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Homotopy perturbation technique to solve nonlinear systems of Volterra integral equations of 1st kind

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ABSTRACT Integral equations are topics of major interest and can found in a wide range of engineering and industrial applications. The analytical solutions of the integral equations is restricted to few range of applications, but in a general most authors tend to approximate or numerical methods due to the advances in the numerical methods and techniques. In the present paper, the He 'perturbation method will be modified to solve Volterra integral equations(VIE). In the present paper, He's homotopy perturbation(HPM) with a proposed technique was developed to" solve system of" Volterra integral equations of 1st type". Three different test problems were solved using the proposed technique and their results gave the impression that it is efficient for dealing with the Volterra integral equations.

KEYWORDS: Integral equations, approximate techniques, homotopy perturbation method.

1. INTRODUCTION

Integral equations are topics of major interest and can found in a wide range of engineering and industrial applications. The analytical solutions of the integral equations is restricted to few range of applications, but in a general most authors tend to approximate or numerical methods due to the advances in the numerical methods and techniques. He in 1999 introduced the HMP. His method was applied by both researchers in the science and engineering fields for solving a wide range of linear and nonlinear problems. Between 2000 and 2004, the He's method had been modified to resolve different kinde of integral equations with miscellaneous Differences

In the present paper,' the He's perturbation technique will be modified to solveVIE. In the current paper, the modified version of the perturbed homotopy method was developed to resolve the system of VIE of 1st kind having the following form:

$$\int_0^x \zeta_i(x,t)k_i(y_1(x), y_2(x), \dots\dots, y_n(x))dt = f_i(x) \tag{1}$$

In equation (1), $f_i(x)$are prior recognized functions, $\zeta_i(x,t)$are called Kernels, and $k_i(y_1(x), y_2(x),\dots, y_n(x))$.

Exact and The sacrificial solution of integral equ. is of great Relevance because it has wide significance implementation in scientific research." Many researchers" [1,10,32,33] have solved various types of integral equations by several technique. By making use of operational matrix with block-pulse functions [11-14], this type of equations had been solved. The system of VIE of the 1st kind had been solved by AD technique [15,30,34], also by making some modification on homotopy perturbation method [16,31] solved nonlinear integral equations. In 2010, Masouri introduced' numerical solution of VIE. of the 1st kind by introducing an expansion-iterative technique". In the present paper, He's homotopy perturbation with a proposed technique was developed to solve system of VIE. of 1st type. Three different test problems were solved using the proposed technique and their results gave the impression that it is efficient for dealing with the VIE." treat An operator that is integral or differential such that ":

$$\ell(\mathrm{u}) = 0 \quad (2)$$

accordingly to homotopy technique, we can construct a homotopy $v(\mathrm{r};\rho): \Omega \times [0,1] \to \Re$ and the domain Ω satisfies:

$$\mathrm{H}(\mathrm{r};\rho) = (1-\rho)F(v) + p\ell(v) = 0 \quad (3)$$

In equation (3), ρ is called embedding parameter.

The term $F(\mathrm{v})$ is known as 'functional operator with known solution u_0,' which can be gained easily. obviously, we have:

$$\mathrm{H}(\mathrm{v};0) = F(\mathrm{v}) \quad (4)$$

And

$$\mathrm{H}(\mathrm{v};1) = \ell(\mathrm{v}) \quad (5)$$

We show that this process of changing the include 'parameter from zero to unit is just a process of mutable from solution'. This is known as deformation and also in topology and is called symmetry. Therefore, we may postulate that the solution to eq. (4) and (5) can be 'expressed as': $\mathrm{v} = \mathrm{v}_0 + \rho \mathrm{v}_1 + \rho^2 \mathrm{v}_2 + \ldots..$ (6)

Putting $\rho = 1$, we get the approximate solution of equation (1) as follows:

$$\mathrm{u} = \lim_{n\to 1} v = v_0 + v_1 + v_2 + \cdots \quad (7)$$

"Let us consider the system of VIE of first kind. In the new proposed approach, we split $f_i(\mathrm{x})$ to infinite sums as follows":

$$f_i(\mathrm{x}) = \sum_{j=0}^{\infty} K_{ij}(x), i = 1,2,3,\ldots.,n \quad (8)$$

Defining;

$$\varphi_i(\mathrm{x};\rho) = \sum^{\infty} K_{ij}(x)\rho^j, i = 1,2,3,\ldots.,n \quad (9)$$

