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Araştırma Makalesi / Research Article

Risk Distribution Among Uncorrelated Risk Factors: Diversified Risk Parity

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Abstract

This paper aims to distribute the risk among equity risk, interest rate risk and inflation risk, in a portfolio to prevent a risk concentrated portfolio by employing diversified risk parity (DRP) strategy. Principal component analysis and minimum linear torsion models are used to obtain DRP strategies which are compared with other risk based models and tested on five different asset classes whose prices are collected between January 1988 and December 2017. For attaining a thorough analysis, we include mean-variance optimization whose results are compared with both risk-based and DRP strategies in the out-of-sample testing using Sharpe ratio and uncorrelated risk factors. The results demonstrate that DRP strategies have better performance than other models. Specifically, DRP based on the minimum linear torsion model yields the highest Sharpe and risk diversification ratios. Thus, this strategy may guide the investors to construct risk diversified portfolios, especially, during financial crises.

Keywords: *Diversified Risk Parity, Principal Component Analysis, Minimum Linear Torsion Model, Risk Diversification.*

İlişiksiz Risk Faktörleri Arasında Risk Dağılımı: Çeşitlendirilmiş Risk Paritesi

Öz

Bu makale, risk yoğunlaştırılmış portföy oluşturmayı önlemek için toplam riski "Çeşitlendirilmiş Risk Paritesi" (DRP) kullanarak portföydeki piyasa, faiz ve enflasyon gibi risk faktörleri arasında dağıtmayı amaçlamaktadır. Temel bileşenler analizi ve minimum torsiyon modeli aracılığıyla, beş farklı varlık sınıfının Ocak 1988 ile Aralık 2017 arasındaki aylık fiyatları üzerinde yapılan uygulama ile DRP stratejileri risk bazlı stratejilerle karşılaştırılmaktadır. Kapsamlı bir karşılaştırma için, ortalama varyans optimizasyonunu sonuçları, örneklem dışı testlerde hem risk temelli stratejiler hem de DRP stratejileri Sharpe oranına ve ilintisiz risk faktörlerinin sayısı göstergelerine göre karşılaştırılmıştır. Bu çalışmanın sonuçları, DRP stratejilerinin diğer modellere göre daha iyi performansa ve en yüksek sayıda ilintisiz risk faktörüne sahip olduğunu göstermektedir ve yatırımcıların finansal krizlerde bile risk çeşitlendirilmiş portföyler oluşturmasına yardımcı olacağı belirlenmiştir.

Anahtar Kelimeler: *Çeşitlendirilmiş Risk Paritesi, Temel Bileşenler Analizi, Minimum Doğrusal Torsiyon Modeli, Risk Çeşitlendirme.*

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INTRODUCTION

Asset allocation plays an essential role in investment management at which how to invest the capital among different asset classes. The construction of the optimal portfolio is based on the optimization of the trade-off between risk and reward. As one of the well-known quantitative techniques, Markowitz mean-variance optimization aims to find the best asset allocation considering risk-return trade-off. One of the main drawbacks of Markowitz's strategy is high estimation errors, especially errors in estimated mean (Braga, 2016; Chopra and Ziemba, 2013). Avoding errors in estimated means results in minimum variance portfolio that gives more weights to low volatility assets, hence, such portfolio generally consists of a few assets, and results in the lack of diversification benefits. These problems lead us to very sensitive and skewed Markowitz portfolios which especially, demonstrated poor performance during the 2008 financial crisis due to its drawbacks.

The crises drive the reseachers and investors finding new strategies that are not based on the mean-variance approach. In the literature, they are known as risk based asset allocation strategies or risk parity (RP) strategies whose main characteristics is excluding expected return and focuses on only diversifying or minimizing risk. Since these strategies only use covariance matrix, they are also known as "μ-free strategies". Risk based asset allocation strategies aim to allocate the risk among asset classes and construct well balanced portfolios in terms of risk (Maillard et al., 2010). However, the ability of diversification of risk based strategies depends on the characteristics of the underlying assets. If chosen assets are highly correlated and dependent on the same underlying risk factors, the aim of diversification may not be achieved and the portfolio may have a concentrated risk structure. Especially, this problem can arise during the financial crisis times, since the correlations generally increase when economy goes bad.

The correlation problem among asset classes may generate ill-diversified portfolios. To overcome this problem, the targeted risk factors should be less correlated, if possible, uncorrelated. To obtain uncorrelated risk sources, one of quantitative method is principal component analysis (PCA) whose results yield principal portfolios (Partovi and Caputo, 2004). This approach is criticized for being unstable over time, lacking of economic interpretation and not having unique eigenvectors (Poddig and Unger, 2012; Kind, 2013). Meucci et al. (2015) propose minimum linear torsion (MLT) model that extracts uncorrelated variables closely following the original variables and is expected to be more robust than the PCA model. Lohre et al. (2014) and Bernardi et al. (2018) propose "Diversified Risk Parity" strategy based on PCA and MLT following theories by Meucci et al. (2015).

This study contributes to the literature a comprehensive risk based asset allocation strategies focusing on the elimination of interdependencies and create well diversified portfolios in terms of risk factor allocations. The findings of this study enables researchers to identify which of risk factors are in charge regarding to the volatility of overall portfolio in terms of volatilities of their components. To emphasize on the proposed methodology, we make a thorough comparison of these strategies with Markowitz mean-variance approach. In the empirical analyses, the asset classes consist of bonds, equities, commodity indices are used to illustrate the proposed approach. Additionally, the out-of sample performances of these strategies in different time intervals to capture the economic changes in the market are performed in line with the literature (Lohre et al. (2014) and Bernardi et al. (2018)). As being different from them, by following the work of Qian (2005), we only focus on three main risk sources: equity risk, interest rate risk, and inflation risk which are taken the most influential factors in financial markets.

The organization of the paper is designed in three parts. The second section presents literature review which is followed by the information about the risk-based asset allocation and DRP strategies. Section 3 presents the application of proposed approach on a real life portfolio with selected strategies and their comparisons in achieving the diversification together with the detailed discussion about the comparison of selected strategies. The final section gives the comments and conclusion.

1. LITERATURE REVIEW

The risk parity approach introduced by Qian (2005), starts with explaining how 60/40 portfolio demonstrates high equity risk which is almost 93%. The 40% of the portfolio includes bonds but its risk contribution to the portfolio is only 7%. Thus, he claims that risk parity strategies generate true diversified portfolios. Another risk based model, Equal Risk Contribution (ERC), proposed by Maillard et al. (2010) claims that any component does not have a dominant role on the whole portfolio risk. They compare ERC with minimum variance portfolio and equally weighted portfolio by using data based on equity and commodity portfolios between 1973 and 2008. It is stated that ERC method has the best Sharpe ratio compared to other used methods. The theoretical foundation for risk budget theory (Bruder and Roncalli, 2012) illustrates its volatility place at between volatility of equally weighted portofolio and global minimum variance portfolio. Kazemi (2012) states that Risk Parity method depicts close results to Markowitz mean- variance strategy with using the assets, HFRI Fund Weighted Composite, MSCI World index, and the Barclays Capital Global Aggregate between 1990 and 2011. His ERC portfolio demonstrates a good performance in terms of Sharpe-ratio compared to both the 10/50/40 portfolio and the equally weighted portfolio.

