq t \leq T\}$), and $\overline{H}_1 : S_\tau \to \Re$

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($S_\tau := [0,T] \times [-\tau, T-\tau]$) are weight functions.
We introduce the transformation

$$x(t) = y(t) - w(t), \tag{1.2}$$

where $w(t) = \begin{cases} 0, & 0 \le t \le T \\ \Phi(t), & -\tau \le t \le 0 \end{cases}$.
Then, the equivalent problem of (1.1) can be written as

$$\begin{cases} x'(t) - \lambda x(t) - \mu x(t-\tau) = f(t) + \int_0^t H_1(t,s)x(s)ds + \int_{t-\tau}^t \overline{H}_1(t,s)x(s)ds, \ t \in [0,T], \\ \\ x(t) = 0, \ t \in [-\tau, 0], \end{cases} \tag{1.3}$$

where

$$f(t) = \begin{cases} g(t) + \mu\Phi(t-\tau) + \int_{t-\tau}^0 \overline{H}_1(t,s)\Phi(s)ds, & 0 \le t \le \tau, \\ g(t), \quad \tau < t \le T. \end{cases} \tag{1.4}$$

The theory of reproducing kernels was first proposed by Zaremba [1]. This theory has played an important role in a number of successful applications in numerical analysis and has successfully been used for constructing approximate solutions to several linear and nonlinear problems such as singular nonlinear second-order periodic boundary value problems [9], nonlinear system of second order boundary value problems [10], one-dimensional variable-coefficient Burgers equation [4], the coefficient inverse problem [6], nonlinear age-structured population equation [3], the generalized regularized long wave equation [14], nonlinear delay differential equations of fractional order [11], variational problems depending on indefinite integrals [8], nonlocal initial-boundary value problems for parabolic and hyperbolic integro-differential equations [7] and the generalized Black-Scholes equation [15]. Cui and Lin in [5] provide an excellent overview of the existing reproducing kernel methods for solving various model problems such as integral and integro-differential equations.
The present work outlines reliable numerical strategies for solving these equations based on the reproducing kernel theory.
The advantages of the current method lie in the following facts:

- The method is mesh-free, easily implemented and capable of treating various boundary conditions
- The method is based on the reproducing kernel theory and does not need the Gram-Schmidt orthogonalization process.

This paper is organized as follows. A brief review of the reproducing kernel theory is given in Section 2. In Section 3, a numerical method is presented to construct our numerical solutions for the Volterra DIDEs. Some test examples are solved and the results are shown in Section 4. The paper ends with conclusions in Section 5.

2. Preliminaries

In this section, we provide some fundamental definitions and then we introduce some reproducing kernel Hilbert spaces which will be used later in the paper.

Definition 2.1. (Page 3/Section 1.1/Chapter 1 in [5]). Let H be a real Hilbert space of functions on a set U. Denote by $\langle x, y\rangle_H$ the inner product and let $\|x\| = \sqrt{\langle x, x\rangle_H}$ be the norm in H, for $x, y \in H$. The real valued function $R : U \times U \longrightarrow \Re$ is called a reproducing kernel of H if the followings are satisfied
(i) For every t, $R_t(s) = R(t,s)$ as a function of s belongs to H.
(ii) The reproducing property: $x(t) = \langle x(.), R_t(.)\rangle_H$ for all $x \in H$ and all $t \in U$.

Definition 2.2. (Page 3/Section 1.1/Chapter 1 in [5]). A Hilbert space H of functions on a set U is called a reproducing kernel Hilbert space if there exists a reproducing kernel R of H.

Definition 2.3. (Page 4/Section 1.2/Chapter 1 in [5]). If a Hilbert space H of functions on a set U possesses a reproducing kernel, then the reproducing kernel R is uniquely determined by the Hilbert space H.

