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Research Paper

# On Generalized Mannheim curves in the Equiform Geometry of the Galilean 4-Space

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**Abstract:** In this paper on curve, we give a definition of generalized Mannheim curves that we will depict over the Equiform differential geometry of Galilean 4-space. We show some characterizations of generalized Mannheim curves. A new characterization has been procured among curvatures by employing the Mannheim curve overview

**Key words:** Manheimm curve, Galilean space, Equiform geometry

## 4-boyutlu Galileo Uzayının Equform Geometrisindeki Genelleştirilmiş Mannheim Eğrileri

**Öz:** Eğri hakkında ki bu makale de, Galileo 4-uzayının Equform diferansiyel geometrisi üzerinde tanımlayacağımız genelleştirilmiş Mannheim eğrinin bir tanımını veriyoruz. Genelleştirilmiş Mannheim eğrilerin bazı karekterizasyonlarını gösteriyoruz. Mannheim eğrilerine genel bir bakış kullanılarak eğrilikler arasında yeni bir karekterizasyon sağlandı.

**Anahtar kelimeler:** Mannheim eğri, Galileo uzayı, Equform geometri

## 1. Introduction

Although many issues are taken into consideration in differential geometry, curves have an important place for this geometry. Curves have been studied in several spaces and continue to be studied. The topic of curves has been studied in many areas and is still being studied. The characterizations of curves is grand circumstance for curves in curves geometry. The articles concerning Equiform geometry are few. Many scientists studied, joining different perspectives on the curves in Mannheim. Mannheim presented in 1878 for the first time that space curves, whose principal normal are binormal of another curve, are called Mannheim curves [1]. Using the relationships between curvature and torsion of curves, some custom curves can be called. Some researchers attained Mannheim curves by identifying conditions between curvature of the curves [2]. Onder at al. have presented new characterizatons for these curves by examining the Mannheim curves for the spacelike and and timelike conditions [3,4]. Novel characterizations of Mannheim partner curves are granted in Minkowski 3-space by Kahraman and at all [5]. Matsuda and Yorozu had given some characterizations and examples of generalized Mannheim curve [6]. Orbay and Kasap gave [7] novel characterizations of Mannheim partner curves in Euclidean 3-space. Indeed, Mannheim curves for discrepant spaces appear to be able to be depicted Mannheim curves, so new qualifications for these curves have been offered to the literature [8-15]. Mannheim curve have an important place in place in differential geometry and new studies are still being added to the literature on Mannheim curves [16-21]. The Mannheim-B curve is among the new topics presented to the literature [22]. Differential geometry has a large are, and curves are an open area in differential geometry.

In this paper, we introduce a definition of generalized Mannheim curves in the Equiform differential geometry of Galilean 4-space. In this space, new characterizations for the generalized Mannheim curves are obtained. We also give properties of Mannheim curves for the space we work.

## 2. Preliminaries

Here, we have given some basic information about curves in 4D Galilean space, necessary for our study [23]. The Galilean scalar product between any two points is expressed in Affine coordinates as follows.

$$\mathbb{P}_{\mathrm{i}} = (\mathbb{P}_{\mathrm{i}1}, \mathbb{P}_{\mathrm{i}2}, \mathbb{P}_{\mathrm{i}3}, \mathbb{P}_{\mathrm{i}4}),\ \mathrm{i} = 1,2, \tag{2.1}$$

is designated by

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$$\mathrm{l}(\mathbb{P}_1,\mathbb{P}_2)=\begin{cases}|\mathbb{P}_{21}-\mathbb{P}_{11}|, & \text{if } \mathbb{P}_{21}\neq\mathbb{P}_{11},\\ (|(\mathbb{P}_{22}-\mathbb{P}_{12})^2+(\mathbb{P}_{23}-\mathbb{P}_{13})^2+(\mathbb{P}_{24}-\mathbb{P}_{14})^2|)^{1/2}, & \text{if } \mathbb{P}_{21}=\mathbb{P}_{11}.\end{cases} \tag{2.2}$$

The Galilean cross product in G4 for the vectors $\vec{\mathbb{p}}=(\mathbb{p}_1,\mathbb{p}_2,\mathbb{p}_3,\mathbb{p}_4)$, $\vec{\mathbb{r}}=(\mathbb{r}_1,\mathbb{r}_2,\mathbb{r}_3,\mathbb{r}_4)$, $\vec{\mathbb{q}}=(\mathbb{q}_1,\mathbb{q}_2,\mathbb{q}_3,\mathbb{q}_4)$, is depicted by

$$\vec{\mathbb{p}}\Lambda\vec{\mathbb{r}}\Lambda\vec{\mathbb{q}}=-\begin{vmatrix}0 & \mathrm{e}_2 & \mathrm{e}_3 & \mathrm{e}_4\\ \mathbb{p}_1 & \mathbb{p}_2 & \mathbb{p}_3 & \mathbb{p}_4\\ \mathbb{r}_1 & \mathbb{r}_2 & \mathbb{r}_3 & \mathbb{r}_4\\ \mathbb{q}_1 & \mathbb{q}_2 & \mathbb{q}_3 & \mathbb{q}_4\end{vmatrix}, \tag{2.3}$$

where $\mathrm{e_m}, 1\leq \mathrm{m}\leq 4,$ are the standard basis vectors.