Constructing uncorrelated portfolios mostly employs PCA (Partovi and Caputo (2004)) whose conditional version is firstly employed by Meucci (2009) to obtain number of uncorrelated bets in a portfolio consisting of 30 liquid mid-cap stocks from Russel Index. The results show that the conditional PCA gives better results compared to PCA in terms of risk distribution. On the other hand, Lohre et al. (2014) proposes “Diversifying Risk Parity” model that is based on the application of Meucci (2009) is directly applied to multi-asset classes (JPM Global Bond Index, MSCI World, MSCI Emerging Markets, Barclay US Aggregate Credit Index, and US 3- months U.S. T-Bills) from 1987 to 2011 to obtain maximum diversified portfolio. The uncorrelated risk sources are measured with Shannon entropy and Gini coeefficients. They demonstrate that proposed strategy outperforms the ERC, global minimum variance and equally weighted portfolio. Similarly, the data US Treasury Bond index, US Corporate Bond index, US Large cap stock index, US Private Equity index, and an international equity index, real estate, and commodity indices from 1992 to 2012 are studied by Deguest et al. (2013) whose results support the earlier studies. Kind and Poonia (2014) illustrate that the portfolios generated by PCA may not outperform the nominal strategies, such as minimum variance and maximum diversification, in terms of Sharpe ratio. Despite this low performance, PCA portfolios significantly reduce downside risk and provide low turnover ratio. Later on, Meucci et al. (2014) propose a minimum linear torsion (MLT) model that extracts uncorrelated variables closely following the original variables and it is expected to be more robust than the PCA. The distribution of whole portfolio risk among the uncorrelated portfolios (Partovi and Caputo, 2004:

5; Meucci, 2010: 75 and Meucci et al., 2014: 3) can be adopted to portfolio risk at which the total risk is distributed among risk factors to prevent risk concentration. Bernardi et al. (2018) following the work by Meucci et al. (2014) directly construct the uncorrelated portfolios with applying risk parity strategy from MLT. They examine risk-based strategies using the 24 commodities included in the S&P Goldman Sachs Commodity Index (GSCI) from January 1983 to December 2014. They claim that DRP based on MLT outperforms the DRP based on PCA approach.

2. RISK-BASED ASSET ALLOCATION STRATEGIES

The equally weighted, global minimum variance, and risk parity strategies are the commonly used risk based strategies to distribute risk in a portfolio.

In equally weighted (EW) strategy, investors hold equal weights from each asset in their portfolio, which does not require either a target return or complex performance skills (Braga, 2016). In more detail, suppose that the number of assets included in a portfolio determines the weights such that w_i, $i = 1, \dots, n$. In the case of *n* number of securities in a portfolio, each asset weight is taken as equally likely between *n* items. Therefore, the more assets are hold in a portfolio, the lower is the weight allocation. Then, portfolio's return (average return), *R*, and the risk, σ, are

$$R = \frac{1}{n}\sum_i^n R_i \text{and } \sigma = \sqrt{\frac{1}{n}^T \sum \frac{1}{n}} = \frac{1}{n}\sqrt{1\sum 1},$$

respectively. Hence, the marginal (MRC) and total risk contributions (TRC) of the asset *i* become

$$MRC_i = \frac{w_i}{\sqrt{w_i^T \sum w_i}} \quad \text{and} \quad TRC_i = w_i \frac{w_i}{\sqrt{w_i^T \sum w_i}},$$

respectively. Despite of its simplicity, this approach yields some drawbacks such as lack of economic interpretations. However, some researchers claim that EW strategy outperforms the mean-variance model and its sophisticated extended versions based on the Sharpe ratio and it demonstrates better out of sample results than advanced models (Demiguel, 2007).

Global minimum variance (GMV) strategy aims to construct a portfolio with a lowest possible variance. GMV portfolio lies on the the most left of the Markowitz's efficient frontier. Despite of being on the efficient frontier, it does not rely on the expected mean and the covariance matrix is the only input parameter. More specifically, the quadratic optimization problem of the strategy to obtain the optimal asset weights goals a portfolio with the minimum risk. The input parameters are the correlations and volatilities of assets. If the GMV portfolio is subject to long only assets and budget constraints, the optimization problem is expressed as

$$w = \underset{w \in R^n}{argmin} \frac{1}{2} w^T \sum w \tag{1}$$

$$w^T 1 = 1 \quad 0 \le w_i \le 1.$$

The marginal and total risk contributions of asset *i* are given as

$$MRC_i = \frac{w_i}{\sqrt{w_i^T \sum w_i}} \quad \text{and} \quad TRC_i = w_i \frac{w_i}{\sqrt{w_i^T \sum w_i}},$$

respectively. Some researchers conclude that GMV strategy outperforms the market-weighted portfolio (Braga, 2016; Haugen and Baker, 1991) due to its low volatility and high return result in high Sharpe ratio, because low risk assets outperform the high risk assets in terms of returns in the long period.

Risk parity (RP) is an asset allocation strategy that allocates the weights according to risk characteristics of asset classes. Covariance matrix is the only input parameter. This approach distributes the whole portfolio risk equally based on the volatility of included asset classes. The risk contribution of asset class *i* to overall risk is the center interest of the RP. Two different approaches exist in RP strategy inverse volatility and equal risk contribution.

Inverse volatility (IV) strategy is also known as naive risk parity allocating the weights of assets inversely to their risk. The volatilities of the components determine the component weights. Investors apply this method assuming the uniform correlations among all asset classes, i.e., correlations do not have a role in this strategy. The optimal weights of the components are given by

$$w_i = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}, i, j = 1,2, \ldots, n. \tag{2}$$

The asset class with higher volatility has low weight in IV strategy. If there are more than two asset classes, the portfolio becomes very sensitive to the correlation of the assets. The ignorance of the relationship between different securities will thus lead to the potential undiversible portfolio risk problem.

On the other hand, equal risk contribution strategy (ERC) at which the risk contributions of each asset are equal to assure that any component does not have a dominant role on the whole portfolio risk so that the same risk budget or contribution should be evenly distributed to each component (Maillard et al., 2010). Then the objective function is the minimization of the square of the difference between risk contributions of all pairs of components, given as following

$$f(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(w_i \frac{\partial(w)}{\partial w_i} - w_j \frac{\partial(w)}{\partial w_i} \right)^2 \tag{3}$$

$$w_{ERC} = \underset{w \in R^n}{argmin} f(w)$$

$$w^T 1 = 1 \;\; 0 \leq w_i \leq 1.$$

Contrary to the IV strategy, ERC strategy considers the correlation among asset classes. However, it still underestimates the well-diversified portfolio with considering risk from asset classes and ignores the underlying risk factors.

Contrary to the RP strategies based on asset classes mentioned above, we focus on the RP approach that aims diversification based on the main risk sources driving the asset returns. Diversified Risk Parity (DRP) strategy in principal component analysis (PCA) and minimum linear torsion (MLT) approaches are explained as following.

