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A nice copy of a degenerate Lorentz-Marcinkiewicz space that implies the failure of the fixed point property

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Abstract. Introducing the notion of asymptotically isometric copies inside Banach spaces, Dowling, Lennard and Turett made easier to detect failure of the fixed point property for nonexpansive mappings. Their tool was very usefull for indicating the failure. Since then, researchers have investigated alternative tools. Recently, Nezir introduced the notion of asymptotically isometric copies of $\ell^{1\boxplus 0}$. He noticed that a renorming of ℓ^1 turns out to be a degenerate Lorentz-Marcinkiewicz space and using its structure he introduced his notion which implies the failure of the fixed point property for nonexpansive mappings. In this study, we introduce another notion which is derived from the structure of another degenerate Lorentz-Marcinkiewicz space and we show that detecting our new tool in Banach spaces will indicate the failure of the fixed point property for nonexpansive mappings.

1. Intoduction and Preliminaries

While Dowling and Lennard initially wanted to prove that nonreflexive subspaces of $L^1[0,1]$ fail the fixed point property, they introduced the concept of a Banach space containing an asymptotically isometric copy of ℓ^1 and then used this notion to prove that every equivalent renorming of $\ell^1(\Gamma)$, for Γ uncountable, fails the fixed point property [4].

The notion of asymptotically isometric copies of the classical Banach spaces ℓ^1 has applications in metric fixed point theory because they arise naturally in many places. For example, every non-reflexive subspace of $(L_1[0,\ 1],\ \|.\|_1)$, every infinite dimensional subspace of $(\ell^1\ \|.\|_1)$, and every equivalent renorming of ℓ^∞ contains an asymptotically isometric copy of ℓ^1 and so all of these spaces fail the fixed point property [4, 6]. The concept of containing an asymptotically isometric copy ℓ^1 also arises in the isometric theory of Banach spaces in an intriguing way: a Banach space X contains an asymptotically isometric copy ℓ^1 if and only if X^* contains an isometric copy of $(L_1[0,\ 1],\ \|.\|_1)$ [6].

In 1996, Dowling, Lennard and Turett investigated Banach spaces containing asymptotically isometric copies of ℓ^1 deeply and they reached important results which has leaded researchers to test the failure of the fixed point property for nonexpansive mappings in Banach spaces they have studied. In fact, Lin was impressed with their work [5] which proved ℓ^1 with a norm does not contain any asymptotically isometric

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copy of ℓ^1 and then he later showed using a special version of the norm in [7] that ℓ^1 has the fixed point property.

Thus, importance of detecting nice copies of ℓ^1, after witnessing their applications especially, suggests the researchers to investigate alternative properties which will help explore the failure of the fixed point property for nonexpansive mappings. For example, recently, Álvaro, Cembranos and Mendoza [1] introduced another nice property, which they called N1, to detect failure of fixed point property for nonexpansive mappings. Their notion was more general than the concept of a Banach space containing an asymptotically isometric copy of c_0.

In 2019, the first author explored a new renorming of ℓ^1 and noticed that his renorming was actually yielding a degenerate Lorentz-Marcinkiewicz space. He investigated fixed point properties for the dual and predual of his renorming and obtained the results of their failure of the fixed point property for nonexpansive mappings [10]. Later, using the structure of these spaces, in [11], he introduced the notion of asymptotically isometric copies of $\ell^{1 \boxplus 0}$ which implies failure of the fixed point property for nonexpansive mappings for nonexpansive mappings. One can say that detecting Nezir's construction in Banach spaces is a sign of detecting a nice copy of a degenerate Lorentz-Marcinkiewicz space.

In this study, we introduce another notion which is derived from the structure of another degenerate Lorentz-Marcinkiewicz space and we show that detecting our new tool in Banach spaces will indicate the failure of the fixed point property for nonexpansive mappings.

Now, we give some preliminaries for our study.

Definition 1.1. *[3] A Banach space $(X, \|\cdot\|)$ is said to contain an asymptotically isometric copy of ℓ^1 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sum_{n=1}^{\infty}|a_n| \ ,$$

for all $(a_n)_n \in \ell^1$.

The usefulness of this notion can be found in the next result.

