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# A Note on the Division of Multivariate Polynomial Matrices and Grobner Basis

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Multivariate polynomial matrices,
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Ideal,
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**Abstract**

This paper has two main objectives, the first objective is to define an algorithm for the right (left) division of multivariate polynomial matrices while the second is to generalize the concept of Grobner basis to ideals generated by a finite set of multivariate polynomial matrices.

## 1. Introduction

In many problems in systems theory, we encounter matrices called "Polynomial matrices" whose elements are polynomials over the field of rationals or over the ring of integers, in an indeterminate $x$ or several indeterminate $x, y, z, ...$ [1]. These matrices constitute one of the most attractive research area in matrices theory.

In this article, we try to make a link between all the matrices whose elements are multivariate polynomials and one of the most powerful tools in the resolution of polynomial systems, it is Grobner's Bases [2]. First, we introduce an algorithm to calculate the quotient and the remainder produced by running a right or left division of $A$ by a finite set of multivariate polynomial matrices $A$. Nevertheless, this algorithm present some pathologies linked essentially to the dependence of remainder on how we order the polynomial matrices inside $A$, "The remainder is not unique".

This situation led us to reproduce the Buchberger's technique developed in his PhD thesis by defining a prototype of Grobner basis in the context of polynomial matrices.

This paper is organized as follow: In the section 2, we give some notations and some auxiliary result needed in sequel. In the section 3, we give an algorithm for the right division of two or more multivariate polynomial matrices. Some problems linked to this concept such as the well known membership ideal problem are also investigated. In order to tackling these problems and pathologies, we present in the section 4, the concept of Grobner basis for right ideal generated by a set of multivariate polynomial matrices. For this, we will show at first the notions of monomial ideal and leading terms ideal, than we give the definition of a Grobner basis and some basic and elementary properties of this notion, this section was achieved by a generalization of some of Buchberger's work, such as the Buchberger's criterion and the Buchberger's algorithm.

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## 2. Preliminaries and Notations

Let $\mathbb{K}$ be a field and let $X$ be a sequence of $n$ algebraically independent variables $x_1, x_2, ..., x_n$. Each product of the form $x_1^{\alpha_1}...x_n^{\alpha_n}$ where $\alpha_1, ..., \alpha_n \in \mathbb{N}$ is called a monomial and it will be abbreviated by $X^\alpha$ such that $\alpha = (\alpha_1, ..., \alpha_n)$. The set of all monomial over $\mathbb{K}$ will be denoted by $\mathcal{M}(X)$. It is well known that we can sort $\mathcal{M}(X)$ by some special types of orderings so called monomial orderings. Recall that a total ordering $\prec$ on $\mathcal{M}(X)$ is called monomial ordering wherever for each $X^\alpha, X^\beta$ and $X^\gamma$ in $\mathcal{M}(X)$, we have:

1. $X^\alpha \prec X^\beta \Rightarrow X^\gamma X^\alpha \prec X^\gamma X^\beta$,
2. $\prec$ is well-ordering.

Absolutely, there exists many monomial orderings, each one is convenient for a special type of problems. Among them, we point to the pure and graded reverse lexicographic ordering denoted respectively by $\prec_{lex}$ and $\prec_{grelex}$.

Each $\mathbb{K}-$linear combinations of monomials in $\mathcal{M}(X)$ is called a polynomial on $x_1, x_2, ..., x_n$ over $\mathbb{K}$. The set of all polynomials on $x_1, x_2, ..., x_n$ over $\mathbb{K}$ will be denoted by $\mathbb{K}[x_1, x_2, ..., x_n]$ or shortly $\mathbb{K}[X]$. Clearly, $\mathbb{K}[X]$ equipped with the usual polynomial addition and multiplication, has the rings structure.

Let $f \in \mathbb{K}[X]$ and $\prec$ be a monomial ordering on $\mathcal{M}(X)$. Then:
- The greatest monomial with respect to $\prec$ contained in $f$ is called the leading monomial of $f$ and we write $lm(f)$.
- The coefficient of $Lm(f)$ is called the leading coefficient of $f$, it is denoted by $lc(f)$.
- The leading term $Lt(f)$ of $f$ is the product $lc(f).lm(f)$.
- We call the multidegree of $f$ and we write $multideg(f)$, the power of the leading monomial $lm(f)$ of $f$.

Now, we will introduce the concept of multivariate polynomial matrices.

**Definition 2.1.** *Let $p, q \in \mathbb{N}^*$ and let $(f_{ij})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}}$ be a double sequence of $p \times q$ polynomials in $\mathbb{K}[X]$. A multivariate polynomial matrix is a matrix $A(X)$ of the form*

$$A(X) = (f_{ij}) = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1q} \\ f_{21} & f_{22} & \cdots & f_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ f_{p1} & f_{p2} & \cdots & f_{pq} \end{pmatrix}.$$

*The set of all multivariate polynomial matrices of $p$ rows and $q$ columns over $\mathbb{K}$ will always denoted by $\mathbb{K}^{p\times q}[X]$.*

A multivariate polynomial matrix $A(X)$ may obviously be considered as a polynomial in $x_1, x_2, ..., x_n$ whose coefficients are $p \times q$ constant matrices:

$$A(X) = A^{(1)}X^{\alpha_1} + A^{(2)}X^{\alpha_2} + ... + A^{(s)}X^{\alpha_s}. \tag{2.1}$$

**Definition 2.2.** *A diagonal matrix of order $n$ in $\mathbb{K}^{n\times n}[X]$ is said to be monomial matrix if all the diagonal coefficients equal to the same monomial $X^\alpha$ such that $\alpha \in \mathbb{N}^n$. Throughout this paper all monomial matrices will be denoted by $\mathcal{J}_\alpha$.*

Clearly, Each square multivariate polynomial matrix of order $n$ can be written as a linear combination of monomial matrices in $\mathbb{K}^{n\times n}[X]$:

$$A(X) = \sum_{i=1}^{s} A^{(i)}\mathcal{J}_{\alpha_i}.$$

**Notations:** For all multivariate polynomial matrix $A(X) \in \mathbb{K}^{p\times q}[X]$ we have:

1. The leading monomial $lm(A(X))$ of $A(X)$ is: $lm(A(X)) = \max_{i,j}(lm(f_{ij}))$.
2. The leading monomial matrix of $A(X)$ is: $LM(A(X)) = \mathcal{J}_\alpha$ such that $X^\alpha = lm(A(X))$.