Where

$\rho \in [0,1]$ is an embedding parameter, now at $\rho = 0$, $\varphi_i(\mathrm{x};0) = K_{i0}$, and at $\rho = 1$ $\varphi_i(\mathrm{x};1) = f_i(x)$. Let us now construct the homotopy' $Y_i(\mathrm{x};\rho): \Re \times [0,1] \to \Re$ ', which satisfies the following eq.

$$Y_i(\mathrm{x};\rho) = \varphi_i(\mathrm{x};\rho) + \rho Y_i(\mathrm{x};\rho) - \rho \int_0^N c_i(x,t) g_i(Y_i(\mathrm{x};\rho)) dt i = 1,2,3,\ldots.,n \quad (10)$$

Where

$$Y_i(\mathrm{x};\rho) = Y_1(\mathrm{x};\rho), Y_2(\mathrm{x};\rho), Y_3(\mathrm{x};\rho), \ldots\ldots, Y_n(\mathrm{x};\rho) \quad (11)$$

$$Y_i(\mathrm{x};\rho) = Y_{i0} + \rho Y_i + \rho^2 Y_{i2} + \cdots.$$

exchange Eq. (10) into Eq. (9), 'and equating the terms with equal powers of', we can obtain a series of linear eq. and solving them, we can get the approximate solutions. next,

"The proposed approach has been applied to obtain accurate solutions to some linear and nonlinear systems of the first-type integrated Volterra equations"

Definition 1-1:

Any functional equation in which the unknown function appears under the sign of integration is called" **integral equation"**.

The general form of non-linear integral equations may be written as follows:

$$\vartheta(\chi)\mu(\chi) = \zeta(\chi) + \lambda \int_a^{b(\chi)} \kappa\big(\chi, \tau, \mu(\tau)\big)d\tau \qquad \chi\epsilon\mathrm{I} = [a, b]$$

where the forcing function $\zeta(\chi)$ and the kernel function $\kappa\big(\chi, \tau, \mu(\tau)\big)$ are prescribed while $\mu(\chi)$ is the unknown function to be determined. The parameter λ is often omitted; it is, however, of importance in certain theoretical investigations (e.g. stability) and in the eigenvalue problem.

Definition 1-2:

The IE (1-1) is called **LIE** if the kernel $\big(\chi, \tau, \mu(\tau)\big) = \kappa(\chi, \tau)\mu(\tau)$, i.e.

$$\vartheta(\chi)\mu(\chi) = \zeta(\chi) + \lambda \int_a^{b(\chi)} \kappa(\chi, \tau)\mu(\tau)d\tau \qquad \chi\epsilon\mathrm{I} = [a, b]$$

otherwise it is called **non-linear.**

Definition 1-3:

The IE (1-1) is said to be an equation of the **first kind,** if $\vartheta(\chi) = 0$,i.e

$$\zeta(\chi) = \lambda \int_a^{b(\chi)} \kappa\big(\chi, \tau, \mu(\tau)\big)d\tau$$

Definition 1-4:

The IE (1-1) is called **non-linear Volterra integral equation(NLVIE)** if $b(\chi) = \chi$, i.e

$$\mu(\chi) = \zeta(\chi) + \lambda \int_a^{\chi} \kappa\big(\chi, \tau, \mu(\tau)\big)d\tau$$

Example

'Consider the system of linear VIE of 1st kind':

$$\int_0^x \big((1 - x^2 + t^2)y_i(\mathrm{t}) - (2x - t)y_2(\mathrm{t})\big)dt = -\frac{1}{3}x^3 - \frac{2}{15}x^5 \qquad (12)$$

$$\int_0^x \big((x + t^2)y_i(\mathrm{t}) - (2 + x - t)y_2(\mathrm{t})\big)dt = -x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 + \frac{1}{5} \qquad (13)$$

Having

$$y_1(\mathrm{t}) = x^2, y_2(\mathrm{t}) = x$$

as the exact solutions.