Theorem 1.2. *[4] If a Banach space X contains an asymptotically isometric copy of ℓ^1, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

Moreover, Dowling, Lennard and Turett provided the following theorem which shows an alternative way of detecting an asymptotically isometric copy of ℓ^1 in Banach spaces.

Theorem 1.3. *[3] A Banach space X contains an asymptotically isometric copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X such that there are constants $0 < m \leq M < \infty$ so that for all $(t_n)_n \in \ell^1$,*

$$m\sum_{n=1}^{\infty}|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M\sum_{n=1}^{\infty}|t_n| \ ,$$

and $\lim_{n\to\infty}\|x_n\| = m$.

Now, let's recall the definition of Lorentz-Marcinkiewicz space and the degenerate one, the space Nezir introduced in [10].

First of all, we note that our reference for Lorentz spaces is [8, 9].

Now, we recall the construction of Lorentz-Marcinkiewicz spaces.

Let $w \in (c_0 \setminus \ell^1)^+$, $w_1 = 1$ and $(w_n)_{n\in\mathbb{N}}$ be decreasing; that is, consider a scalar sequence given by $w = (w_n)_{n\in\mathbb{N}}$, $w_n > 0, \forall n \in \mathbb{N}$ such that $1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_n \geq w_{n+1} \geq \ldots, \forall n \in \mathbb{N}$ with $w_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sum_{n=1}^{\infty} w_n = \infty$. This sequence is called a weight sequence. For example, $w_n = \frac{1}{n}, \forall n \in \mathbb{N}$.

Definition 1.4.

$$\ell_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{w,\infty} := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} x_j^\star}{\sum_{j=1}^{n} w_j} < \infty \right\}.$$

Here, $x^\star$ represents the decreasing rearrangement of the sequence x, which is the sequence of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in non-increasing order, followed by infinitely many zeros when $|x|$ has only finitely many non-zero terms.

This space is non-separable and an analogue of ℓ_∞ space.

Definition 1.5.

$$\ell^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \limsup_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} x_j^\star}{\sum_{j=1}^{n} w_j} = 0 \right\}.$$

This is a separable subspace of $\ell_{w,\infty}$ and an analogue of c_0 space.

Definition 1.6.

$$\ell_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{w,1} := \sum_{j=1}^{\infty} w_j \, x_j^\star < \infty \right\}.$$

This is a separable subspace of $\ell_{w,\infty}$ and an analogue of ℓ^1 space with the following facts: $(\ell^0_{w,\infty})^\star \cong \ell_{w,1}$ and $(\ell_{w,1})^\star \cong \ell_{w,\infty}$ where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic.

Now, we will introduce Nezir's construction.

For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define $|||x||| := \|x\|_1 + \|x\|_\infty = \sum_{n=1}^{\infty} |x_n| + \sup_{n\in\mathbb{N}} |x_n|$. Clearly $|||\cdot|||$ is an equivalent norm on ℓ^1 with $\|x\|_1 \le |||x||| \le 2\|x\|_1$, $\forall x \in \ell^1$.

We shall call $|||\cdot|||$ the $1 \boxplus \infty$-norm on ℓ^1.

Note that $\forall x \in \ell^1$, $|||x||| = 2x_1^* + x_2^* + x_3^* + x_4^* + \cdots$ where z^* is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}$, $\forall z \in c_0$.

Let $\delta_1 := 2, \delta_2 := 1, \delta_3 := 1, \cdots, \delta_n := 1, \ \forall n \ge 4$.

We see that $(\ell^1, |||\cdot|||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0 \backslash \ell^1$ (the usual Lorentz situation). This suggests that $\ell^0_{\delta,\infty} = (c_0, \|\cdot\|)$ is an isometric predual of $(\ell^1, |||\cdot|||)$ where for all $z \in c_0$, $\|z\| := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} z_j^*}{\sum_{j=1}^{n} \delta_j}$.

In our study, we will consider the degenerate Lorentz-Marcinkiewicz space generated by the weight sequence $\delta = \left(1+1, 1+\frac{1}{2}, 1+\frac{1}{4}, 1+\frac{1}{8}, 1+\frac{1}{16}, \cdots, 1+\frac{1}{2^n}, \cdots\right)$.