3. The (matrix) coefficient of $LM(A(X))$ is the leading coefficient of $A(X)$, it will be denoted by $LC(A(X))$.

4. The leading term of $A(X)$ is:

$$LT(A(X)) = LC(A(X))lm(A(X)) = LC(A(X))LM(A(X)) = LC(A(X))\mathcal{J}_\alpha.$$

5. The multidegree of $A(X)$ is : $multideg(A(X)) = \max_{i,j}(multideg(f_{ij}))$.

The following lemma is very easy to prove.

**Lemma 2.1.** *Let $X^\alpha, X^\beta \in \mathcal{M}(X)$. Then,*

1. $multideg(\mathcal{J}_\alpha) = \alpha$.
2. *If $\alpha = 0$, then $\mathcal{J}_0 = Id$, $Id$ is the identity matrix of order $n$.*
3. $\mathcal{J}_\alpha \times \mathcal{J}_\beta = \mathcal{J}_{\alpha+\beta}$.
4. *If $X^\beta$ divides $X^\alpha$, then $\mathcal{J}_\beta$ divides $\mathcal{J}_\alpha$ and we have:*

$$\mathcal{J}_\alpha = \mathcal{J}_{\alpha-\beta}\mathcal{J}_\beta.$$

**Definition 2.3.** *If $A(X)$ is a square multivariate polynomial matrix, then we say that $A(X)$ is regular if $LC(A(X))$ is invertible.*

**Example 2.1.** *Let $A(x, y)$ be a square multivariate polynomial matrix in $\mathbb{K}^{2\times 2}[x, y]$ defined by $A(x, y) = \begin{pmatrix} x^2 + xy & 3x^2 - xy + y^2 \\ x^2 + y + 1 & -xy + 2 \end{pmatrix}$. Then,*

$$\begin{aligned} A &= \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix} x^2 + \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} xy + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y^2 + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}. \\ &= \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}. \end{aligned}$$

$lm(A) = x^2, LM(A) = \mathcal{J}_{(2,0)} = \begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}$, $LC(A) = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$ *and $multideg(A) = (2, 0)$, $A$ is regular because $LC(A)$ is invertible.*

In the following proposition, we present some elementary properties linked to the sum and product of multivariate polynomial matrices.

**Proposition 2.1.** *Let $A(X), B(X) \in \mathbb{K}^{n\times n}[X]$ be non-zero multivariate polynomial matrices and let $\prec$ be any monomial ordering on $\mathcal{M}(X)$. Then*

1. $multideg(A(X)B(X)) = multideg(A(X)) + multideg(B(X))$.
2. *If $A(X) + B(X) \neq 0$, then*

$$multideg(A(X) + B(X)) \leq \max(multideg(A(X)), multideg(G(X))).$$

*If, in addition, $multideg(A(X)) \neq multideg(B(X))$, then equality occurs.*

For the rest of the paper we will use the letters $A$, $B$ to indicate the multivariate polynomial matrices. The letters of the form $\mathcal{A}$, $\mathcal{B}$ stands for the constant matrices

## 3. Division of Square Multivariate Polynomial Matrices

This section is intended to define a division algorithm for matrices with multivariate polynomials entries.

### 3.1. The Division's Algorithm

The problem of the determination of the right (left) quotient and the right (left) remainder of the division of polynomial matrices was the main point of interest in a large number of papers [3], [4], [5], [6], because of the large number of its applications in linear system theory. In this section, we give a right (left) division's algorithm of multivariate polynomial matrices.

**Theorem 3.1.** *Let $\prec$ be a monomial ordering on $\mathcal{M}(X)$. Then for all matrices $A$ and $B$ in $\mathbb{K}^{n\times n}[X]$ such that $B$ is regular, there exist $Q_r$ and $R_r$ in $\mathbb{K}^{n\times n}[X]$ such that*

$$A = Q_r B + R_r,$$

*and $R_r = 0$, or $R_r$ is a $\mathbb{K}^{n\times n}-$ linear combination of monomial matrices which are not r-divisible by $LM(B)$. $Q_r$ and $R_r$ are respectively called the right quotient of $A$ and the right remainder of $A$ on the right division by $B$. Similarly, there exist $Q_l$ and $R_l$ defined as the left quotient and left remainder of $A$ on the left division by $B$ satisfy*

$$A = BQ_l + R_l,$$

*with $R_l = 0$, or $R_l$ is a $\mathbb{K}^{n\times n}-$linear combination of monomial matrices which are not l-divisible by $LM(B)$.*