2.1. Diversified Risk Parity Portfolios Using PCA

The PCA constructs uncorrelated portfolios, namely principal portfolios, and they are realizable whenever there is no constraint on short-selling in the portfolios (Partovi and Caputo, 2004). Furthermore, these portfolios can be evaluated as uncorrelated risk sources. Assuming that a portfolio of *n* assets, the decomposition of covariance matrix of asset returns is provided as $E^T \sum E = \Lambda = diag(\lambda_1, \dots, \lambda_n)$which is equivalent to $E^{-T} \Lambda E^{-1} = \sum$. Here, the columns of *E* are the principal portfolios. Then, unique vector w_{PP} (the principal portfolio weight) satisfies the condition

$$\widetilde{w}_{PP} = E^{-1} w = E^T w, \tag{4}$$

at which *w* gives the original weight of asset returns. Therefore, the returns of the principal portfolio, $\tilde{R}_{PP}$ are given by

$$\tilde{R}_{PP} = E^{-1} R = E^T R. \tag{5}$$

Here, *R* stands for the original returns of the assets. The marginal risk contribution of each principal portfolio is then equal to

$$MRC_{PP} = \frac{\partial \sigma(\tilde{R}_{PP})}{\partial w_i} = \frac{\widetilde{w}_{PP,i} \sigma(\lambda_i)}{\sigma(\tilde{R}_{PP})}. \tag{6}$$

Since covariances in the principal space are equal to zero, the risk contribution of each principal portfolio is given by

$$TRC_{PP} = \frac{\widetilde{w}_{PP,i}^2 \sigma(\lambda_i)}{\sigma(\tilde{R}_{PP})}. \tag{7}$$

DRP is obtained by applying ERC optimization which is

$$\widetilde{w}_{PP}^* = \underset{w \in R^n}{\operatorname{argmin}} f(\widetilde{w}_{PP}) \tag{8}$$

$$\widetilde{w}_{PP}^T \mathbb{1} = 1$$

$$0 \leq \widetilde{w}_{PP} \leq 1$$

where $f(\widetilde{w}_{PP}) = \sum_{i=1}^n \sum_{j=1}^n (\widetilde{RC}_i - \widetilde{RC}_j)^2$. Here, $\widetilde{RC}$ represents the risk contribution of each principal portfolio. Due to zero covariances in the principal space, the weights can be calculated from a closed-form solution as in IV strategy. Then, the optimal weights of the principal portfolios, $\widetilde{w}_{PP,i}^*$, are given by Partovi and Caputo (2004).

$$\widetilde{w}_{PP,i}^* = \frac{\left(\sqrt{\lambda_i}\right)^{-1}}{\sum_{i=1}^n \left(\sqrt{\lambda_i}\right)^{-1}} \tag{9}$$

where λ_i gives the variance of each principal portfolio. Therefore, $\widetilde{w}_{PP,i}^*$ provides the equal risk contribution from each risk factor.

2.2. Diversified Risk Parity Strategy Using MLT

Minimum linear torsion (MLT) model is another way to extract uncorrelated risk factors. The MLT approach guarantees synthetic variables that represent the nearest uncorrelated representation of original data. The decomposition of covariance matrix of asset returns, $\sum$, is given by MLT such that

$$\sum = (t')^{-1}\sum_t t^{-1}$$

where$\sum_t$consists of only diagonal entries, i.e. σ_t= $\sigma_{t,1}, \sigma_{t,2}, \ldots, \sigma_{t,n}$. Here, t represents the minimum linear torsion transformation matrix that is given as

$$t = \underset{Corr(tX)=I_{n\times n}}{argmin} \sqrt{\frac{1}{n}\sum_{i=1}^{n} Var\left(\frac{(tX)_i - X_i}{\sigma_i}\right)} \tag{10}$$

where σ represents the volatility of original data. Transformation matrix *t* ensures that the new variables are uncorrelated. The minimum linear torsion transformation requires the minimization of the net tracking errors between the generated variables and original variables (Meucci et al., (2014: 5)). Each column of *t* matrix, $t_1, t_2, \ldots, t_n$ for *n* number of assets, is called minimum linear torsion portfolio (MTP). Then, unique vector $\widetilde{w}_{MTP}$ satisfies

$$\widetilde{w}_{MTP} = t'^{-1}w,$$

where w_{MTP} is called principal portfolio weight; *w* gives the original weight of asset returns. In this set up, the returns of the principal portfolio, $\tilde{R}_{MTP}$, in terms of original returns, *R*, becomes

$$\tilde{R}_{MTP} = t'^{-1}R. \tag{11}$$

Then, the marginal risk contribution of each principal portfolio becomes

$$MRC_{MTP} = \frac{\partial\sigma(\tilde{R}_{MTP})}{\partial w_i} = \frac{\widetilde{w}_{MTP,i}\sigma_{t,i}}{\sigma(\tilde{R}_{MTP})}. \tag{12}$$

Since covariances in the principal space are equal to zero, the risk contribution of each torsion portfolio is given by

$$TRC_{MTP} = \frac{\widetilde{w}_{MTP,i}^2\sigma_{t,i}^2}{\sigma(\tilde{R}_{MTP})}. \tag{13}$$

DRP is obtained by applying ERC optimization such that (Meucci et al., 2014)

$$\widetilde{w}_{MTP}^* = \underset{w\in R^n}{\operatorname{argmin}} f(\widetilde{w}_{MTP}) \tag{14}$$

$$\widetilde{w}_{MTP}^T\mathbb{1} = 1$$

$$0 \le \widetilde{w}_{MTP} \le 1$$

where $f(\widetilde{w}_{MTP}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\widetilde{RC}_i - \widetilde{RC}_j\right)^2$. Here, $\widetilde{RC}$ represents the risk contribution of each torsion portfolio. Based on the solutions, it can be intuitively said that each minimum torsion portfolio should affect the portfolio risk equally. Since the MTPs are uncorrelated, a well-diversified portfolio requires investing these portfolios to achieve the uniform diversification distribution, which leads to the minimum torsion portfolios to have the same exposures to the shocks.

3. EMPIRICAL ANALYSIS

We examine the proposed strategies to depict the difference of risk allocation among asset classes versus uncorrelated risk factors. In addition, we include mean-variance optimization and compare it with both risk-based strategies and DRP strategies in the out-of-sample performance.

Seven broad asset classes representing equity, bond and commodity indices are chosen to construct the asset allocation strategies. Monthly prices between January 1988 and December 2017 are retrieved from Bloomberg database. The analyses use logarithmic returns based on the closing prices at the end of each month using Matlab. The portfolios consist of two equities, four bonds and one commodity indices due to their broad and widely use in financial markets and their abbreviations given in parenthesis are kept consistent to Bloomberg tickers. Since we focus on the risk distribution among risk factors, two equities are chosen to represent the equity risk factor. High yield demonstrates a different structure from other bonds representing equity risk factor since it generally has high correlation with equity (Qian, 2013). Remaining bonds are chosen as they are sensitive to the interest rate risk. Commodity index is selected due to its sensitivity to the inflation risk (Qian, 2013, Lohre et al., 2014). Equity risk comes from developed and emerging markets, MSCI World Total Return Index (M1WO) and MSCI Emerging Markets Total Return Index (M1EF), respectively. MSCIs of developed and emerging countries represent the whole world equity risk factor. High yield index (H0A0) that is a sort of fixed income tracks the performance of US dollar denominated below investment grade rated corporate debt publicly issued in the US domestic market. The chosen bonds are Citi WGBI Currency Hedged USD for world government bonds hedged (SBWGC), Citi WBGI USD for world government bonds (SBWGU) and Barclay's U.S. Aggregate for U.S. aggregate bonds (LBUSTRUU). The selected commodity is given by S&P GSCI index (SPGSCITR).