That is, we will consider the degenerate Lorentz-Marcinkiewicz space given by the following definition.

Definition 1.7.

$$\ell_{\delta,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \,\middle|\, \|x\|_{\ell,1} := \sum_{j=1}^{\infty} |x_j| + \sum_{j=1}^{\infty} \frac{x_j^\star}{2^{j-1}} < \infty \right\}.$$

Here, one can notice that for $x \in \ell_{\delta,1}$,

$$\begin{aligned} \|x\|_{\ell,1} &= \sum_{j=1}^{\infty} |x_j| + \sum_{j=1}^{\infty} \frac{x_j^\star}{2^{j-1}} \\ &= \sum_{j=1}^{\infty} x_j^\star + \sum_{j=1}^{\infty} \frac{x_j^\star}{2^{j-1}} \\ &= \sum_{j=1}^{\infty} \left(1 + \frac{1}{2^{j-1}}\right) x_j^\star . \end{aligned}$$

Inspired by the construction of degenerate Lorentz-Marcinkiewicz Nezir introduced in [10], in [11], Nezir introduced the structure of asymptotically isometric copies of $\ell^{1\boxplus 0}$. Then, he proved that if a Banach space contains an asymptotically isometric copies of $\ell^{1\boxplus 0}$, it fails the fixed point property for nonexpansive mappings. This was an alternative property to the concept of Banach spaces' containing an asymptotically isometric copies of ℓ^1. Now, we will recall this notion and the consequences it yields in fixed point theory.

Definition 1.8. *[11] A Banach space $(X, \|\cdot\|)$ is said to contain an asymptotically isometric copy of $\ell^{1\boxplus 0}$ if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\frac{1}{2}\left[\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|a_n| + \sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n|\right] \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \frac{1}{2}\left[\sup_{n\in\mathbb{N}}|a_n| + \sum_{n=1}^{\infty}|a_n|\right],$$

for all $(a_n)_n \in \ell^1$.

Then, as we previously stated that he obtained the following result.

Theorem 1.9. *[11] If a Banach space X contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

He also showed that the above result could be given as the consequence of the following theorem.

Theorem 1.10. *[11] If a Banach space X contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then X contains an asymptotically isometric copy of ℓ^1 but the converse is not true.*

2. Main Results

In this section, we define two new properties that imply the failure of the fixed point property for nonexpansive mappings. That is, we show that if a Banach space has any of the properties we introduce then it fails to have the fixed point property for nonexpansive mappings. We find alternative ways of detecting our properties. Then, we show that a Banach space contains an asymptotically isometric copy of ℓ^1 if and only if it has any of the properties we introduce. Moreover, we show that the degenerate Lorentz-Marcinkiewicz space we introduce in the earlier section has any of the properties we introduce in this section but we show that a Banach space isomorphic to the degenerate Loretz-Marcinkiewicz space we introduce in the previous section does not contain any asymptotically isometric copy of ℓ^1 while oviously it has the properties we introduce in this section. Now, let's introduce those new properties and the results we have mentioned.

First of all, we give the definitions of our properties as follows:

Definition 2.1. *We will say that a Banach space $(X, \|\cdot\|)$ has property NM-1 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| + \sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}},$$

for all $(a_n)_n \in \ell^1$.

Definition 2.2. *We will say that a Banach space $(X, \|\cdot\|)$ has property NM-2 if there is a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that*

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2} \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sqrt{\left[\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2},$$

for all $(a_n)_n \in \ell^1$.

First, we give an alternative ways of detecting our properties NM-1 and NM-2 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has one of the properties NM-1 and NM-2.

Theorem 2.3. *A Banach space $(X, \|\cdot\|)$ has property NM-1 if and only if there is a sequence $(x_n)_n$ in X such that*

1. *there exists $M \in [1, \infty)$ so that for any $(a_n)_n \in \ell^1$,*

$$\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq M \left[\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \right] \tag{1}$$

and

2.

$$\lim_{n\to\infty} \|x_n\| = 1. \tag{2}$$

Proof. Suppose that $(X, \|\cdot\|)$ has property NM-1. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and a sequence $(x_n)_n$ in X so that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n)|a_n| + \sum_{n=1}^{\infty} \frac{(1 - \varepsilon_n)|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} , \tag{3}$$

for all $(a_n)_n \in \ell^1$.