*Proof.* We prove this theorem by giving an algorithm for evaluating the right quotient and right remainder:
Let $A = \sum_{i=1}^{p} A^{(i)}\mathcal{J}_{\alpha_i}$ and $B = \sum_{j=1}^{q} B^{(j)}\mathcal{J}_{\beta_j}$ be in $\mathbb{K}^{n\times n}[X]$ with $\alpha_i, \beta_j \in \mathbb{N}^n$. Let $LT(A) = A^{(p)}\mathcal{J}_{\alpha_p}$ and $LT(B) = B^{(q)}\mathcal{J}_{\beta_q}$, suppose also that $B$ is regular (that is $\det(B^{(q)}) \neq 0$).
If $\mathcal{J}_{\beta_q}$ does not divide any monomial matrix in $A$, we put:

$$Q_r = 0 \text{ and } R_r = A.$$

If $\mathcal{J}_{\beta_q}$ divides one or more monomials in $A$, we choose from them the monomial of the higher multidegree. Without loss of generality, we suppose that $\mathcal{J}_{\beta_q}$ divides $\mathcal{J}_{\alpha_p}$, then:
Compute

$$\begin{aligned} \mathcal{A}_1 &= LC(A)\,[LC(B)]^{-1} \\ A_1 &= A - \mathcal{A}_1 \mathcal{J}_{\alpha_p-\beta_q} B \end{aligned}$$

If $\mathcal{J}_{\beta_q}$ does not divide any monomial matrix in $A_1$, then:

$$\begin{aligned} &Q_r(X) = \mathcal{A}_1 \mathcal{J}_{\alpha_p-\beta_q}, \\ &R_r = A_1. \end{aligned}$$

If $\mathcal{J}_{\beta_q}$ divides one or more monomial matrices in $A_1$, we choose from them the monomial of the higher multidegree. Without loss of generality, we suppose that $\mathcal{J}_{\beta_q}$ divides $LM(A_1)$, then: We put $LM(A_1) = \mathcal{J}_{\alpha^{(1)}}$ and we calculate:

$$\begin{aligned} \mathcal{A}_2 &= LC(A_1)\,[LC(B)]^{-1} \\ A_2 &= A_1 - \mathcal{A}_2 \mathcal{J}_{\alpha^{(1)}-\beta_q} B \end{aligned}$$

If $\mathcal{J}_{\beta_q}$ does not divide any monomial matrix in $A_2$, then:

$$\begin{aligned} &Q_r = \mathcal{A}_1 \mathcal{J}_{\alpha_p-\beta_q} + \mathcal{A}_2 \mathcal{J}_{\alpha^{(1)}-\beta_q}, \\ &R_r = A_2. \end{aligned}$$

If not, we repeat this operation until we get a matrix $A_s$ for which $\mathcal{J}_{\beta_q}$ is not a divisor of any monomial in $A_s$ and then

$$\begin{aligned} &Q_r = \mathcal{A}_1 \mathcal{J}_{\alpha_p-\beta_q} + \mathcal{A}_2 \mathcal{J}_{\alpha^{(1)}-\beta_q} + \ldots + \mathcal{A}_s \mathcal{J}_{\alpha^{(s-1)}-\beta_q}, \\ &R_d(X) = A_s. \end{aligned}$$

with $\alpha^{(i)} = multideg(A_i(X))$ for all $i = 1, \ldots, s$.

To determine the left quotient and the left remainder, just perform the right division of the transpose of $A$ by the transpose of $B$, then by taking the transposes of the obtained quotient and remainder we get $Q_l$ and $R_l$. □

**Example 3.1.** *Let $A, B \in \mathbb{R}^{2\times 2}[x,y]$ defined respectively by*

$$A = \begin{pmatrix} x^2y+1 & 2x^2y+y-1 \\ x-y-2 & x^2y+2 \end{pmatrix} \text{ and } B = \begin{pmatrix} xy+x+1 & x-y-1 \\ x-2 & xy+2 \end{pmatrix}.$$

*We have,* $LT(A) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^2y & 0 \\ 0 & x^2y \end{pmatrix}$ *and* $LT(B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix}$.
*Since $LM(B) = \mathcal{J}_{(1,1)}$ divides $LM(A) = \mathcal{J}_{(2,1)}$, we put:*

$$\begin{aligned} \mathcal{A}_1 &= LC(A)\,[LC(B)]^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\ A_1 &= A - \mathcal{A}_1 \times \mathcal{J}_{((2,1)-(1,1))}B \\ &= \begin{pmatrix} -3x^2+3x+1 & -x^2+xy-3x+y-1 \\ -x^2+3x-y-2 & 2-2x \end{pmatrix} \end{aligned}$$

*$LM(B) = \mathcal{J}_{(1,1)}$ is not a divisor of $LM(A_1) = \mathcal{J}_{2,0}$, but $A_1$ contains a monomial matrix $\mathcal{J}_{(1,1)}$ which is divisible by $LM(B)$. Since the coefficient of $\mathcal{J}_{(1,1)}$ in $A_1$ is the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we put*

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

*and*

$$A_2 = A_1 - \mathcal{A}_2\mathcal{J}_{(0,0)}B = \begin{pmatrix} -3x^2+2x+3 & -x^2-3x+y-3 \\ -x^2+3x-y-2 & 2-2x \end{pmatrix}.$$

*$A_2$ does not contains any monomial matrix divisible by $\mathcal{J}_{(1,1)}$. Hence,*

$$Q_r = \mathcal{A}_1.\mathcal{J}_{(1,0)} + \mathcal{A}_2.\mathcal{J}_{(0,0)} = \begin{pmatrix} x & 2x+1 \\ 0 & x \end{pmatrix} \text{ and } R_r = A_2.$$

**Remark 3.1.** *If $R_r = 0$ (respectively $R_l = 0$), then we say that $B$ a right (respectively left) divisor of $A$.*

**Proposition 3.1.** *The right quotient $Q_r$ of $A$ and the right remainder $R_r$ of $A$ on the right division by $B$ are unique.*