Definition 2.4. (Page 38/Section 2.3/Chapter 2 in [5]). The inner product space $\mathsf{H}^2[0,T]$ is defined as $\mathsf{H}^2[0,T] = \{x(t)|x^{'}(t)$ is one-variable absolutely continuous real-valued function in $[0,T]$, $x^{''} \in L^2[0,T]$ and $x(0) = 0\}$. The inner product space in $\mathsf{H}^2[0,T]$ is given by

$$\langle x(t), y(t)\rangle_{\mathsf{H}^2[0,T]} = x^{'}(0)y^{'}(0) + \int_0^T x^{''}(t)y^{''}(t)dt, \tag{2.1}$$

and the norm $\|x\|^2_{\mathsf{H}^2[0,T]}$ is denoted by

$$\|x\|^2_{\mathsf{H}^2[0,T]} = \langle x(t), x(t)\rangle_{\mathsf{H}^2[0,T]},$$

where $x, y \in \mathsf{H}^2[0,T]$.

It can be proved that the inner product space $\mathsf{H}^2[0,T]$ is a reproducing kernel Hilbert space and its reproducing kernel function $M(t,s)$ is given by [5]

$$M_t(s) = M(t,s) = \begin{cases} st + \frac{st^2}{2} - \frac{t^3}{6}, & t \le s, \\ st + \frac{ts^2}{2} - \frac{s^3}{6}, & s < t. \end{cases} \tag{2.2}$$

Definition 2.5. (See [14]) The inner product space $\mathsf{H}^{2,\sigma}[-\tau,T]$ is defined as $\mathsf{H}^{2,\sigma}[-\tau,T] = \{x(t)|x(t) \in \mathsf{H}^2[0,T]\ \forall t \in [0,T]$ and $x(t) = 0\ \forall t \in [-\tau,0]\}$. The inner product in $\mathsf{H}^{2,\sigma}[-\tau,T]$ is given by

$$\langle x(t), y(t)\rangle_{\mathsf{H}^{2,\sigma}[-\tau,T]} = x^{'}(0)y^{'}(0) + \int_0^T x^{''}(t)y^{''}(t)dt, \tag{2.3}$$

and the norm $\|x\|_{\mathsf{H}^{2,\sigma}[-\tau,T]}$ is denoted by

$$\|x\|_{\mathsf{H}^{2,\sigma}[-\tau,T]} = \langle x(t), x(t)\rangle_{\mathsf{H}^{2,\sigma}[-\tau,T]},$$

where $x, y \in \mathsf{H}^{2,\sigma}[-\tau,T]$.

We can prove that the space $\mathsf{H}^{2,\sigma}[-\tau,T]$ is a reproducing kernel Hilbert space [16] and its reproducing kernel function $R(s,t)$ can be written as

$$R_t(s) = R(t,s) = \begin{cases} M_t(s), & t \in [0,T], \\ 0, & t \in [-\tau,0]. \end{cases} \tag{2.4}$$

3. Description of the method

In this section, a linear differential operator and a complete system of the space $\mathsf{H}^{2,\sigma}[-\tau,T]$ are introduced. Then an iterative method is represented to obtain the analytical solution of (1.3) in the space $\mathsf{H}^{2,\sigma}[-\tau,T]$.
By defining the linear operator $Ł : \mathsf{H}^{2,\sigma}[-\tau,T] \longrightarrow L^2[0,T]$ as

$$Ł[x(t)] = x^{'}(t) - \lambda x(t) - \mu x(t-\tau) - \int_0^t H_1(t,s)x(s) - \int_{t-\tau}^t \overline{H}_1(t,s)x(s)ds, \tag{3.1}$$

model problem (1.3) takes the following form

$$Ł[x(t)] = f(t). \tag{3.2}$$

We assume that (1.3) has a unique solution. In order to represent the approximate solution of (1.3), it is easy to show that $Ł : \mathsf{H}^{2,\sigma}[-\tau, T] \to L^2[0, T]$ is a bounded linear operator [5].