We may assume $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence since we may replace that with $\xi_j := \max_{k\geq n} \varepsilon_k$, for all $j \in \mathbb{N}$. Let $z_k = (1 - \varepsilon_k)^{-1} x_k$ for each $k \in \mathbb{N}$. Then, for all $(a_k)_k \in \ell^1$,

$$\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n z_n \right\| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{1 - \varepsilon_n} + \sum_{n=1}^{\infty} \frac{|a_n|}{(1 - \varepsilon_n)2^{n-1}} .$$

Let $M = \frac{1}{1-\varepsilon_1}$. Then, condition (1) is achieved for the sequence $(z_n)_n$ in X. Also, it is clear to see the condition (2) is achieved for the sequence $(z_n)_n$ too since in inequality (3), taking $(a_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim_{n\to\infty} \|x_n\| = 1$ and so $\lim_{n\to\infty} \|z_n\| = 1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $M \in [1, \infty)$ so that for all $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq M \left[\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \right] \tag{4}$$

and $\lim_{n\to\infty} \|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0, 1)$. Since $\lim_{k\to\infty} \|x_k\| = 1$, and $\|x_k\| \geq 1$ for all $k \in \mathbb{N}$, by passing to subsequences, if necessary, we may suppose that $1 \leq \|x_k\| \leq 1 + \varepsilon_k$ for all $k \in \mathbb{N}$. Define $z_k = \frac{x_k}{1+\varepsilon_k}$ for every $k \in \mathbb{N}$. Then, since $\|z_k\| \leq 1$, we have

$$\left\| \sum_{n=1}^{\infty} a_n z_n \right\| \leq \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}} \quad \text{for every } (a_k)_k \in \ell^1 .$$

Also, from the left hand side inequality of (4), we have

$$\left\| \sum_{n=1}^{\infty} a_n z_n \right\| = \left\| \sum_{n=1}^{\infty} a_n \frac{x_n}{(1 + \varepsilon_n)} \right\| \geq \sum_{n=1}^{\infty} \frac{|a_n|}{1 + \varepsilon_n} + \sum_{n=1}^{\infty} \frac{|a_n|}{(1 + \varepsilon_n)2^{n-1}} \geq \sum_{n=1}^{\infty} (1 - \varepsilon_n)|t_n| + \sum_{n=1}^{\infty} (1 - \varepsilon_n) \frac{|a_n|}{2^{n-1}} .$$

□

Now, we show the alternative way of detecting NM-2 property.

Theorem 2.4. *A Banach space $(X, \|\cdot\|)$ has property NM-1 if and only if there is a sequence $(x_n)_n$ in X such that*

1. *there exists $M \in [1, \infty)$ so that for any $(a_n)_n \in \ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \le \left\|\sum_{n=1}^{\infty}a_nx_n\right\| \le M\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \quad (5)$$

and

2. $$\lim_{n\to\infty}\|x_n\| = 1. \quad (6)$$

Proof. Assume that $(X, \|\cdot\|)$ has property NM-2. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ and a sequence $(x_n)_n$ in X so that

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|\right]^2+\left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2} \le \left\|\sum_{n=1}^{\infty}a_nx_n\right\| \le \sqrt{\left[\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}|a_j|\right]^2+\left[\sum_{n=1}^{\infty}\frac{\sum_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2}, \quad (7)$$

for every $(a_n)_n \in \ell^1$.

Now for every $n\in\mathbb{N}$, define $z_n := x_n - x_{n-1}$ with $x_0 = 0$. So there exist a null sequence $(\varepsilon_n)_n$ in $(0, 1)$ so that for all $(a_n)_n \in \ell^1$

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n|\right]^2+\left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)|a_n|}{2^{n-1}}\right]^2} \le \left\|\sum_{n=1}^{\infty}a_nz_n\right\| \le \sqrt{\left[\sum_{n=1}^{\infty}|a_n|\right]^2+\left[\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right]^2}, \quad (8)$$