The monthly asset prices and their cumulative returns are plotted in Figure 1. Commodity and developed equity demonstrate a volatile price pattern over the period. Another important note on the movements of these prices, they do have relatively expected variability till 2004 at which all the returns show significant increasing trend within a cycle of around each four years. The remaining asset prices present more stable pattern. As for the cumulative returns, emerging equities give the highest return, which is followed by returns of high yield index. Their reactions to 2008 financial crisis illustrate that the most of the prices decline sharply except bonds. The returns of the emerging equity decrease most among all asset classes. The returns of high yield, commodity and developed equity indices also decrease. There is not remarkable decrease in the bond returns.

Table 1. Descriptive statistics of selected assets

Abbrev.	Index	Return (%)	Risk (%)	Sharpe ratio	MDD (%)
M1WO	MSCI World Total Return	6.83	15.63	0.22	14.00
M1EF	MSCI World Total Return	12.43	24.70	0.36	16.03
H0A0	ICE BofAML US High Yield Master II Index Value	8.78	8.79	0.61	8.67
SBWGC	Citi WGBI Currency-Hedged USD	6.84	3.17	1.09	1.33
SBWGU	Citi WGBI USD	6.93	6.79	0.52	2.34
LBUSTRUU	U.S. Aggregate Bonds	7.34	3.87	1.02	1.34
SPGSCITR	S&P GSCI Total Return CME	6.52	21.27	0.15	17.65

The descriptive statistics summarized in Table 1 shows that the developed equity has annual return of 6.83% at a risk of 15.63% which yields lower return and volatility compared to emerging equity. High yield index has similar return and volatility as equity indices, but, shows higher return and volatility compared to bonds which illustrate the lowest volatility and return in all asset classes. The commodity draws a similar figure with developed countries with respect to return and risk and gives the lowest ratio (0.15), which may be due to the high volatility in oil prices with low return. The unexpected high sharpe ratios in bond indices can be the consequence of 2008 and subsequent financial crises, since bonds generally give the high performance during the bad times (Braga, 2016). Developed equity presents the poor Sharpe ratio (0.22) compared to emerging equity. The highest drawdowns belong to commodity and equity indices. High yield also has high drawdown compared to bond indices (between 1% and 2%).

Table 2. Correlation Matrix of Asset Classes for two Different Time Periods

January 1988 – December 2017							
Type	**Asset**	**M1WO**	**M1EF**	**H0A0**	**SBWGC**	**SBWGU**	**LBUSTRUU**
Equity	M1EF	**0.74**	1				
High yield	H0A0	**0.61**	**0.58**	1			
Bond	SBWGC	0.01	-0.11	0.01	1		
Bond	SBWGU	0.26	0.06	0.09	**0.55**	1	
Bond	LBUSTRUU	0.12	0.01	0.24	**0.85**	**0.57**	1
Commodity	SPGSCITR	0.24	0.28	0.22	-0.16	1.12	-0.02
August 2008 – February 2009							
Type	**Asset**	**M1WO**	**M1EF**	**H0A0**	**SBWGC**	**SBWGU**	**LBUSTRUU**
Equity	M1EF	**0.95**	1				
High yield	H0A0	**0.92**	0.87	1			
Bond	SBWGC	-0.20	-0.15	-0.14	1		
Bond	SBWGU	0.19	0.28	0.29	**0.68**	1	
Bond	LBUSTRUU	**0.45**	**0.46**	**0.50**	**0.72**	**0.79**	1
Commodity	SPGSCITR	**0.62**	**0.64**	**0.62**	-0.54	-0.04	0.02

To depict the association among these assests, the correlations (Table 2) are calculated for two different periods: January 1988 - December 2017 and August 2008 - February 2009. High dependence is observed between equity and high yield index (0.74), whereas, bond indices illustrate low correlations with the remaining asset classes. The influence of crisis is significant on the correlations causing increase in the correlations among asset i.e. equity indices and high yield (0.95), for developed and emerging equities ranging from 0.61 to 0.92. As for the bonds, only SBWGC index shows increasing negative correlation with other asset classes whereas, the other bond indices show increasing positive correlations with other asset classes. The high yield and commodity indices react to subprime crises three times, which increased the correlation between them. Briefly, over the bad economic times, they demonstrate highly correlated behavior contrary to the expectation.

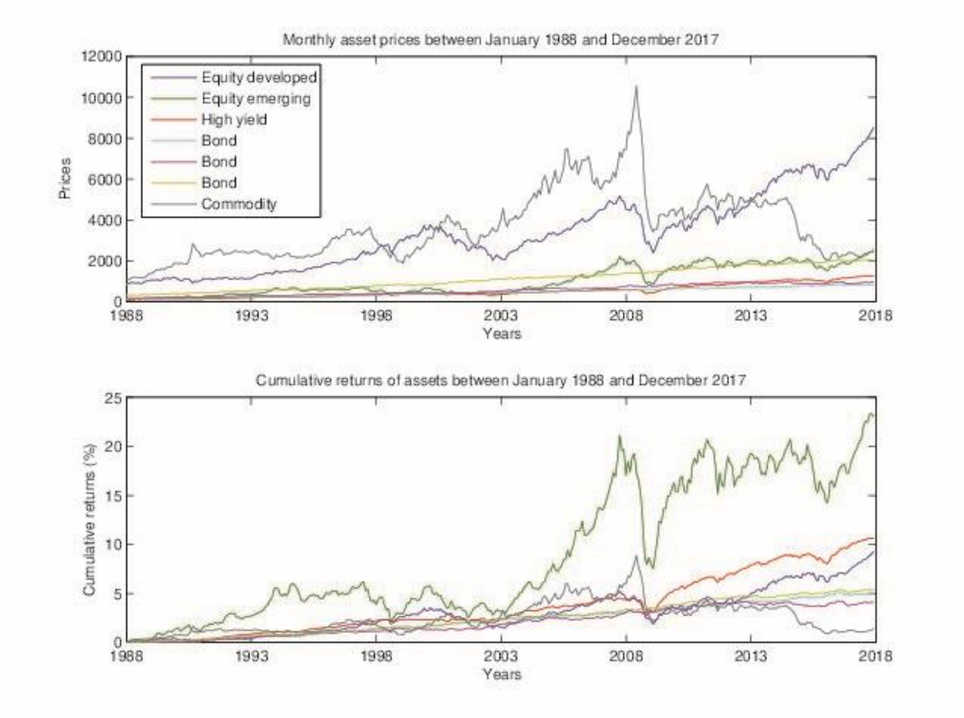
3.1. Constructing Uncorrelated Portfolios

To extract the uncorrelated risk factors hidden in the multi-asset classes, PCA and MTP are used. The economic interpretation of the principal portfolios is based on the coefficients of the asset classes in the eigenvectors. The asset that yields high coefficients in absolute value drives the volatility of the eigenvector. There are seven eigenvectors that are also uncorrelated principal portfolios (PPs). The economic interpretation of each eigenvector is presented with their variances (Table 3). High coefficients (in bold face) in the first principal portfolio (PP1) are dominated by both equity and commodity risks with the weights of 0.44, 0.79 and 0.38 for developed equity, emerging equity and commodity indices, respectively. It has the highest variance (4.22%), as the first eigenvector is driven by high volatile assets, i.e. equities and commodity. The second principal portfolio (PP2) is purely driven by commodity index with weight 0.92 and corresponds to the inflation risk with variance of 2.60%. The developed and emerging equities have the highest weights, hence PP3 represents the equity risk that accounts for 1.19% of the total variance. The fourth principal portfolio (PP4) and the fifth principal portfolio (PP5) can not be interpreted properly in economic aspects since high weights do not belong to any of single asset class. The sixth principal portfolio (PP6) is dominated by bonds, and therefore it represents the interest rate risk with a volatility of 0.38%. The seventh principal portfolio (PP7) is not defined since there is no one type of asset.