*Proof.* Suppose that there exist $Q_r, R_r$ and $Q'_r, R'_r$ in $\mathbb{K}^{n\times n}[X]$, such that $A = Q_rB + R_r = Q'_rB + R'_r$. Thus, $R_r - R'_r = (Q'_r - Q_r)B$. If $R_r \neq R_r$ then $R_r - R'_r$ contains one or more monomial matrices divisible by $LM(B)$ which impossible because neither $R_r$ nor $R'_r$ contains monomial matrices divisible by $LM(B)$. Hence $R_r(X) = R'_r(X)$, therefore $Q_r = Q'_r$ because $B$ is regular. □

The definition of right quotient and remainder can easily be extended to dividends $A$, which are $p \times q$ where the divisor $B$ is a regular $q \times q$ matrix. The uniqueness property is preserved and $Q_r, R_r$ are also $p \times q$ matrices. In a similar manner the definition of left quotient and remainder may be extended to $q \times p$ matrices, resulting in unique multivariate polynomial matrices $Q_l$ and $R_l$ which are also $q \times p$.

**Corollary 3.1.** *If $A$ and $B$ are two multivariate polynomial matrices commuting in each other, then $Q_l = Q_r$ and $R_l = R_r$.*

Now, we focus our intention to study the divisibility of a non null multivariate polynomial matrix $A \in \mathbb{K}^{n\times n}[X]$ by a set of regular multivariate polynomial matrices $A_1, A_2, ..., A_s$ from the same ring $\mathbb{K}^{n\times n}[X]$.

**Theorem 3.2.** *Let $\mathbf{A} = \{A_1, A_2, ..., A_s\}$ be a set of regular multivariate polynomial matrices in $\mathbb{K}^{n\times n}[X]$. Then for all $A \in \mathbb{K}^{p\times n}[X]$, there exist $Q_{r1}, Q_{r2}, ..., Q_{rs}$ in $\mathbb{K}^{p\times n}[X]$ such that:*

$$A = \sum_{i=1}^{s} Q_{ri}A_i + R_r.$$

*where $R_r = 0$, or $R_r$ is a combination of terms in $\mathbb{K}^{n\times n}[X]$ which are not divisible by any $LC(A_i)$ for all $i \in \{1, 2, ..., s\}$.*

*Proof.* We proceed by induction on $s$, and then the theorem occur. □

### 3.2. The Ideal Membership Problem

To make some context let us consider the following example.

**Example 3.2.** *Let* $A = \begin{pmatrix} xy^2 - x & 0 \\ 0 & xy^2 - x \end{pmatrix}$, $B = \begin{pmatrix} xy + 1 & 0 \\ 0 & xy + 1 \end{pmatrix}$ *and* $C = \begin{pmatrix} y^2 - 1 & 0 \\ 0 & y^2 - 1 \end{pmatrix}$. *It is so easy to verify that:*

$$\begin{aligned} A &\overset{B,C}{\to} -\begin{pmatrix} x + y & 0 \\ 0 & x + y \end{pmatrix}, \\ A &\overset{C,B}{\to} 0. \end{aligned}$$

This example shows that the right remainder produced by the right division algorithm when run on a multivariate polynomial matrix $A$ depends of the choices performed in the run.

This situation leads us to wonder about the following ideal membership problem:
Sometimes, for a right ideal $I$ in $GL_n(\mathbb{K})[X]$ generated by a finite set of matrices $\mathbf{A} = \{A_1, A_2, ..., A_s\}$ in $GL_n(\mathbb{K})[X]$, we can find some matrices $A$ belongs to $I$ but $A$ do not reduce to zero modulo $A_1, ..., A_s$.This seems contradictory.

In order to tackling this problem, we introduce the concept of right (left) Grobner basis for right (left) ideal in $GL_n(\mathbb{K})[X]$.

## 4. Right and Left Grobner Basis

### 4.1. Leading Terms Ideal, Monomial Ideal in $GL_n(\mathbb{K})[X]$

**Definition 4.1.** *Let* $I \subseteq GL_n(\mathbb{K})[X]$ *be a right (left) ideal other than* $\{0\}$, *and fix a monomial ordering* $\prec$ *on* $\mathcal{M}(X)$.
*(1) We denote by* $LT(I)$ *the set of leading terms of non-zero elements of I with respect to* $\prec$.

$$LT(I) = \{A^{(\alpha)}\mathcal{J}_\alpha : \exists A \in I - \{0\} \text{ such that } LT(A) = A^{(\alpha)}\mathcal{J}_\alpha\}.$$

*(2) We denote by* $\langle LT(I)\rangle$ *the right (left) ideal generated by the elements of* $LT(I)$.

Let $I = \langle A_1, A_2, ..., A_s\rangle$ be a right (left) ideal in $GL_n(\mathbb{K})[X]$, then for all $i = 1, ..., s$ we have, $LT(A_i \in LT(I) \subset \langle LT(I)\rangle$, hence:

$$\langle LT(A_1), LT(A_2), ..., LT(A_s)\rangle \subseteq \langle LT(I)\rangle.$$

This inclusion is strict in general. However, sometimes it is possible to find a set of generators $\{G_1, G_2, ..., G_t\}$ of $I$, for which we have

$$\langle LT(I)\rangle = \langle LT(G_1), ..., LT(G_t)\rangle.$$

In what follow, we will focus on the determination of such set of generators. For this, we need to introduce a prototype to the concept of monomial ideal in $GL_n(\mathbb{K})[X]$.

**Definition 4.2.** *A right (left) ideal I in* $GL_n(\mathbb{K})[X]$ *is called monomial if it is generated by monomial matrices, that is:*

$$I = \langle\{\mathcal{J}_\alpha,\ \alpha \in \mathcal{F} \subset \mathbb{N}^n\}\rangle.$$

**Lemma 4.1.** *Every right or left monomial ideal* $I \subset GL_n(\mathbb{K})[X]$ *has a finite monomial matrices generating set* $\{\mathcal{J}_{\alpha_1}, \mathcal{J}_{\alpha_2}, ..., \mathcal{J}_{\alpha_t}\}$.