We may assume $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence since we may replace that with $\xi_j := \max_{k\ge n} \varepsilon_k$, for all $j \in \mathbb{N}$. Let $y_k = (1-\varepsilon_k)^{-1}z_k$ for each $k \in \mathbb{N}$. Then, for all $(a_k)_k \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \le \left\|\sum_{n=1}^{\infty}a_ny_n\right\| \le \sqrt{\left(\sum_{n=1}^{\infty}\frac{|a_n|}{1-\varepsilon_n}\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{(1-\varepsilon_n)2^{n-1}}\right)^2}.$$

Let $M = \frac{1}{1-\varepsilon_1}$. Then, condition (5) is achieved for the sequence $(z_n)_n$ in X. Also, it is clear to see the condition (6) is achieved for the sequence $(z_n)_n$ too since in inequality (3), taking $(a_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim_{n\to\infty}\|x_n\| = 1$ and so $\lim_{n\to\infty}\|z_n\| = 1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $M \in [1, \infty)$ so that for all $(a_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \le \left\|\sum_{n=1}^{\infty}a_nx_n\right\| \le M\sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \quad (9)$$

and $\lim_{n\to\infty}\|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0, 1)$. Since $\lim_{k\to\infty}\|x_k\| = 1$, and $\|x_k\| \ge 1$ for all $k \in \mathbb{N}$, by passing to subsequences, if necessary, we may suppose that $1 \le \|x_k\| \le 1 + \varepsilon_k$ for all $k \in \mathbb{N}$. Define $z_k = \frac{x_k}{1+\varepsilon_k}$ for every $k \in \mathbb{N}$. Then, since $\|z_k\| \le 1$, we have

$$\left\|\sum_{n=1}^{\infty}a_nz_n\right\| \le \sum_{n=1}^{\infty}|a_n| \le \sqrt{\left(\sum_{n=1}^{\infty}|a_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right)^2} \quad \text{for every } (a_n)_n \in \ell^1.$$

Also, from the left hand side inequality of (9), we have

$$\left\|\sum_{n=1}^{\infty} a_n z_n\right\| = \left\|\sum_{n=1}^{\infty} a_n \frac{x_n}{(1+\varepsilon_n)}\right\| \geq \sqrt{\left(\sum_{n=1}^{\infty} \frac{|a_n|}{1+\varepsilon_n}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{(1+\varepsilon_n)2^n}\right)^2}$$
$$\geq \sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n|\right)^2 + \left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|a_n|}{2^{n-1}}\right)^2}\ .$$

Now for each $n \in \mathbb{N}$, define $y_n := \sum_{j=1}^{n} z_j$. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that for all $(a_n)_n \in \ell^1$,

$$\sqrt{\left[\sum_{n=1}^{\infty}(1-\varepsilon_n)\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{(1-\varepsilon_n)\sum\limits_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2} \leq \left\|\sum_{n=1}^{\infty} a_n y_n\right\| \leq \sqrt{\left[\sum_{n=1}^{\infty}\sum_{j=n}^{\infty}|a_j|\right]^2 + \left[\sum_{n=1}^{\infty}\frac{\sum\limits_{j=n}^{\infty}|a_j|}{2^{n-1}}\right]^2}\ .$$

□

Now, we give important results for properties NM-1 and NM-2, one by one.

Theorem 2.5. *Let $(X, \|.\|)$ be a Banach space. Then, X has property NM-1 if and only if X contains an asymptotically isometric copy of ℓ^1.*

Proof. Suppose that X has property NM-1. Then, there is a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty} \|x_n\| = 1$ and there exists a constant $M \in [1, \infty)$ so that for any $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M\left[\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right].$$

Thus, letting $R := 2M$ we have

$$\sum_{n=1}^{\infty}|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq R\sum_{n=1}^{\infty}|a_n|\ .$$

Hence, by Theorem 1.3, X contains an asymptotically isometric copy of ℓ^1.