Lemma 3.1. *(see [6]) If $\{t_i\}_{i=1}^{\infty}$ is dense in $[0, T]$, then $\{\rho_i(t)\}_{i=1}^{\infty} = \{Ł_s R(t, s)|_{s=t_i}\}_{i=1}^{\infty}$ is the complete system in $H^{2,\sigma}[-\tau, T]$, where the subscript s of the operator Ł indicates that the operator Ł applies to the function of s.*

Remark 3.2. Completeness of system $\{\rho_i\}_{i=1}^{\infty}$ can be stated as: if $< x, \rho_i >_{\mathsf{H}^{2,\sigma}} = 0, i = 1, \cdots$, and some $x \in \mathsf{H}^{2,\sigma}[-\tau, T]$ then $x = 0$.

The exact solution and approximate solution can be obtained using the following theorem.

Theorem 3.3. *Suppose that $x(t) \in H^{2,\sigma}[-\tau, T]$, then*

$$x(t) = \sum_{i=1}^{\infty} \alpha_i \rho_i(t), \tag{3.3}$$

where

$$\sum_{i=1}^{m} \alpha_i Ł\rho_i(t_k) = f(t_k), \ \ k = 1, 2, ..., m. \tag{3.4}$$

Proof. System $\{\rho_i(t)\}_{i=1}^{\infty}$ is complete in $\mathsf{H}^{2,\sigma}[-\tau, T]$, then

$$x(t) = \sum_{i=1}^{\infty} \alpha_i \rho_i(t). \tag{3.5}$$

Now, by the n-term intercept of (3.5), the approximate solution is presented by

$$P_m x(t) = x_m(t) = \sum_{i=1}^{m} \alpha_i \rho_i(t). \tag{3.6}$$

where $P_m : \mathsf{H}^{2,\sigma}[-\tau, T] \to \mathrm{Span}\{\rho_i(t)\}_{i=1}^{m}$ is an orthogonal projection operator. Since

$$\begin{aligned} Ł[x_m(t_k)] &= \langle x_m, \rho_k \rangle_{\mathsf{H}^{2,\sigma}} = \langle P_m x, \rho_k \rangle_{\mathsf{H}^{2,\sigma}} \\ &= \langle x, \rho_k \rangle_{\mathsf{H}^{2,\sigma}} = Ł[x(t_k)], k = 1, 2, ..., m. \end{aligned}$$

Therefore, we must have

$$Ł[x_m(t_k)] = Ł[x(t_k)], \ \ k = 1, 2, ..., m. \tag{3.7}$$

Therefore, we must have

$$Ł[x_m(t_k)] = \sum_{i=1}^{m} \alpha_i Ł\rho_i(t_k) = f(t_k), \ \ k = 1, 2, ..., m. \tag{3.8}$$

Then, the approximate solution $x_m(t)$ can be obtained by

$$x_m(t) = P_m x(t) = \sum_{i=1}^{m} \alpha_i \rho_i(t), \tag{3.9}$$

where the coefficients $\alpha_i, \ i = 1, 2, ..., m$, can be determined by (3.8). Combining (3.9) and (1.2) leads to the approximate solution of (1.1)

$$y_m(t) = x_m(t) + w(t) = \sum_{i=1}^{m} \alpha_i \rho_i(t) + w(t). \tag{3.10}$$

$\square$

3.1. Convergence and error estimation

In the following, we will consider the convergence and error estimation of the approximate solution is given by (3.10).
First, we discuss the convergence of the approximate solution $x_m(t)$ and its derivative $x_m^{'}(t)$. We set $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$.