Table 3. Eigenvector Matrix in Constructing Uncorrelated Portfolios Using PCA

Type	Asset	PP1	PP2	PP3	PP4	PP5	PP6	PP7
Equity	M1WO	**0.44**	-0.18	**-0.74**	0.22	**0.41**	-0.08	-0.01
Equity	M1EF	**0.79**	-0.33	**0.48**	-0.20	-0.03	-0.01	0.00
High yield	H0A0	0.19	-0.06	-0.23	**0.42**	**-0.84**	0.17	0.07
Bond	SBWGC	-0.01	-0.02	-0.12	-0.27	-0.15	**-0.53**	**0.78**
Bond	SBWGU	0.03	0.02	-0.26	**-0.76**	-0.17	**0.50**	-0.01
Bond	LBUSTRUU	0.01	-0.01	-0.16	-0.28	-0.28	**-0.65**	**-0.62**
Commodity	SPGSCITR	**0.38**	**0.92**	0.01	0.01	0.02	-0.03	0.01
	Risk type	**Equity + Comm.**	**Inflation**	**Equity**	**Undef.**	**Undef.**	**Int. rate**	**Undef .**
	Variance	4.22%	2.59%	1.19%	1.04%	0.81%	0.38%	0.17%

Figure 1. Monthly Asset Prices and Their Cumulative Returns



class with high coefficients to dominate this eigenvector, i.e., two bonds have high absolute coefficients (0.78, 0.62), the rest having quite low coefficients prevents the domination of bonds. Therefore, we extract three main uncorrelated risk sources as the result of PCA which are equity, inflation and interest rate. The variances of those risks are: $\sigma^2_{PP,equity} = 1.19\%$, $\sigma^2_{PP,inflation} = 2.59\%$, $\sigma^2_{PP,interestrate} = 0.38\%$. It can be seen that the interest rate risk is less significant compared to equity and inflation.

Table 4. Torsion Matrix in Constructing Uncorrelated Portfolios Using MLT

Type	Asset	MTP1	MTP2	MTP3	MTP4	MTP5	MTP6	MTP7
Equity	M1WO	**1.32**	-0.03	-0.50	-0.02	-0.45	0.20	-0.02
Equity	M1EF	-0.75	**1.28**	-0.61	0.27	0.15	0.30	-0.11
High yield	H0A0	-0.16	-0.08	**1.20**	0.38	0.10	-0.64	-0.02
Bond	SBWGC	0.00	0.00	0.05	**1.55**	-0.10	-0.68	0.02
Bond	SBWGU	-0.08	0.01	0.58	-0.47	**1.16**	-0.38	-0.03
Bond	LBUSTRUU	0.01	0.00	-0.12	-1.01	-0.12	**1.57**	-0.01
Commodity	SPGSCITR	-0.04	-0.08	-0.12	0.96	-0.30	-0.21	**1.04**
	Risk type	**Equity**	**Equity**	**Equity**	**Interest rate**	**Interest rate**	**Interest rate**	**Inflation**
	Variance	0.14%	0.35%	0.05%	0.06%	0.03%	0.07%	0.34%

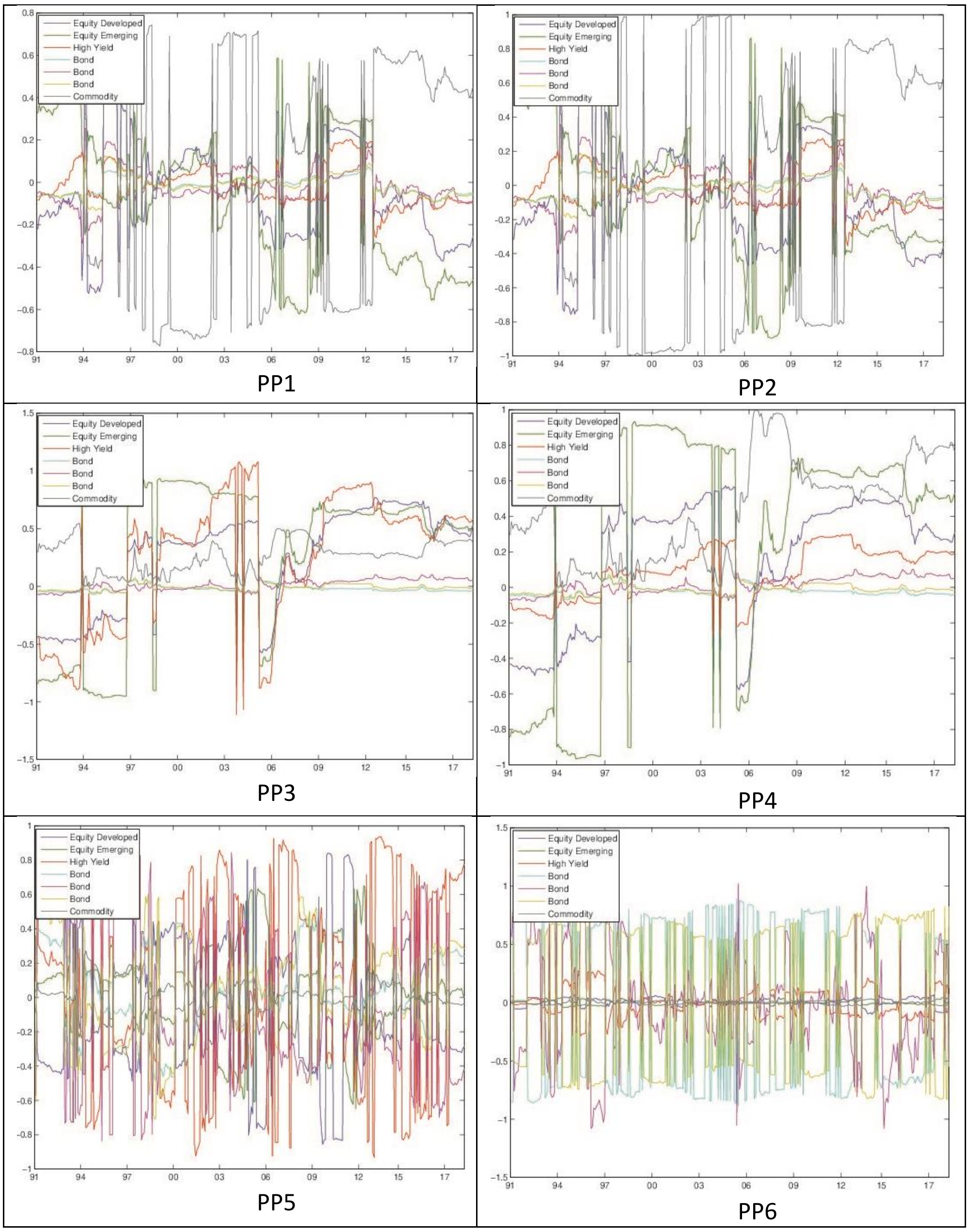
We see that the MLT model gives the uncorrelated risk factors which closely track the original factors. This property helps us to extract and interpret the torsion portfolios straightforwardly. Seven minimum torsion portfolios (MTP), presented in Table 4, depict the torsion matrix of monthly returns based on the sample period from January 1988 to December 2017 at which the bold face values showing the high coefficients. Seven MTP portfolios given in the columns of Table 4 have the highest score for only one asset making much easier to match the risk sources for asset types. MTP1 and MTP2 represent the equity indices with the variances 0.14% and 0.35%, respectively. Third column has the highest score for the high yield that is assessed as the equity risk. Therefore, first three columns represent the equity risk. Forth, fifth and sixth columns are for the bond indices with the volatility of 0.06%, 0.03% and 0.07%, respectively. These columns denote the interest rate risk. The last column with the volatility of 0.34% presents the commodity risk, i.e., inflation risk. This leaves us with three main uncorrelated risk sources: equity, inflation and interest. The variance of each risk is the aggregate variances of the associated columns and found to be $\sigma^2_{MTP,equity} = 0.54\%$, $\sigma^2_{MTP,inflation} = 0.16\%$, $\sigma^2_{MTP,interest} = 0.34\%$.