Conversely, suppose that a Banach space X contains an asymptotically isometric copy of ℓ^1. Then, by Theorem 1.3, there is a sequence $(x_n)_n$ in X with $\lim_n \|x_n\| = 1$ and there exists a constant $M \in [1, \infty)$ such that for all $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty}|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M\sum_{n=1}^{\infty}|a_n|\ .$$

Now, define $z_n =: \left(1 + \frac{1}{2^{n-1}}\right)x_n$ for each $n \in \mathbb{N}$ and let $K = 2M$, then we have

$$\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq M\left(\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}\right).$$

Hence,

$$\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}} \leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq K\sum_{n=1}^{\infty}|a_n| \leq K\sum_{n=1}^{\infty}|a_n| + \sum_{n=1}^{\infty}\frac{|a_n|}{2^{n-1}}$$

and $\lim_n \|z_n\| = 1$.

Hence, by Theorem 2.4, X has property NM-2 and we are done. □

Therefore, we can give the following corollary using Theorem 1.2.

Corollary 2.6. *If a Banach space X has property NM-1, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

Theorem 2.7. *Let $(X, \|.\|)$ be a Banach space. Then, X has property NM-2 if and only if X contains an asymptotically isometric copy of ℓ^1.*

Proof. Suppose that X has property NM-2. Then, there is a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty} \|x_n\| = 1$ and there exists a constant $M \in [1, \infty)$ so that for any $(a_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2} \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M \sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2} .$$

Thus, letting $R := \sqrt{2}M$ we have

$$\sum_{n=1}^{\infty} |a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq R \sum_{n=1}^{\infty} |a_n| .$$

Hence, by Theorem 1.3, X contains an asymptotically isometric copy of ℓ^1.

Conversely, suppose that a Banach space X contains an asymptotically isometric copy of ℓ^1. Then, by Theorem 1.3, there is a sequence $(x_n)_n$ in X with $\lim_n \|x_n\| = 1$ and there exists a constant $M \in [1, \infty)$ such that for all $(a_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq M \sum_{n=1}^{\infty} |a_n| .$$

Now, define $z_n =: \left(1 + \frac{1}{2^{n-1}}\right) x_n$ for each $n \in \mathbb{N}$ and let $K = 2M$, then we have

$$\begin{aligned}\sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2} &\leq \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^{n-1}} \\ &\leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| \leq M \left(\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{n-1}\right)^2} \leq \left\|\sum_{n=1}^{\infty} a_n z_n\right\| &\leq K \sum_{n=1}^{\infty} |a_n| \\ &\leq K \sqrt{\left(\sum_{n=1}^{\infty} |a_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|a_n|}{2^{n-1}}\right)^2}\end{aligned}$$

and $\lim_n \|z_n\| = 1$.

Hence, by Theorem 2.4, X has property NM-2 and we are done. □

Therefore, we can give the following corollary using Theorem 1.2.

Corollary 2.8. *If a Banach space X has property NM-2, then X fails the fixed point property for nonexpansive mappings on closed bounded convex subset of X.*

3. Some Examples and Remarks

In this section, we will give examples that will show some utilization of our property in the fixed point theory.

As we mentioned in the introduction section, our construction appears in the structure of some degenerate Lorentz-Marcinkiewicz spaces. We have been impressed by the first author's solely works [10, 11] that introduce a degenerate Lorentz-Marcinkiewicz space and later give the definition of the concept of Banach spaces containing asymptotically isometric copies of $\ell^{1\boxplus 0}$. Now, let's consider the degenerate Lorentz-Marcinkiewicz space $\ell_{\delta,1}$ that we had talked about in the introduction section where its weight sequence δ is given by $\delta = \left(1 + \frac{1}{2^{n-1}}\right)_n$. In a recent, unpublished study by the authors of this paper, it was shown that $\ell_{\delta,1}$ contains an asymptotically isometric copy of ℓ_1 and so we can say by Theorem 2.5 that it has property NM-1 and so equivalently it has property NM-2. Then, this would prove that $\ell_{\delta,1}$ fails the fixed point property for nonexpansive mappings.

On the other hand, by Theorem 1.10, we know that if a Banach space contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then it contains an asymptotically isometric copy of ℓ_1. Thus, we can conclude by Theorem 2.5 and Theorem 2.7 that if a Banach space contains an asymptotically isometric copy of $\ell^{1\boxplus 0}$, then it has properties NM-1 and NM-2. However, the following example shows that there exists a Banach space that has these properties but it does not contain any asymptotically isometric copy of $\ell^{1\boxplus 0}$. We have to note that the first author showed in [11] that there exists a Banach space that contains an asymptotically isometric copy of ℓ_1 but it does not contain any asymptotically isometric copy of $\ell^{1\boxplus 0}$. Thus, his example also verifies our remark but here we provide a different example.