In eq.(12) and (13):

$$f_1(x) = -\frac{1}{3}x^3 - \frac{2}{15}x^5$$

$$f_2(x) = -x^2 - \frac{1}{6}x^3 + \frac{1}{2}x^4 + \frac{1}{5}x^5$$

Now splitting $f_1(x)$ as:

$$f_1(x) = -\frac{1}{3}x^3 - \frac{2}{15}x^5 = \sum_{j=0}^{\infty} K_{1j}(x)$$

With

$K_{10}(x) = x^2$

$$K_{11}(x) = -x^2 - \frac{1}{3}x^3 - \frac{2}{15}x^5$$

$$K_{21}(x) = -x - x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 + \frac{1}{5}x^5$$

$$K_{2j}(x) = 0,\ j > 1$$

Now construct the homotopy" $Y_1(\mathrm{x};\rho): \Re \times [0,1] \to \Re$ & $Y_2(\mathrm{x};\rho): \Re \times [0,1] \to \Re$ ", which satisfies the adjective equation:

$$Y_1(\mathrm{x};\rho) = \varphi_1(\mathrm{x};\rho) + \rho Y_1(\mathrm{x};\rho) - \rho \int_0^x \left((1 - x^2 + t^2)y_i(\mathrm{t}) - (2x - t)y_2(\mathrm{t})\right)dt \quad (13)$$

$$Y_2(\mathrm{x};\rho) = \varphi_2(\mathrm{x};\rho) + \rho Y_2(\mathrm{x};\rho) - \rho \int \left((x + t^2)y_i(\mathrm{t}) - (2 + x - t)y_2(\mathrm{t})\right)dt \quad (14)$$

(15)

exchange "Eq. (11) into Eq. (14) and (15), and then collecting terms of same power of p", we get:

$$\rho^0: \begin{cases} Y_{10} = K_{10} \\ Y_{20} = K_{20} \end{cases} \quad (16)$$

$$\rho^1: \begin{cases} Y_{11} = K_{11} + Y_{10} - \int_0^x \left((1 - x^2 + t^2)y_i(\mathrm{t}) - (2x - t)y_2(\mathrm{t})\right)dt \\ Y_{21} = K_{21} + Y_{20} - \int_0^x \left((x + t^2)y_i(\mathrm{t}) - (2 + x - t)y_2(\mathrm{t})\right)dt \end{cases} \quad (17)$$

Knowing that

$$Y_{10} = x^2 \quad Y_{20} = x \quad (18)$$

$$Y_{11} = Y_{12} \dots\dots = 0; Y_{21} = Y_{22} \dots\dots = 0 \quad (19)$$

And

$$y_1(x) = x^2\ y_2(x) = x \quad (20)$$

Example

Let us consider the integral 1st kind of NOVIE system ':

$$\int_0^x \left(y_i(\mathrm{t}) + (x - t)y_2(\mathrm{t})\right)dt = -\frac{3}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 + e^x - \frac{1}{4}e^{2x} \quad (21)$$

$$\int_0^x \left(y_2(\mathrm{t}) + (x - t)y_1(\mathrm{t})y_2(\mathrm{t})\right)dt = \frac{5}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x} \quad (22)$$

In these two equations; $y_i = x + e^x\ and y_i = x - e^x$ represent the exact solutions.

In the present example:

$$f_1(x) = -\frac{3}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 + e^x - \frac{1}{4}e^{2x}$$

$$f_2(x) = \frac{5}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x}$$

As had been occurred in the previous example, let us start by splitting functions and let us start by splitting $f_1(x)$ as follows:

$$f_1(x) = -\frac{3}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 + e^x - \frac{1}{4}e^{2x} = \sum_{j=0}^{\infty} K_{1j}(x)$$

with

$K_{10} = x + e^x$

$K_{11} = -\frac{3}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 + e^x - \frac{1}{4}e^{2x}$

$K_{1j} = 0, j > 1$

Similarly|

$$f_2(x) = \frac{5}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x} = \sum_{j=0}^{\infty} K_{2j}(x)$$

With

$K_{10} = x - e^x$

$K_{21} = \frac{5}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x}$

$K_{2j} = 0, j > 1$

Now construct the homotopy $Y_1(\mathrm{x};\rho): \Re \times [0,1] \to \Re \ \& \ Y_2(\mathrm{x};\rho): \Re \times [0,1] \to \Re$, which satisfies the adjective eq.:

$$Y_1(\mathrm{x};\rho) = \varphi_1(\mathrm{x};\rho) + \rho Y_1(\mathrm{x};\rho) - \rho \int_0^x \big(y_i(\mathrm{t};\rho) + (x-t)y_i(\mathrm{t};\rho)y_2(\mathrm{t};\rho)\big)dt \tag{22}$$

$$Y_2(\mathrm{x};\rho) = \varphi_2(\mathrm{x};\rho) + \rho Y_2(\mathrm{x};\rho) - \rho \int_0^x \big(y_2(\mathrm{t};\rho) + (x-t)y_1(\mathrm{t};\rho)y_2(\mathrm{t};\rho)\big)dt \tag{23}$$