*Proof.* This is a simple reformulation of the well known Dickson's Lemma. □

**Lemma 4.2.** *Let* $J_1 \subseteq J_2 \subseteq J_3 \subseteq ...$ *be a sequence monomial ideals in* $GL_n(\mathbb{K})[X]$. *For some* $j \in \mathbb{N}$ *we must have* $J_j = J_{j+1} = J_{j+2} = ....$

*Proof.* We consider the ideal $\mathbf{J} = \bigcup_i J_i$ generated by all monomial matrices in all $J_i$. By the above Lemma, $\mathbf{J}$ has a finite generating set $\mathbf{A}$. For each $\mathcal{J}_i \in \mathbf{A}$ there exists a $j_i \in \mathbb{N}$ such that $\mathcal{J}_j \in J_{j_i}$. For $j = \max_i(j_i)$ we have $\mathbf{A} \subseteq J_j$ , implying $\mathbf{J} \subseteq J_j$. Since $J_i \subseteq \mathbf{J}$ for all $i$ we get the desired equality. □

**Theorem 4.1.** *Let $I$ a right (left) in $GL_n(\mathbb{K})[X]$ with $I \neq \{0\}$. Then,*

1. *The ideal $\langle LT(I)\rangle$ is monomial in $GL_n(\mathbb{K})[X]$.*
2. *There exists a finite set of multivariate polynomial matrices $\{G_1, G_2, ..., G_t\}$ in $I$ such that $\langle LT(I)\rangle = \langle LT(G_1), ..., LT(G_t)\rangle$.*

*Proof.* (1) Consider the ideal generated by the leading monomial matrices of all non zero multivariate polynomial matrices in $I$,

$$\langle \{LM(A),\ A \in I - \{0\}\rangle.$$

Since $LT(A) = LC(A).LM(A)$, then $LT(A)$ and $LM(A)$ differ only by a multiplicative factor in $GL_n(\mathbb{K})$. Hence,

$$\langle LT(T)\rangle = \langle LT(A),\ A \in I - \{0\}\rangle = \langle LM(A),\ A \in I - \{0\}\rangle.$$

Thus $\langle LT(I)\rangle$ is monomial in $GL_n(\mathbb{K})[X]$.
(2) Since $\langle LT(I)\rangle = \langle LM(A),\ A \in I - \{0\}\rangle$, it follows from the above lemma that there exist $G_1, ..., G_t \in I$ such that $\langle LT(I)\rangle = \langle LM(G_1), ..., LM(G_t)\rangle$. Since for all $i = 1, ..., t$, $LT(G_i)$ and $LM(G_i)$ differ only by a multiplicative factor in $GL_n(\mathbb{K})$, we obtain:

$$\langle LT(I)\rangle = \langle LT(G_1), ..., LT(G_t)\rangle$$

□

### 4.2. Right (Left) Grobner Basis

**Definition 4.3.** *Fix a monomial ordering $\prec$ on $\mathcal{M}(X)$. A finite subset $\mathcal{G} = \{G_1, ..., G_t\}$ of non-zero multivariate polynomial matrices of a right (left) ideal $I \subseteq GL_n(\mathbb{K})[X]$ different from $\{0\}$ is said to be a right (left) Grobner basis with respect to $\prec$ if*

$$\langle LT(I)\rangle = \langle LT(G_1), ..., LT(G_t)\rangle.$$

**Example 4.1.** *In $GL_2(\mathbb{R})[x, y]$, we consider the right ideal generated by $B = \begin{pmatrix} xy+1 & 0 \\ 0 & xy+1 \end{pmatrix}$ and $C = \begin{pmatrix} y^2+1 & 0 \\ 0 & y^2+1 \end{pmatrix}$, and let*

$$A = \begin{pmatrix} xy^2 - x & 0 \\ 0 & xy^2 - x \end{pmatrix} \in \mathbb{R}[x, y].$$

*We have shown above that*

$$\begin{array}{rcl} A & = & Q_r B + 0C + R_r \\ A & = & Q'_r C + 0B, \end{array} \text{, such that}$$

$Q_r = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$, $R_r = \begin{pmatrix} -x-y & 0 \\ 0 & -x-y \end{pmatrix}$ *and* $Q'_r = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. *The second equation shows that $A \in I$, and from the first one we get:*

$$R_r = A - Q_r B \in I.$$

*Thus, $LT(R_r) \in \langle LT(I)\rangle$, but $LT(R_r) \notin \langle LT(B), LT(C)\rangle$ because $LM(R_r)$ it is not divisible neither by $LM(B)$ nor by $LM(C(X))$.*
*Consequently, $\{B, C\}$ is not a right Grobner basis of $I$.*

**Proposition 4.1.** *Each non-zero right (left) ideal in $GL_n(\mathbb{K})[X]$ has a right (left) Grobner basis with respect to every monomial order .*

*Proof.* The result derives immediately from the theorem 4.1. □

**Lemma 4.3.** *If $\mathcal{G} = \{G_1, ..., G_t\}$ is a Grobner basis for a right (left) ideal $I \subseteq GL_n(\mathbb{K})[X]$ with respect to a monomial order $\prec$, then $\mathcal{G}$ is a basis of $I$.*

*Proof.* We need to show that $I \subseteq \langle \mathcal{G} \rangle$, so we pick $A \in I$. Let $R_r$ be the remainder produced by a run of the right division algorithm.
Notice that $R_r \in I$. If $R_r \neq 0$, then $LT(R_r) \in \langle LT(I) \rangle = \langle LT(G_1), ..., LT(G_t) \rangle$. This means that some $LT(G_i)$ divides $LT(R_r)$. This contradicts the properties of the right remainder. Hence $R_r = 0$, which implies $f \in \langle \mathcal{G} \rangle$. □