Theorem 3.4. *Let $\{t_i\}_{i=1}^{\infty}$ be dense in $[0,1]$ and $x(t)$ be the solution of (3.2), then the approximate solution $x_m(t)$ and $x_m^{'}(t)$, converge uniformly to the exact solution $x(t)$ and its derivative $x^{'}(t)$, respectively.*

Proof. Since $R_t(t)$ and $\frac{\partial^2}{\partial t_1 \partial t_2} R_{t_2}(t_1)|_{t_1=t_2=t}$ are continuous function with respect to t in $[0,T]$, we get

$$\begin{aligned} |x(t) - x_m(t)| &= |< x - x_m, R_t(.) >_{\mathsf{H}^{2,\sigma}}| \\ &\leq \sqrt{R_t(t)}\|x - x_m\|_{\mathsf{H}^{2,\sigma}}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} |x^{'}(t) - x_m^{'}(t)| &= |< x - x_m, \frac{\partial}{\partial\, t} R_t(.) >_{\mathsf{H}^{2,\sigma}}| \\ &\leq \sqrt{\frac{\partial^2}{\partial t_1 \partial t_2} R_{t_2}(t_1)|_{t_1=t_2=t}}\|x - x_m\|_{\mathsf{H}^{2,\sigma}}. \end{aligned} \tag{3.12}$$

It follows from

$$\|x - x_m\|_{\mathsf{H}^{2,\sigma}} = \| \sum_{i=m+1}^{\infty} \alpha_i \rho_i \|_{\mathsf{H}^{2,\sigma}}. \tag{3.13}$$

that $\|x - x_m\|_{\mathsf{H}^{2,\sigma}} \to 0$ as $m \to \infty$. Thus the approximate solution $x_m(t)$ and $x_m^{'}(t)$ converge uniformly to the exact solution $x(t)$ and its derivative $x^{'}(t)$ respectively. □

In the following we will obtain the error estimates for the approximate solution of (3.2) in $\mathsf{H}^{2,\sigma}[-\tau, T]$.

Theorem 3.5. *Let $\Delta_m = \{0 = t_1 < t_2 < ... < t_m = T\}$, be a partition of interval $[0,T]$ and also $x_m(t)$ be the approximate solution of (3.2) in the space $\mathsf{H}^{2,\sigma}[-\tau, T]$. The following relation holds,*

$$\|x - x_m\|_\infty \leq c\ h, \quad h = \max_{1\leq i\leq m-1}(t_{i+1} - t_i), \tag{3.14}$$

where c is a positive constant.

Proof. In each subinterval $[t_i, t_{i+1}]$, we can write

$$x(t) - x_m(t) = x(t) - x(t_i) + x_m(t_i) - x_m(t) + x(t_i) - x_m(t_i). \tag{3.15}$$

By means of the mean value Theorem and the continuity of $x^{'}$, one can show that

$$|x(t) - x(t_i)| \leq a\ h. \tag{3.16}$$

We now have

$$|x_m(t) - x_m(t_i)| \leq \int_{t_i}^{t} |x_m^{'}(s)| ds, \tag{3.17}$$

and since $x_m(t) \in \mathsf{H}^{2,\sigma}[-\tau, T]$, it follows that

$$|x_m(t) - x_m(t_i)| \leq b\ h. \tag{3.18}$$

Using Theorem 3.4, for large m we have

$$|x(t_i) - x_m(t_i)| \leq \varepsilon. \tag{3.19}$$

Given the fact that ϵ is arbitrary, after combining Eqs. (3.15)-(3.19) and choosing sufficiently large value of m, we must have

$$\|x - x_m\|_\infty \leq c\ h, \quad h = \max_{1\leq i\leq m-1}(t_{i+1} - t_i). \tag{3.20}$$

□

Theorem 3.6. *Suppose that the conditions of Theorem 3.5 hold. The approximate solution* $y_m(t)$ *satisfies*

$$\|y - y_m\|_\infty \leq c\ h, \quad h = \max_{1 \leq i \leq m-1}(t_{i+1} - t_i). \tag{3.21}$$

Proof. In view of

$$y_m(t) = x_m(t) + w(t) = \sum_{i=1}^{m} \alpha_i \rho_i(t) + w(t), \ y(t) = x(t) + w(t), \tag{3.22}$$

by Theorem 3.5, one can see that

$$\|y - y_m\|_\infty \leq c\ h, \quad h = \max_{1 \leq i \leq m-1}(t_{i+1} - t_i). \tag{3.23}$$