Similarly|

$$f_2(x) = \tfrac{5}{4} + \tfrac{x}{2} + \tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 - e^x - \tfrac{1}{4}e^{2x} = \textstyle\sum_{j=0}^{\infty} K_{2j}(x) \tag{24}$$

With

$K_{10} = x - e^x$

$K_{21} = \frac{5}{4} + \frac{x}{2} + \frac{1}{2}x^2 + \frac{1}{12}x^4 - e^x - \frac{1}{4}e^{2x}$

$K_{2j} = 0, j > 1$

'exchange Eq. (11) into Eq. (23) and (24)', and then collecting terms of same power of p, we

$$\rho^0: \begin{cases} Y_{10} = K_{10} \\ Y_{20} = K_{20} \end{cases} \tag{25}$$

$$\rho^1: \begin{cases} Y_{11} = K_{11} + Y_{10} - \int_0^x \big(Y_{10}(\mathrm{t}) + (x-t)Y_{10}(\mathrm{t})Y_{20}(\mathrm{t})\big)dt \\ Y_{21} = K_{21} + Y_{20} - \int^{\mathrm{x}} \quad \big(Y_{20}(\mathrm{t}) + (x-t)Y_{10}(\mathrm{t})Y_{20}(\mathrm{t})\big)dt \end{cases} \tag{26}$$

(

$Y_{10} = x + e^x$

$Y_{20} = x - e^x$ (27)

$Y_{11} = Y_{12} \dots \dots = 0; Y_{21} = Y_{22} \dots \dots = 0$ (28)

And

$$\begin{aligned} y_1(x) &= x + e^x \\ y_2(x) &= x - e^x \end{aligned} \tag{29}$$

Example

Let us consider the system of nonlinear VIE having $y_i(\mathrm{x}) = x^2$ and $y_2(\mathrm{x}) = x$ as the exact solutions

$$\int_0^x (1 - x^2 + t^2)(y_i(t) + y_2^3(t))dt = -\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 \tag{30}$$

$$\int_0^x (5 + x - t)(y_1^3(t) - y_2(t))dt = \frac{1}{56}x^8 + \frac{5}{7}x^7 - \frac{1}{6}x^3 - \frac{5}{2}x^2 \tag{31}$$

In the present example:

$$f_1(x) = -\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 \tag{32}$$

$$f_2(x) = \frac{1}{56}x^8 + \frac{5}{7}x^7 - \frac{1}{6}x^3 - \frac{5}{2}x^2 \tag{33}$$

As had been occurred in the previous example, let us start by splitting functions and let us start by splitting $f_1(x)$ as follows:

$$f_1(x) = -\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 = \sum_{j=0}^{\infty} K_{1j}(x) \tag{34}$$

With

$$\begin{aligned} K_{10} &= x^2 \\ K_{11} &= -\frac{1}{12}x^6 - \frac{2}{15}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^3 \\ K_{1j} &= 0, j > 1 \end{aligned} \tag{35}$$

Similarly

$$f_2(x) = \frac{1}{56}x^8 + \frac{5}{7}x^7 - \frac{1}{6}x^3 - \frac{5}{2}x^2 = \sum_{j=0}^{\infty} K_{2j}(x) \tag{36}$$

With

$$\begin{aligned} K_{10} &= x \\ K_{21} &= \frac{1}{56}x^8 + \frac{5}{7}x^7 - \frac{1}{6}x^3 - \frac{5}{2}x^2 - x \\ K_{2j} &= 0, j > 1 \end{aligned} \tag{37}$$