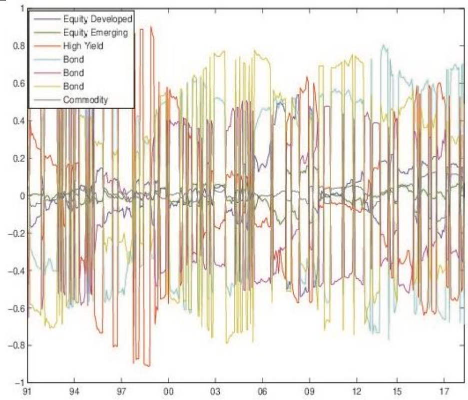
3.2. Portfolio Performances Based On Strategies

To extract the uncorrelated risk factors hidden in the multi-asset classes, PCA and MTP are used. Then, as a next step, we check whether PCA and MLT give the same economic interpretations over the time. Figure 2 and Figure 3 demonstrate the weights of each portfolio from PCA and MTP for three-year rolling window estimations, left and right columns, respectively. It should be noticed that the *x*-axis in Figure 2 refers to the years coded with respect to their last two digits.

The first principal portfolio (PP1) is mostly dominated by commodity and equity risks over the period, these risks demonstrate high volatile structure concluding inconsistency. PP2 is dominated by the commodity risk, in a short period equity risk vaguely demonstrates itself, as a result, it represents the commodity risk but it is not strictly stable over the time. Third principal portfolio is obviously equity risk; however, the commodity risk is shown only over 2008 for a short time. Forth and fifth principal portfolios do not demonstrate a clear pattern, thus they are not defined. Sixth principal portfolio exhibits the interest rate risk. The last portfolio is not clearly distinguished well. In general, principal portfolios do not demonstrate consistent pattern according to three-year rolling windows.

Figure 2. Weights of PPs





PP7

As for the torsion portfolios, they have the most robust results and each torsion portfolio clearly tracks the original corresponding factor. Therefore, it makes easy for economic interpretation. First three torsion portfolios present the equity risk; following three torsion portfolios represent the interest rate risk and the remaining exhibits the commodity risk.

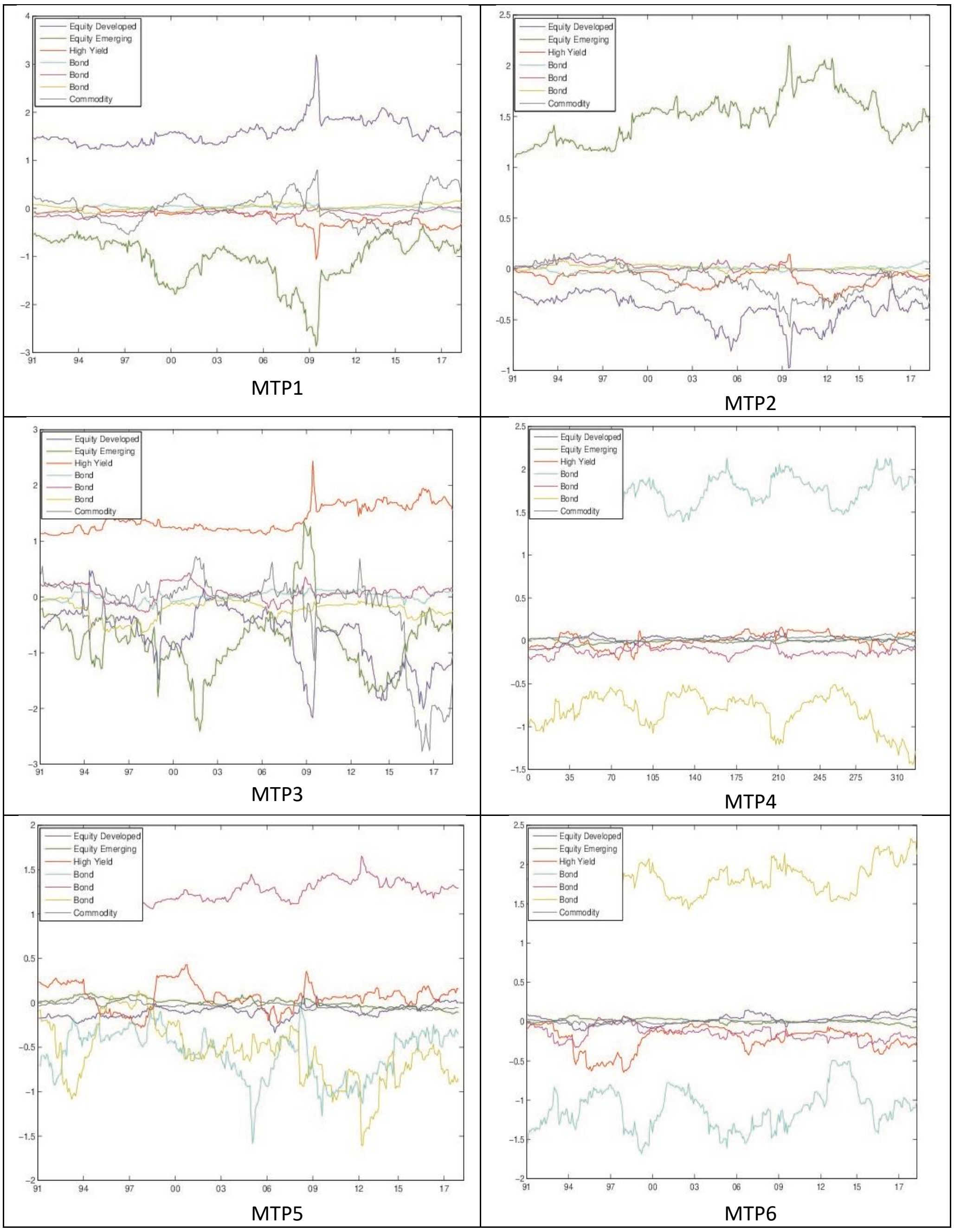
Table 5. Performance Results of Asset Allocation Strategies