□

Theorem 3.7. *Let the partition of the interval* $[0, 1]$*, denoted by* $\Delta_m = \{0 = t_1 < t_2 < ... < t_m = T\}$*, also suppose that* $x_m(t)$ *be the approximate solution of (3.2) in the space* $H^{2,\sigma}[-\tau, T]$ *such that* $\|x_m^{'}\|_\infty$ *then bounded. If* $x \in C^2[0, T]$*, then following relations hold,*

$$\|x - x_m\|_C \leq c\ h^2, \quad \|x^{'} - x_m^{'}\|_C \leq d\ h, \quad h = \max_{1 \leq i \leq m-1}(t_{i+1} - t_i), \tag{3.24}$$

where c and d are positive constants.

Proof. In each subinterval $[t_i, t_{i+1}]$, we can write

$$x^{'}(t) - x_m^{'}(t) = x^{'}(t) - x^{'}(t_i) + x_m^{'}(t_i) - x_m^{'}(t) + x^{'}(t_i) - x_m^{'}(t_i). \tag{3.25}$$

According to the mean value theorem, there exists $\xi_i \in (t_i, t_{i+1})$ such that

$$x^{'}(t) - x^{'}(t_i) = (t - t_i)x^{''}(\xi_i). \tag{3.26}$$

Since $x(t) \in C^2[0, T]$ then for some $a > 0$

$$|x^{'}(t) - x^{'}(t_i)| \leq a\ h. \tag{3.27}$$

Note that

$$|x_m^{'}(t) - x_m^{'}(t_i)| \leq \int_{t_i}^{t} |x_m^{''}(t)|dt. \tag{3.28}$$

Hence

$$|x_m^{'}(t) - x_m^{'}(t_i)| \leq b\ h. \tag{3.29}$$

Using Theorem 3.4 for large m we have

$$|x(t_i) - x_m(t_i)| \leq \epsilon, \ |x^{'}(t_i) - x_m^{'}(t_i)| \leq \epsilon. \tag{3.30}$$

Given the fact that ϵ is arbitrary, after combining Eqs. (3.25)-(3.30) and choosing sufficiently large value of m, we must have

$$\|x^{'} - x_m^{'}\|_\infty \leq d\ h, \quad h = \max_{1 \leq i \leq m-1}(t_{i+1} - t_i). \tag{3.31}$$

We know

$$x(t) - x_m(t) = x(t_i) - x_m(t_i) + \int_{t_i}^{t} (x^{'}(s) - x_m^{'}(s))ds, \tag{3.32}$$

By using (3.30)- (3.32) for large m it is straightforward to see that

$$\|x - x_m\|_\infty \leq c\ h^2, \tag{3.33}$$

and this completes the proof. □

Similarly, one obtains Theorem 3.8.

t	$\lvert y(t)-y_{40}(t)\rvert$	$\lvert y(t)-y_{50}(t)\rvert$	$\lvert y(t)-y_{60}(t)\rvert$	Bellour and Bousselsal [2] (m=4,N=4)
0.5	1.5781e-5	4.7476e-6	2.2112e-6	1.04e-6
1.0	2.5459e-5	8.2134e-6	3.6462e-6	2.13e-6
1.5	4.1868e-5	1.3609e-5	6.0291e-6	3.03e-6
2.0	6.1929e-5	2.2479e-5	9.9413e-6	4.12e-6
2.5	1.1410e-4	3.6847e-5	1.6423e-5	6.19e-6
3.0	1.7962e-4	5.8169e-5	2.5756e-5	1.00e-5
CPU-time (s)	10.098(s)	14.714(s)	16.915(s)	
t	$\lvert y'(t)-y'_{40}(t)\rvert$	$\lvert y'(t)-y'_{50}(t)\rvert$	$\lvert y'(t)-y'_{60}(t)\rvert$	
0.5	8.3546e-5	6.2223e-5	2.6121e-5	
1.0	5.7278e-4	8.4822e-5	3.6834e-5	
1.5	6.2124e-4	1.4388e-4	6.0556e-5	
2.0	6.9134e-4	2.2878e-4	8.9617e-5	
2.5	7.2623e-3	3.8112e-4	9.1024e-5	
3.0	7.8409e-3	5.8421e-4	9.5439e-5	