**Proposition 4.2.** *Let $\mathcal{G} = \{G_1, ..., G_t\}$ be a right (left) Grobner basis for a right (left) ideal $I \subseteq GL_n(\mathbb{K})[X]$ with respect to a monomial ordering $\prec$. The remainder produced by the right division algorithm when run on a multivariate polynomial matrix $A$ is independent of the choices performed in the run.*

*Proof.* Suppose that one run gave $R_r$ and another gave $R'_r$. Then,

$$A = \sum_{i=1}^{t} Q_{ri} G_i + R_r = \sum_{i=1}^{t} Q'_{ri} G_i + R'_r$$

Thus, $R_r - R'_r = \sum_{i=1}^{t} (Q'_{ri} - Q_{ri}) G_i$, which means $R_r - R'_r \in I$. If $R_r \neq R'_r$ then there would be a leading term $LT(R_r - R'_r) \in \langle LT(I) \rangle$ which is not divisible by any $LT(G_i)$ for $i = 1, ..., t$. This contradicts the fact that $\mathcal{G}$ is a Grobner basis of $I$. Hence $R_r = R'_r$. □

It follows from the above proposition that if $\mathcal{G}$ is a right (left) Grobner basis for a right (left) ideal $I \subseteq GL_n(\mathbb{K})[X]$ with respect to a monomial ordering $\prec$. A multivariate polynomial matrix $A$ belongs to $I$ if and only if the remainder produced by the right division algorithm is 0.

### 4.3. Buchberger's Criterion

In this paragraph we will show how to construct a right (left) Grobner basis for a right (left) ideal $I$ generated by a finite subset in $GL_n(\mathbb{K})[X]$.

**Definition 4.4.** *Let $\prec$ be a monomial order on $\mathcal{M}(X)$ and $A, B$ be two non-zero multivariate polynomial matrices in $GL_n(\mathbb{K})[X]$ with $multideg(A) = \alpha$ and $multideg(B) = \beta$, $(\alpha, \beta \in \mathbb{N}^n)$. We define the $S_r-$polynomial matrix of $A$ and $B$:*

$$S_r(A, B) = \mathcal{J}_{\gamma-\alpha}(LC(A))^{-1} A - \mathcal{J}_{\gamma-\beta}(LC(B))^{-1} B,$$

*such that $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n$ with $\gamma_i = \max(\alpha_i, \beta_i)$. $\mathcal{J}_\gamma$ is least common multiple $LM(A)$ and $LM(B)$. Similarly, the $S_l-$polynomial matrix of $A$ and $B$ is defined by:*

$$S_l(A, B) = A\mathcal{J}_{\gamma-\alpha}(LC(A))^{-1} - B\mathcal{J}_{\gamma-\beta}(LC(B))^{-1}$$

We observe that the leading terms of the two parts of the $S_r-$polynomial matrix (respectively $S_l-$polynomial matrix) cancel. In particular, every monomial matrix of $S_r(A, B)$ (respectively $S_l(A, B)$) is $\prec$ $-$smaller than $\mathcal{J}_\gamma$.

**Lemma 4.4.** *Let $\{A_1, ..., A_s\}$ be a set of multivariate polynomial matrices in $GL_n(\mathbb{K})[X]$ and let $A = \mathcal{C}_1 A_1 + ... + \mathcal{C}_s A_s$ with $\mathcal{C}_i \in GL_n(\mathbb{K})$ such that for all $i = 1, ..., s$, $multideg(A_i) = \delta \in \mathbb{N}^n$. If $multideg(A) < \delta$ Then,*

1. *$A$ is a $GL_n(\mathbb{K})-$linear combination of $S_r(A_i, A_j),\ i, j \in \{1, ..., s\}$.*
2. *$\forall i, j \in \{1, ..., s\} : multideg(S_r(A_i, A_j)) < \delta$.*

*Proof.* Let us prove the first assertion. For all $i \in \{1, ..., s\}$ we have

$$lC(\mathcal{C}_i A_i) = \mathcal{C}_i LC(A_i) \text{ and } multideg(\mathcal{C}_i A_i) = multideg(A_i) = \delta.$$

From the assumption $multideg(\mathcal{C}_1 A_1 + ... + \mathcal{C}_s A_s) < \delta$, we certainly obtain:

$$\mathcal{C}_1 LC(A_1) + ... + \mathcal{C}_s LC(A_s) = 0. \tag{4.1}$$

We have

$$\begin{array}{rcl}\sum_{i=1}^{s}\mathcal{C}_iA_i & = & \sum_{i=1}^{s}\mathcal{C}_iLC(A_i)[LC(A_i)]^{-1}A_i\\ & = & \mathcal{C}_1LC(A_1)\left[(LC(A_1))^{-1}A_1-(LC(A_2))^{-1}A_2\right]\\ & & +[\mathcal{C}_1LC(A_1)+\mathcal{C}_2LC(A_2)][(LC(A_2))^{-1}A_2-\\ & & (LC(A_3))^{-1}A_3]+\cdots+[\mathcal{C}_1LC(A_1)+\cdots+\\ & & +\mathcal{C}_{s-1}LC(A_{s-1})][(LC(A_{s-1}))^{-1}A_{s-1}-(LC(A_s))^{-1}A_s]\\ & & +[\mathcal{C}_1LC(A_1)+\cdots+\mathcal{C}_sLC(A_s)](LC(A_s))^{-1}A_s.\end{array}$$