Now construct the homotopy

$Y_1(\mathrm{x};\rho): \Re \times [0,1] \to \Re \& Y_2(\mathrm{x};\rho): \Re \times [0,1] \to \Re$, which satisfies the

$$Y_1(\mathrm{x};\rho) = \varphi_1(\mathrm{x};\rho) + \rho Y_1(\mathrm{x};\rho) - \rho \int_0^x (1 - x^2 + t^2)\big(y_i(\mathrm{t};\rho) + y_2^3(\mathrm{t};\rho)\big)dt \tag{37}$$

$$Y_2(\mathrm{x};\rho) = \varphi_2(\mathrm{x};\rho) + \rho Y_2(\mathrm{x};\rho) - \rho \int (5 + x - t)\big(y_1^3(\mathrm{t};\rho) - y_2(\mathrm{t};\rho)\big)dt \tag{38}$$

Substituting Equation (11) into Equations (38) and (39), and then collecting terms of same power of p, we get:

$$\rho^0:\begin{cases} Y_{10} = K_{10} & (39) \\ Y_{20} = K_{20} & (40) \end{cases}$$

$$\rho^1:\begin{cases} Y_{11} = K_{11} + Y_{10} - \int_0^x (1 - x^2 + t^2)\big(Y_{10}(\mathrm{t}) + Y_{20}^3(\mathrm{t})\big)dt & (41) \\ Y_{21} = K_{21} + Y_{20} - \int_0^x (5 + x - t)\big(Y_{10}^3(\mathrm{t}) - Y_{20}(\mathrm{t})\big)dt & (42) \end{cases}$$

With

$$Y_{10} = x^2$$
$$Y_{20} = x$$
$$Y_{11} = Y_{12} \ldots \ldots = 0; Y_{21} = Y_{22} \ldots . . = 0 \qquad (43)$$

And

$$y_1(x) = x^2$$
$$y_2(x) = x \qquad (44)$$

2. CONCLUSIONS

Drawing on HPM, an analytical approach has been developed to solve the Volterra Type I integrated system of equations." When evaluating the examples", we noted that the proposed process is unpretentious in computation and highly effective in both linear and nonlinear situations. In addition, in most cases, 'it gives accurate solutions at a first rough estimation.

REFERENCES

[1] Armand A, Gouyandeh Z (2014). Numerical solution of the system of Volterra integral equations of the first kind. Int. J. Ind. Math. 6:1.

[2] Babolian E, Masouri Z (2008). Direct method to solve Volterra integral equation of the first kind using operational matrix with block-pulse functions. J. Comput. Appl. Math. 220:51-57.

[3] Biazar J, Babolian E, Islam R (2003). Solution of a system of Volterra integral equations of the first kind by Adomian method. Appl. Math. Comput. 139:249-258.

[4] Biazar J, Eslami M (2010a). Analytic solution for Telegraph equation by differential transforms method. Physics Letters A 374:2904-2906

[5] Biazar J, Eslami M (2010b). Differential transform method for quadratic Riccati differential equation. Intl. J. Nonlinear Sci. 9(4):444-447.

[6] Biazar J, Eslami M (2010c). Differential transform method for systems of Volterra integral equations of the first kind. Nonl. Sci. Lett. A, 1(2):173-181.

[7] Biazar J, Eslami M (2011). Differential transform method for nonlinear fractional gas dynamics equation. Intl. J. Physical Sci. 6(5):12031206.

[8] Biazar J, Eslami M, Aminikhah H (2009). Application of homotopy perturbation method for systems of Volterra integral equations of the first kind. Chaos, Solitons and Fractals 42:3020-3026.

[9] Biazar J, Eslami M, Ghazvini H (2008). Exact solutions for systems of Volterra integral equations of the first kind by homotopy perturbation method. Appl. Math. Sci. 2(54):2691-2697.

[10] Biazar J, Eslami M, Islam MR (2012). Differential transform method for special systems of integral equations. J. King Saud UniversityScience 24(3):211-24.

[11] Biazar J, Ghazvini H, Eslami M (2009). He's homotopy perturbation method for systems of integro-differential equations. Chaos, Solitons Fractals 39:1253-1258.

[12] Biazar J, Mostafa E (2011a). Differential transform method for systems of Volterra integral equations of the second kind and comparison with homotopy perturbation method. Int. J. Phy. Sci. 6(5):1207-1212.

[13] Biazar J, Mostafa E (2011b). A new homotopy perturbation method for solving systems of partial differential equations. Comput. Math. Appl. 62:225-234.

[14] Eslami M (2014a). New homotopy perturbation method for a special kind of Volterra integral equations in two-dimensional space. Comput. Math. Modeling 25(1).