Table 1. The absolute errors $|y(t)-y_m(t)|$ and $|y'(t)-y'_m(t)|$ obtained by using the proposed method.

Theorem 3.8. *Suppose that the conditions of Theorem 3.7 hold. The approximate solution* $y_m(t)$ *satisfies*

$$\|y-y_m\|_\infty \leq c\, h^2, \quad h=\max_{1\leq i\leq m-1}(t_{i+1}-t_i). \tag{3.34}$$

4. Numerical examples

In this section, we consider two numerical examples that demonstrate the performance and efficiency of the proposed method for solving Volterra DIDEs. We calculate the absolute error for different values of m between the exact solution $y(t)=x(t)+w(t)$ and the approximate solution $y_m(t)=x_m(t)+w(t)$. The computations were performed by means of the symbolic software Maple 16 on a PC with a CPU of 2.4 GHz. Results obtained by the proposed method are compared with the exact solution for each example. This comparison shows that the approximate solution is in good agreement with the exact solution.

Example 5.1. Let's first consider the delay integro-differential equation

$$y'(t)=g(t)+\int_{t-1}^{t}(\cos(t+s+1)+2)y(s)ds,\ t\in[0,3], \tag{4.1}$$

$$\begin{aligned} g(t) &= 3\cos(t)-\frac{1}{4}\cos(3t-1)-2+\frac{1}{2}\sin(t+1) \\ &\quad +\sin(2t)-2\cos(t-1)+\frac{1}{4}\cos(3t+1)-\sin(2t+1), \end{aligned}$$

with the initial function $\Phi(t)=\sin(t)+1$, $t\in[-1,0]$. The exact solution is $y(t)=\sin(t)+1$ for all $t\in[0,3]$.

We introduce the transformation $x(t)=y(t)-w(t)$, where

$$w(t)=\begin{cases} 0, & 0\leq t\leq 3 \\ \sin(t)+1, & -1\leq t\leq 0. \end{cases}$$

Using the transformation, the equivalent problem of (4.1) can be written as:

$$x'(t)=f(t)+\int_{t-1}^{t}(\cos(t+s-1)+2)x(s)ds,\ t\in[0,3], \tag{4.2}$$

t	$\lvert y(t)-y_{55}(t)\rvert$	$\lvert y(t)-y_{65}(t)\rvert$	$\lvert y(t)-y_{75}(t)\rvert$
0.2	2.1113e-5	6.9389e-6	2.6992e-6
0.4	2.0578e-5	7.3023e-6	3.3541e-6
0.6	2.5853e-5	8.8828e-6	4.0669e-6
0.8	3.1309e-5	1.0880e-5	4.9478e-6
1.0	3.8117e-5	1.3290e-5	6.0345e-6
1.2	4.6447e-5	1.6128e-5	7.3647e-6
1.4	5.6519e-5	1.9645e-5	8.9820e-6
1.6	7.0091e-5	2.4047e-5	1.0923e-5
1.8	7.4347e-5	2.8423e-5	1.3465e-5
2.0	1.1834e-4	4.0901e-5	1.8612e-5
CPU-time (s)	13.745(s)	15.970(s)	19.672(s)
t	$\lvert y^{'}(t)-y^{'}_{55}(t)\rvert$	$\lvert y^{'}(t)-y_{65}(t)\rvert$	$\lvert y^{'}(t)-y^{'}_{75}(t)\rvert$
0.2	4.6405e-4	1.1206e-4	3.9665e-5
0.4	4.8038e-4	1.0834e-4	2.4508e-5
0.6	5.0942e-4	4.5869e-4	4.0254e-5
0.8	5.7835e-4	4.0673e-4	4.9111e-5
1.0	6.8530e-4	5.3036e-4	5.9575e-5
1.2	6.3496e-4	5.5947e-4	7.3436e-5
1.4	7.2159e-4	6.9443e-4	8.9462e-5
1.6	9.5500e-4	8.7710e-4	1.0149e-4
1.8	1.0511e-3	9.1117e-4	3.0641e-4
2.0	1.1870e-3	9.0989e-4	4.8686e-4