From the equation (4.1), we get:

$$\begin{array}{rcl}\sum_{i=1}^{s}\mathcal{C}_iA_i & = & \mathcal{C}_1LC(A_1)[(LC(A_1))^{-1}A_1-(LC(A_2))^{-1}A_2]\\ & & +[\mathcal{C}_1LC(A_1)+\mathcal{C}_2LC(A_2)][(LC(A_2))^{-1}A_2-(LC(A_3))^{-1}A_3]+\\ & & +...+[\mathcal{C}_1LC(A_1)+...+\mathcal{C}_{s-1}LC(A_{s-1})][(LC(A_{s-1}))^{-1}A_{s-1}\\ & & -(LC(A_s))^{-1}A_s]\end{array}\tag{4.2}$$

From an other side, we have for all $i\in\{1,...,s\}$ :

$$LT(A_i)=LC(A_i)\mathcal{J}_\delta,$$

and, for all $i,j\in\{1,...,s\}$

$$\begin{array}{rcl}S_r(A_i,A_j) & = & \mathcal{J}_{\delta-\delta}[LC(A_i)]^{-1}A_i-\mathcal{J}_{\delta-\delta}[LC(A_j)]^{-1}A_j\\ & = & [LC(A_i)]^{-1}A_i-[LC(A_j)]^{-1}A_j.\end{array}$$

Hence, the equation (4.2) can be written as follow:

$$\begin{array}{rcl}A=\sum_{i=1}^{s}\mathcal{C}_iA_i & = & \mathcal{C}_1LC(A_1)S_r(A_1,A_2)+[\mathcal{C}_1LC(A_1)+\mathcal{C}_2LC(A_2)]S_r(A_2,A_3)\\ & & +...+[\mathcal{C}_1LC(A_1)+...+\mathcal{C}_{s-1}LC(A_{s-1})]S_r(A_{s-1},A_s).\end{array}$$

The second assertion follows immediately from the first one. □

**Theorem 4.2.** *(Buchberger criterion) Let $\mathcal{G}=\{G_1,...,G_t\}\subseteq GL_n(\mathbb{K})[X]\setminus\{0\}$ and $\prec$ be a monomial order on $\mathcal{M}(X)$. $\mathcal{G}$ is a Grobner basis for $I=\langle\mathcal{G}\rangle$ if and only if for all $i,j$ the multivariate polynomial matrix $S_r(G_i,G_j)$ reduces to zero modulo $\mathcal{G}$.*

*Proof.* If $\mathcal{G}$ is a Grobner basis for the right ideal $I$, then it follows from the above lemma that,

$$S_r(G_i,G_j)\xrightarrow{\mathcal{G}}0$$

Now, let us prove the converse implication, suppose that $\mathcal{G}=\{G_1,...,G_t\}$ is a basis of $I$ such that for all $i,j\in\{1,...,t\}$ with $i\neq j$: $S(G_i,G_j)\xrightarrow{\mathcal{G}}0$. To prove that $\mathcal{G}$ is a Grobner basis for $I$, we need to show that $\langle LT(I)\rangle\overset{?}{=}\langle LT(G_1),\cdots,LT(G_s)\rangle$.
For this, we prove for all $A\in I$ that

$$LT(A)\in\langle LT(G_1),...,LT(G_t)\rangle.$$

Since $\mathcal{G}$ is a basis for $I$, then there exist $A_1,...,A_t\in GL_n(\mathbb{K})[X]$, such that

$$A=A_1G_1+...+A_tG_t.\tag{4.3}$$

Let

$$\delta=max\{multideg(A_1G_1),...,multideg(A_tG_t)\}\tag{4.4}$$

The expansion given in (4.3) is not unique, let's choose an expansion of $A$ for which $\delta$ is minimal. Obviously, $multideg(A) \leq \delta$.
If $multideg(A) = \delta$, then there exist $i \in \{1, ..., t\}$ such that $multideg(A) = multideg(A_iG_i)$, thus $LT(G_i)$ divides $LT(A)$. Hence,

$$LT(A) \in \langle LT(G_1), ..., LT(G_t)\rangle.$$

Let's prove that $multideg(A) = \delta$.
Suppose that $multideg(A) < \delta$ and let $m(i) = multideg(A_iG_i)$, then we have:

$$A = \sum_{m(i)=\delta} A_iG_i + \sum_{m(i)<\delta} A_iG_i.$$

Let

$$A = \sum_{m(i)=\delta} LT(A_i)G_i + \sum_{m(i)=\delta} (A_i - lt(A_i))G_i + \sum_{m(i)<\delta} A_iG_i \tag{4.5}$$

If $m(i) = \delta$, then

$$multideg((A_i - LT(A_i))G_i) < \delta,$$

if $m(i) < \delta$, then

$$multideg(A_iG_i) < \delta.$$

The two last sums in (4.5) have a $multideg$ less than $\delta$.
From the hypothesis $multideg(A) < \delta$, we certainly obtain:

$$multideg\left(\sum_{m(i)=\delta} LT(A_i)G_i\right) < \delta.$$

Put $LT(A_i) = \mathcal{C}_i\mathcal{J}_{\alpha(i)}$ with $\mathcal{C}_i \in GL_n(\mathbb{K})$. We have

$$\sum_{m(i)=\delta} LT(A_i)G_i = \sum_{m(i)=\delta} \mathcal{C}_i\mathcal{J}_{\alpha(i)}G_i,$$

since for all $i$ such that $m(i) = \delta$ we have $multideg\,(LT(A_i)G_i) = \delta$,
the sum $\sum_{m(i)=\delta} \mathcal{C}_i\mathcal{J}_{\alpha(i)}G_i$ is a $GL_n(\mathbb{K})-$linear combination of $S_r-$polynomial matrices $S_r(\mathcal{J}_{\alpha(j)}G_j, \mathcal{J}_{\alpha(k)}G_k)$ :