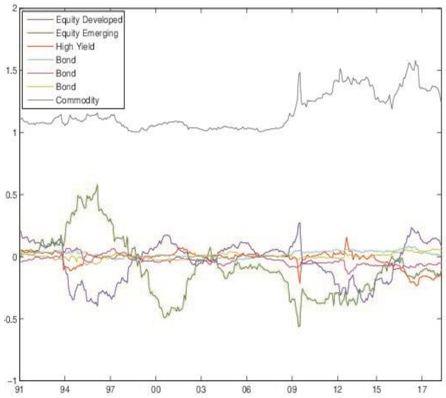
Strategy	Return (%)	Risk (%)	Sharpe ratio	MDD (%)	$Gini_{weight}$	$Gini_{risk}$	Uncorrelated risk
DRP_{MTP}	6.3	5.8	0.63	44.5	0.39	0.00	3.00
DRP_{PP}	5.9	5.7	0.37	48.5	0.56	0.00	3.00
EW	7.9	8.3	0.49	50.7	0.00	0.57	1.08
GMV	5.2	2.9	0.48	22.5	0.91	0.90	1.01
IV	6.5	6.9	0.39	31.7	0.45	0.10	1.90
ERC	6.4	6.3	0.41	34.8	0.47	0.00	2.10

Table 5 presents the performance and risk results of two DRP strategies with riskbased benchmark strategies. The table shows both the performance and risk characteristics results of chosen asset allocation strategies according to the period from January 1988 to December 2017. Return, risk and Sharpe ratio are annualized results. Sharpe ratio is computed with the monthly risk-free rate that is taken from Fama-French website[1]. MDD is reported over one year during the whole sample period. $Gini_{weight}$is calculated with portfolio weights and $Gini_{risk}$is calculated with risk decompositions of asset classes for asset allocation strategies and risk decompositions of uncorrelated risk sources for diversified risk parity strategies. The number of uncorrelated risks gives the result of the uncorrelated risk sources with using the exponential entropy of risk decompositions.

DRP_{MTP} has the return of 6.3% at 5.8% volatility. Given that DRP_{MTP}has the highest Sharpe ratio of 0.63. DRP_{PP} approach gains 5.9% return with 5.7% risk, which gives the Sharpe ratio of 0.37. DRP_{PP} portfolio has the second lowest risk but its return is also relatively low and this results in the lowest Sharpe ratio among all strategies. This meets the expectation of the low risk low return case. As for the benchmark strategies, GMV strategy has the lowest return of 5.2%, yielding the benefit of the lowest volatility (2.3%). Its Sharpe ratio is 0.61, which is a favorable performance and it has the lowest drawdown among all strategies. Remaining strategies place between DRP_{MTP} and GMV strategies according to their risk-return performance.

Figure 3. Weights of MTPs





MTP7

Table 6. Weights and Risk Contributions of Asset Classes Based on Strategies (%)

		EW			GMV			IV		
Type	**Asset**	**Weights**	**MRC**	**RC**	**Weights**	**MRC**	**RC**	**Weights**	**MRC**	**RC**
Equity	M1WO	14.3	156.5	22.4	0.8	131.5	1.0	6.6	248.4	16.5
Equity	M1EF	14.3	253.2	36.2	0.5	147.7	0.7	4.2	353.7	14.9
High yield	H0A0	14.3	74.8	10.7	7.8	100.9	7.8	11.9	133.7	15.9
Bond	SBWGC	14.3	3.2	0.5	85.6	98.6	84.4	31.3	38.6	12.1
Bond	SBWGU	14.3	31.4	4.5	0.6	139.2	0.8	14.9	102.5	15.3
Bond	LBUSTRUU	14.3	12.3	1.8	1.7	112.5	1.9	26.3	59.4	15.6
Commodity	SPGSCITR	14.3	168.6	24.1	3.2	106.1	3.4	4.7	206.3	9.7
		ERC				DRP_{PP}			DRP_{MTP}	
Type	**Asset**	**Weights**	**MRC**	**RC**	**Weights**	**MRC**	**RC**	**Weights**	**MRC**	**RC**
Equity	M1WO	5.8	244.7	14.3	39.5	113.4	44.8	7.8	178.3	13.9
Equity	M1EF	4.1	352.0	14.3	-10.0	25.5	-2.5	8.8	250.5	22.0
High yield	H0A0	10.8	132.2	14.3	-10.6	28.3	-3.0	-4.3	53.5	-2.3
Bond	SBWGC	36.0	39.7	14.3	62.9	31.0	19.5	34.1	34.6	11.8
Bond	SBWGU	13.7	104.4	14.3	-43.2	11.9	-5.2	25.8	59.4	15.3
Bond	LBUSTRUU	23.6	60.4	14.3	73.4	35.5	26.0	15.5	30.0	4.6
Commodity	SPGSCITR	6.0	237.9	14.3	-11.4	-170.9	20.4	12.4	279.5	34.7

The risk contributions of the asset classes for each strategy can be found in the Table 6. DRP_{MTP} has unbalanced weights from asset classes, which is also supported by $Gini_{weight}$coeffient of 0.39. However, $Gini_{risk}$ is zero, thus DRP_{MTP} is well distributed according to risk allocation. Furthermore, this is also supported by the number of uncorrelated risk factors, which is three. The worst performance of the risk allocation is seen in the portfolio of GMV. It is highly bond risk concentrated. $Gini_{weight}$ of 0.91 and $Gini_{risk}$ of 0.90 support the claim. Also, number of uncorrelated risk is one. Remaining strategies risk contributions place between $DRP_{MTP}$and GMV. It can be depicted that DRP strategy based on minimum torsion approach exhibits the best performance among all strategies in terms of both risk/return tradeoff and risk distribution. The DRP based on principal portfolios has the lowest Sharpe ratio contrary to Lohre et al. (2014). Among the benchmark strategies, EW and GMV demonstrate a good reward to volatility ratio but they have a concentrated risk structure. RP strategies are well balanced in terms of risk from asset allocation but they are actually driven by few risk sources, hence they do not meet the expectations, which are also supported by the literature (Kind, 2013; Lohre et al., 2014).