Table 2. The absolute errors $|y(t)-y_m(t)|$ and $|y^{'}(t)-y^{'}_m(t)|$ obtained by using the proposed method.

In (4.2), put $Ł : \mathsf{H}^{2,\sigma}[-1,3] \longrightarrow L^2[0,3]$ such that

$$Ł[x(t)] \equiv x^{'}(t) - \int_{t-1}^{t} (\cos(t+s-1)+2)x(s)ds.$$

It is clear that Ł is a bounded linear operator. The proposed method is tested on this problem with grid points $t_i = \frac{3(i-1)}{m-1}$, $i = 1,\dots,m$. The absolute errors distribution obtained by using the proposed method is provided in Table 1. Also, a comparison is made between the absolute errors obtained by using the proposed method together with the absolute errors obtained by using the method are presented in [2].

Example 5.2. Consider the delay integro-differential equation

$$y^{'}(t) = g(t) - 2y(t) + y(t-1) + \int_{t-1}^{t} y(s)ds + \int_{0}^{t} e^{-st}t^2 y(s)ds, \qquad t \in [0,2], \tag{4.3}$$
$$g(t) = e^{-t^2}(1+t^2) + 1.5,$$

with the initial function $\Phi(t) = t$, $t \in [-1,0]$. The exact solution is $y(t) = t$ for all $t \in [0,2]$. We introduce the transformation $x(t) = y(t) - w(t)$, where

$$w(t) = \begin{cases} 0, & 0 \le t \le 3 \\ x, & -1 \le t \le 0. \end{cases}$$

Using the transformation, the equivalent problem of (4.3) can be written as:

$$x^{'}(t) = f(t) - 2x(t) + x(t-1) + \int_{t-1}^{t} x(s)ds + \int_{0}^{t} e^{-st}t^2 x(s)ds,\ t \in [0,2],\ t \in [0,3]. \tag{4.4}$$

In (4.4), put $Ł : \mathsf{H}^{2,\sigma}[-1,2] \longrightarrow L^2[0,2]$ such that

$$Ł[x(t)] \equiv x^{'}(t) + 2x(t) - x(t-1) - \int_{t-1}^{t} x(s)ds - \int_{0}^{t} e^{-st}t^2 x(s)ds, \ t \in [0,2].$$

The proposed method is tested on this problem with gri d points $t_i = \frac{2(i-1)}{m-1}, \ i = 1, \ldots, m$. The absolute errors distribution obtained by using the proposed method are provided in Table 2.

5. Conclusion

In this paper, an efficient method for reproducing kernel space is developed to solve Volterra DIDEs. The reproducing kernel method was described and tested on two different examples. The method's applicability and accuracy were also assessed by calculating approximate solutions at the selected grid points. Furthermore, the analytical and numerical solutions were obtained with the help of the Maple software package.
We have come up with the following conclusions:

- The proposed method provides the solution in the form of a convergent series with easily computed components.
- The approximate solution and its derivatives converge to the exact solution and its derivatives in a uniform manner.
- It has been discovered that the proposed method produces more accurate numerical results.
- The method is simple to implement, and its algorithm is efficient in providing an approximate solution.
- The numerical examples demonstrate that the proposed method is a reliable numerical method for treating Volterra DIDEs.

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