Example 3.1. *One can easly see that Banach space ℓ^1 with its equivalent renorming given by for any $x = (x_n)_n \in \ell^1$, $|||x|||^{\sim} = \sum_{n=1}^{\infty}\left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)|x_n|$ has the property NM-1 (so NM-2) but we can prove that it does not contain any asymptotically isometric copy of $\ell^{1\boxplus 0}$.*

Proof. We use the similiar ideas expressed in [5] and by contradiction, assume $\left(\ell^1, |||\cdot|||^{\sim}\right)$ does contain an asymptotically isometric copy of $\ell^{1\boxplus 0}$. That is, there exists a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in ℓ^1 such that

$$\begin{aligned}\frac{1}{2}\sup_{n\in\mathbb{N}}(1-\varepsilon_n)|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| &\le \left|\left|\left|\sum_{n=1}^{\infty} t_n x_n\right|\right|\right|^{\sim} \\ &\le \frac{1}{2}\sup_{n\in\mathbb{N}}|t_n| + \frac{1}{2}\sum_{n=1}^{\infty}|t_n|,\end{aligned} \tag{10}$$

for every $(t_n)_{n\in\mathbb{N}} \in \ell^1$.

Without loss of generality we suppose that $(x_n)_n$ is disjointly supported and that by passing to a subsequence, we can assume that (x_n) converges weak* (and so it is pointwise) to some $y \in \ell^1$.

Next, replacing x_n by the $|||\cdot|||^{\sim}$-normalization of $\left(\frac{x_{2n}-x_{2n-1}}{2}\right)_n$ satisfying (10), we can suppose that $y = 0$.

By the proof of the Bessaga-Pełczyński Theorem [2], we may pass to an essentially disjointly supported subsequence of x_n. Hence, when it is normalized and truncated this subsequence appropriately, we get a disjointly supported sequence satisfying (10). Also, by passing to subsequences if necessary, we may suppose that $\varepsilon_n < \frac{1}{3n}$ for all $n \in \mathbb{N}$.

Let $(m(k))_{k\in\mathbb{N}_0}$ with $m(0) = 0$ and $(\xi_k)_{k\in\mathbb{N}}$ a sequence of scalars such that for each $k \in \mathbb{N}$, $y_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j$.

Using the triangular inequality of the norm, for each $K \in \mathbb{N}$, we get

$$\frac{K - K\varepsilon_K}{2} + \frac{K + 1 - \varepsilon_1 - K\varepsilon_K}{2} \leq |||x_1 + Kx_K|||^{\sim}$$
$$\leq \sum_{k=1}^{m(1)} \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)|\xi_k| + K \sum_{k=m(K-1)+1}^{m(K)} \left(\frac{1}{4} + \frac{1}{2^{n+1}}\right)|\xi_k|$$
$$\leq \frac{1}{2}\sum_{k=1}^{m(1)} |\xi_k| + K\left(\frac{1}{4} + \frac{1}{2^{m(K-1)+2}}\right) \sum_{k=m(K-1)+1}^{m(K)} |\xi_k| \, .$$

Therefore, $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K \leq \frac{1}{2} + K\left(\frac{1}{4} + \frac{1}{2^{m(K-1)+2}}\right)$ for all $K \in \mathbb{N}$. But since $\varepsilon_1 < \frac{1}{3}$ and $K\varepsilon_K < \frac{1}{3}$, we have $K + \frac{1-\varepsilon_1}{2} - K\varepsilon_K > K$ and so

$$1 + \frac{1}{2K} - \frac{\varepsilon_1}{2K} - \varepsilon_K \leq \frac{3}{4K} + \left(\frac{1}{4} + \frac{1}{2^{m(K-1)+1}}\right), \text{ for all } K \in \mathbb{N}.$$

Thus, we get a contradiction by letting $K \to \infty$ since we would have $\frac{3}{4} \leq 0$. This completes the proof. □

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