3.2. Out-of Sample Testing

To capture the changes in the economy on asset behavior, we perform analyses with respect to three different time periods separated as test and training samples. These are, 1988-2003, 1988-2008, 1988-2012 to distinguish the influence of crises on the asset behavior. Along with the behavior of price in time, the proportion of data to be chosen for in-sample and out-of-sample is kept consistent with the literature, where 80%-20% is the accustomed choice, respectively. Based on these selected periods, the indicators (return, risk, Sharpe ratio, Gini coefficient and number of uncorrelated risks) are quantified and presented in Table 7. The weights to be used in the out-of-sample periods are based on the test sample period. Percentage estimation error is calculated based on Sharpe ratios following the work by Poddig and Unger (2012). In Period I (2004-2008) the highest estimation errors are observed again by MV (114%) and GMV (70.2%), and lowest one (52.6%) belongs to the DRP_{MTP} leaving non-remarkable difference with the remaining strategies. Compared to the first period, the estimation errors in this period are high. The reason of this increase might be the result of 2008 financial crisis. As for the risk characteristics, the portfolios show the risk concentration on one risk factor except DRP_{MTP} strategy. DRP_{MTP} portfolio distributes the risk among almost three risk factors. In the Period II (2009-2013) results, the highest estimation error (189%) has the same order as the first two periods, but, yields the lowest error (9%) in the EW portfolio. DRP strategies (PP and MTP) also demonstrate favorably low estimation errors (10% and 12.3%, respectively). In this period, despite the large decrease in returns, except MV, we observe no large estimation errors compared to previous periods. After financial crisis, interest rates are reduced to almost zero. Therefore, in the period, the risk-free rate is very low so that the excess returns of the portfolios remain high, which lead to high Sharpe ratios. Except DRP_{MTP} that distributes the risk almost three risk factors, all strategies are concentrated on one risk source. In the last out-of-sample testing, Period III, MV strategy has the highest estimation error of 144.3%. Different from other periods, ERC portfolio has the second highest estimation error of 71%. EW shows the lowest estimation error of 18.3%. DRP strategies also have low estimation errors. As for the risk structure, ERC and DRP_{PP} distribute the risk among almost two risk sources. DRP_{MTP} spreads the risk across almost three risk factors. The remaining portfolios have risk concentrated structure.

To sum up, MV strategy has the highest estimation errors in all out-of-sample results. This drawback of MV optimization is also shown by different researches such as (Jobson and Korkie, 1981: 72; DeMiguel et al., 2007: 1947). The reason of the poor performance is that MV strategy includes the expected mean estimation which leads to large estimation errors. EW portfolio has the lowest estimation errors in all results except in one period. DRP strategies also demonstrate much lower estimation errors than MV strategy. Contrary to the works by Poddig and Unger (2012) and Kind (2013), we obtain good out-of-sample results based on Sharpe ratio for the DRP strategies. As for the risk structures, all strategies have risk concentrated on risk source except

DRP_{MTP} portfolio. The well diversified structure of the DRP_{MTP} model is based on defining each risk properly as shown in Figure 3.

Table 7. Out-of-Sample Performance

Period I	1988 – 2003					2004 - 2008					
	Return (%)	Risk (%)	Sharpe Ratio	$Gini_{risk}$	Uncorr. risk	Return (%)	Risk (%)	Sharpe Ratio	$Gini_{risk}$	Uncorr. risk	Error
MV	10.92	5.10	1.29	0.34	2.47	4.15	6.47	0.63	0.36	1.64	104.4
EW	7.54	8.60	0.37	0.47	1.56	7.58	8.92	0.84	0.64	1.52	55.7
GMV	5.32	3.98	0.25	0.95	1.00	2.07	2.38	0.85	0.95	1.04	70.2
IV	7.33	5.18	0.58	0.11	1.87	8.34	6.49	1.28	0.81	1.39	54.5
ERC	7.33	5.92	0.51	0.10	1.99	8.05	6.82	1.17	0.83	1.35	56.6
DRP_{PP}	6.32	5.57	0.36	0.00	3.00	3.57	4.51	0.78	0.10	2.59	53.8
DRP_{MTP}	7.51	5.61	0.57	0.00	3.00	7.35	6.07	1.20	0.04	2.94	52.6
Period II	1988 – 2008					2009 - 2013					
MV	7.16	2.97	1.58	0.49	2.11	2.80	4.76	0.55	0.78	1.26	189.0
EW	5.96	7.00	0.50	0.67	1.10	2.77	4.69	0.57	0.54	1.16	9.0
GMV	3.95	2.90	0.51	0.90	1.01	2.81	2.70	0.96	0.76	1.03	47.2
IV	7.37	5.19	0.94	0.40	1.90	2.54	3.69	0.78	0.33	1.19	49.0
ERC	7.32	5.10	0.95	0.39	2.00	2.81	3.34	0.78	0.19	1.27	21.6
DRP_{PP}	6.56	5.98	0.68	0.00	3.00	2.56	4.31	0.55	0.13	1.35	10.0
DRP_{MTP}	7.14	6.22	0.75	0.00	3.00	2.94	4.13	0.66	0.07	2.66	13.6
Period III	1988 - 2012					2013 - 2017					
MV	9.51	7.07	0.67	0.56	1.47	4.37	3.57	0.27	0.57	1.41	144.3
EW	9.01	10.52	0.40	0.85	1.32	7.88	9.10	0.49	0.77	1.10	18.3
GMV	5.68	3.05	0.30	0.86	1.21	3.95	2.94	0.19	0.87	1.01	57.6
IV	8.45	3.87	0.95	0.55	1.73	6.73	5.22	0.64	0.58	1.30	48.7
ERC	8.51	3.99	0.94	0.20	2.30	6.59	5.82	0.55	0.29	1.79	71.0
DRP_{PP}	7.85	4.99	0.62	0.00	3.00	6.41	6.08	0.50	0.15	2.06	24.5
DRP_{MTP}	8.74	5.17	0.77	0.00	3.00	6.95	5.67	0.63	0.08	2.94	22.4

4. CONCLUSION

To maximize risk diversification of a portfolio with distributing the risk among the uncorrelated risk factors, we examine the DRP strategies based on PCA and MTP approaches and we compare them with risk based asset allocation strategies using different asset class indices consisting of equities, bonds and commodity. The DRP_{MTP}strategy has a well balanced risk structure with distributing the whole risk among three main risk sources. The result is consistent according to three-year rolling window estimations. The other diversified RP strategy based on principal component analysis also demonstrates the similar result. However, compared to MTP, we observe that principal portfolios are not stable over the time and thus do not give the same economic interpretations. The portfolio may actually concentrate on one or few risk sources. The benchmark strategies create ill-diversified portfolios in terms of risk. Contrary to diversified RP strategies, the risk contribution of these strategies comes from the asset classes instead of the risk factors. As contribution, this study shows that in order to construct well-diversified portfolio for distributing the risk among three factors, DRP strategies, specifically, the one obtained using MTP, demonstrate good performance in both Sharpe ratio and risk diversification in out-of-sample testing. This strategy may help the investors to construct risk diversified portfolios, even in financial crisis. As an extension of this paper, the long-short constraints can be considered. Additionally, along with the variance as risk measure, value at risk, expected shortfall should be analyzed in the frame of the DRP strategies.

NOTES

[1] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

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AUTHOR STATEMENT

Statement of Research and Publication Ethics

This study has been prepared in accordance with the ethical principles of scientific research and publication.

Ethics Committee Approval

This study has been prepared in accordance with the ethical principles of scientific research and publication.

Author Contribution

The authors contributed equally to the study.

Conflict of Interest

There is no conflict of interest for the authors or third parties arising from the study.

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