$$\sum_{m(i)=\delta} \mathcal{C}_i\mathcal{J}_{\alpha(i)}G_i = \sum_{j,k} \mathcal{C}_{jk}S_r(\mathcal{J}_{\alpha(j)}G_j, \mathcal{J}_{\alpha(k)}G_k)$$

for $\mathcal{C}_{jk} \in GL_n(\mathbb{K})$, and

$$\begin{aligned} S(\mathcal{J}_{\alpha(j)}G_j, \mathcal{J}_{\alpha(k)}G_k) &= \mathcal{J}_{\delta-\alpha(j)}[LC(G_j)]^{-1}\mathcal{J}_{\alpha(j)}G_j - \mathcal{J}_{\delta-\alpha(k)}[LC(G_k)]^{-1}\mathcal{J}_{\alpha(k)}G_k \\ &= \mathcal{J}_{\delta}[LC(G_j)]^{-1}G_j - \mathcal{J}_{\delta}[LC(G_k)]^{-1}G_k \\ &= \mathcal{J}_{\delta-\gamma_{jk}}S_r(G_j, G_k) \end{aligned}$$

such that $\gamma_{jk}$ is the $multideg$ of the least common multiple of $LM(G_j)$ and $LM(G_k)$. Therefore,

$$\sum_{m(i)=\delta} LT(A_i)G_i = \sum_{j,k} \mathcal{C}_{jk}\mathcal{J}_{\delta-\gamma_{jk}}S_r(G_j, G_k). \tag{4.6}$$

Or, for all $j, k \in \{1, ..., t\}$ we have

$$S_r(G_j, G_k) \xrightarrow{\mathcal{G}} 0.$$

Thus, $S_r(G_j, G_k) = \sum_{i=1}^{t} D_{ijk} G_i$ such that $D_{ijk} \in GL_n(\mathbb{K})[X]$ satisfy for all $i, j, k$ :

$$multideg(D_{ijk} G_i) \leq multideg(S_r(G_j, G_k)).$$

Hence,

$$\begin{aligned} \sum_{m(i)=\delta} LT(A_i) G_i &= \sum_{j,k} \mathcal{C}_{jk} \mathcal{J}_{\delta-\gamma_{jk}} \left( \sum_{i=1}^{t} D_{ijk} G_i \right) \\ &= \sum_{i=1}^{t} \widetilde{A}_i G_i \end{aligned}$$

where $\widetilde{A_i}$ are multivariate polynomial matrices satisfy $multideg(\widetilde{A_i} G_i) < \delta$, also $multideg \left( \sum_{m(i)=\delta} LT(A_i) G_i \right) < \delta$. So, all the terms in (4.5) have a $multideg$ strictly less than $\delta$ which contradicts the fact that $\delta$ is minimal. Hence, $multideg(A) = \delta$ .

□

### 4.4. Buchberger's Algorithm

**Input:** A generating set $\mathbf{A} = \{A_1, ..., A_s\} \subseteq GL_n(\mathbb{K})[X] \setminus \{0\}$ for a right ideal $I$ and a monomial order $\prec$.
**Output:** A Grobner basis for $I$ with respect to $\prec$.

- $\mathcal{G} = \mathbf{A}$
- While $\exists A_i, A_j \in \mathcal{G}$ such that $S_r(A_i, A_j)$ does not reduce to zero modulo $\mathcal{G}$.
  - Let $R_r$ be a right remainder produced by the right division algorithm run on $S_r(A_i, A_j)$ and $\mathcal{G}$
  - Let $\mathcal{G} := \mathcal{G} \cup \{R_r\}$.

*Proof.* To guarantee that $S_r(A_i, A_j)$ reduces to zero modulo $\mathcal{G}$ we can use the right division Algorithm. (A technical remark: If the remainder is non-zero then it is not clear that $Sr(A_i, A_j)$ does not reduce to zero modulo $\mathcal{G}$. However, it is clear that $\mathcal{G}$ is not yet a Grobner basis and it is safe to add the remainder to $\mathcal{G}$, ensuring that $S_(A_i, A_j)$ now reduces to zero.) If the algorithm terminates, then by Theorem 4.2 the set $\mathcal{G}$ is a Grobner basis for $\langle \mathcal{G} \rangle$. Furthermore $\langle \mathcal{G} \rangle = I$ since we only add elements of $I$ to $\mathcal{G}$. To show that the algorithm terminates we observe that in every step the monomial ideal $\langle LM(A_i) : A_i \in \mathcal{G} \rangle$ keeps getting strictly larger because $LM(R_r)$ is produced from the right division algorithm with the property that no monomial matrix in $A_i$ divides it. By lemma 4.2 this cannot go on forever. □

## 5. Conclusions

In this article, we try to make a link between matrices whose elements are multivariate polynomials and one of the most powerful tools in the resolution of polynomial systems, it is Gröbner's Bases. Firstly, we introduce an algorithm to calculate the quotient and the remainder produced by running a right or left division of multivariate polynomial matrix by a finite set of matrices of the same type. Nevertheless, this algorithm present some problems linked essentially to the dependence of remainder on how we order these polynomial matrices, the remainder is not unique. This situation leads us to reproduce the Buchberger's technique developed in his PhD thesis by defining an algorithm of Gröbner basis in the context of polynomial matrices.

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The author(s) declares that there is no competing financial interests or personal relationships that influence the work in this paper.

## Authorship Contribution Statement

**Kadda Noufa:** Conceptualization, Methodology, Validation, Formal Analysis, Writing Original Draft.
**Fatima Boudaoud:** Methodology, Supervision, Reviewing and Editing.

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