[15] Eslami M (2014b). An efficient method for solving fractional partial differential equations. Thai J. Math. 12(3):601-611.

[16] Eslami M, Mirzazadeh M (2014). Study of convergence of Homotopy perturbation method for two-dimensional linear Volterra integral equations of the first kind. Int. J. Comput. Sci. Math. 5(1):72-80.

[17] Ghorbani A, Saberi-Nadjafi J (2008). Exact solutions for nonlinear integral equations by a modified homotopy perturbation method. Comput. Math. Appl. 56(4):1032-1039.

[18] Golbabai A, Keramati B (2008). Modified homotopy perturbation method for solving Fredholm integral equations, Chaos. Solitons and Fractals 37:1528-1537.

[19] He JH (1999). Comparison of homotopy perturbation method and homotopy analysis method. Appl. Math. Comput. 156(2):527-539.

[20] He JH (2000). A coupling method of homotopy technique and perturbation technique for nonlinear problems. Intl. J. Non-Linear Mech. 35(1):37-43. He JH (2003). A simple perturbation approach to Blasius equation. Appl. Math. Comput. 140:217-222.

[21] Maleknejad K, Najafi E (2001). Numerical solution of nonlinear volterra integral equations using the idea of quasilinearization, Commun Nonlinear Sci Numer Simulat 16:93-100.

[22] Maleknejad K, Shahrezaee M (2004). Using Runge–Kutta method for numerical solution of the system of Volterra integral equation. Appl. Math. Comput. 149:399-410.

[23] Maleknejad K, Sohrabi S, Rostami Y (2007). Numerical solution of nonlinear Volterra integral equations of the second kind by using Chebyshev polynomials. Appl. Math. Comput. 188:123-128.

[24] Masouri Z, Babolian E, Hatamzadeh-Varmazyar S (2010). An expansion–iterative method for numerically solving Volterra integral equation of the first kind. Comput. Math. Appl. 59:1491-1499.

[25] Ngarasta N, Rodoumta K, Sosso H (2009). The decomposition method applied to systems of linear Volterra integral equations of the first kind. Kybernetes. 38(3/4):606-614.

[26] Odibat ZM (2008). Differential transform method for solving Volterra integral equation with separable kernels. Math. Comput. Modelling 48:1144-1149.

[27] Rabbani M, Maleknejad K, Aghazadeh N (2007). Numerical computational solution of the Volterra integral equations system of the second kind by using an expansion method. Appl. Math. Comput. 187:1143-1146

[28] Saeedi L, Tari A, Masuleh SH (2013). Numerical solution of some nonlinear Volterra integral equations of the first kind. Applicat. Appl. Math. 8(1).

[29] Tahmasbi A, Fard OS (2008). Numerical solution of linear Volterra integral equations system of the second kind. Appl. Math. Comput. 201:547-552.

[30] K. K. Abbo, Y. A. Laylani, and H. M. Khudhur, "Proposed new Scaled conjugate gradient algorithm for Unconstrained Optimization," *Int. J. Enhanc. Res. Sci. Technol. Eng.*, vol. 5, no. 7, 2016.

[31] K. K. Abbo and H. M. Khudhur, "New A hybrid Hestenes-Stiefel and Dai-Yuan conjugate gradient algorithms for unconstrained optimization," *Tikrit J. Pure Sci.*, vol. 21, no. 1, pp. 118–123, 2015.

[32] K. K. ABBO, Y. A. Laylani, and H. M. Khudhur, "A NEW SPECTRAL CONJUGATE GRADIENT ALGORITHM FOR UNCONSTRAINED OPTIMIZATION," *Int. J. Math. Comput. Appl. Res.*, vol. 8, pp. 1–9, 2018, [Online]. Available: www.tjprc.org.

[33] K. K. Abbo and H. M. Khudhur, "New A hybrid conjugate gradient Fletcher-Reeves and Polak-Ribiere algorithm for unconstrained optimization," *Tikrit J. Pure Sci.*, vol. 21, no. 1, pp. 124–129, 2015.

[34] H. N. Jabbar, K. K. Abbo, and H. M. Khudhur, "Four--Term Conjugate Gradient (CG) Method Based on Pure Conjugacy Condition for Unconstrained Optimization," kirkuk Univ. J. Sci. Stud., vol. 13, no. 2, pp. 101–113, 2018.