The scalar product of any two vectors $\vec{a}=(a_1,a_2,a_3,a_4)$ and $\vec{b}=(b_1,b_2,b_3,b_4)$ in G4 is designated by

$$\langle\vec{a},\vec{b}\rangle_{G_4}=\begin{cases}a_1b_1, & if\ a_1\neq 0\ or\ b_1\neq 0,\\ a_1b_1+a_2b_2+a_3b_3, & if\ a_1=0\ or\ b_1=0.\end{cases} \tag{2.4}$$

The norm of vector $\vec{u}$ is depicted by

$$\|\vec{u}\|_{G_4}=\sqrt{|\langle\vec{u},\vec{u}\rangle_{G_4}|}\,. \tag{2.5}$$

Let $\alpha: I\subset\mathcal{R}\longrightarrow$ G4, $\alpha(s)=(s,y(s),z(s),w(s))$ be a curve parametrized by arclength $s$. The $t(s)$ tangent vector of α, is depicted by

$$t(s)=\alpha'(s)=(1,y'(s),z'(s),w'(s)). \tag{2.6}$$

Since $t(s)$ is a unit vector, we can phrase

$$\langle\vec{t},\vec{t}\rangle_{G_4}=1 \tag{2.7}$$

Differentiating (2.7) in respect of $s$, we have

$$\langle t',t\rangle_{G_4}=0. \tag{2.8}$$

The $k_1$ real-valued function with the help of the derivative of the tangent vector function,

$$k_1(s)=\|t'(s)\|=\sqrt{(y''(s))^2+(z''(s))^2+(w''(s))^2} \tag{2.9}$$

is stated the as first curvature of the curve α. We assume that $k_1(s)\neq 0$, for all $s\in I$. The $n(s)$ principal vector is defined by

$$n(s)=\frac{t\prime(s)}{k_1(s)}, \tag{2.10}$$

in other words

$$n(s)=\frac{1}{k_1(s)}(0,y''(s),z''(s),w''(s)). \tag{2.11}$$

definition the $k_2$ second curvature function that is depicted by

$$k_2(s)=\|n'(s)\|_{G_4}. \tag{2.12}$$

The $b(s)$ binormal vector field is

$$b(s) = \frac{1}{k_2(s)}\left(0, \left(\frac{y''(s)}{k_1(s)}\right)', \left(\frac{z''(s)}{k_1(s)}\right)', \left(\frac{w''(s)}{k_1(s)}\right)'\right). \tag{2.13}$$

The $e(s)$ fourth unit vector is defined by

$$e(s) = \mu t(s) \Lambda n(s) \Lambda b(s). \tag{2.14}$$

The $k_3$ third curvature of the curve $\alpha$ is depicted by

$$k_3(s) = \langle b', e \rangle_{G_4}. \tag{2.15}$$

Here, as well known, the set $\{t, n, b, e, k_1, k_2, k_3\}$ is Frenet apparatus of the curve $\alpha$. Thus the Frenet equations of the curve in G4 are given by [15]

$$\begin{aligned} t'(s) &= k_1(s)n(s), \\ n'(s) &= k_2(s)b(s), \\ b'(s) &= -k_2(s)n(s) + k_3(s)e(s), \\ e'(s) &= -k_3(s)b(s). \end{aligned} \tag{2.17}$$

## 3. Frenet Formulas in Equiform Geometry of Galilean 4-Space

Here, we confer it in our study in Equiform differential geometry of curves in [24].
Let's take $\alpha: I \subset \mathcal{R} \longrightarrow$ G4 as with arc-length parameter $s$. The equiform parameter of the curve $\alpha(s)$ expressed as

$$\varrho = \int \frac{ds}{\rho}, \tag{3.1}$$

where $\rho = \frac{1}{k_1}$ is radius of curvature of our curve. We can write the above equation as

$$\frac{ds}{d\varrho} = \rho. \tag{3.2}$$

assuming that $h$ is homothety with the center in the origin and the coefficient $\lambda$. Also, we take

$$\tilde{s} = \lambda s \text{ and } \tilde{\rho} = \lambda \rho, \tag{3.3}$$

where $\tilde{s}$ and $\tilde{\rho}$ are the arc-length parameter of $\tilde{\alpha}$ and the radius of curvature of this curve, respectively. So $\varrho$ is an equiform invariant parameter of $\alpha$.
The curvatures $k_1$, $k_2$, $k_3$ of the curve α are not invariants of the homothety group, because from (2.17), we can write

$$\tilde{k}_1 = \frac{1}{\lambda} k_1, \qquad \tilde{k}_2 = \frac{1}{\lambda} k_2, \qquad \tilde{k}_3 = \frac{1}{\lambda} k_3.$$

Now, if we get

$$\mathbb{t} = \frac{d\alpha}{d\varrho} \tag{3.4}$$

and using (3.2), we have

$$\mathbb{t} = \rho t. \tag{3.5}$$

Also, we define the vectors $\mathbb{m}, \mathbb{b}, \mathbb{e}$ by

$$\mathbb{m} = \rho n,\ \ \mathbb{b} = \rho b,\ \ \mathbb{e} = \rho e \tag{3.6}$$

Thus, the frenet formula for the $\mathbb{t}, \mathbb{m}, \mathbb{b}, \mathbb{e}$ vectors in respect of $\varrho$ are follows

$$\begin{aligned}
\mathbb{t}' &= \dot{\rho}\mathbb{t} + \mathbb{m},\\
\mathbb{m}' &= \dot{\rho}\mathbb{m} + \frac{k_2}{k_1}\mathbb{b}\\
\mathbb{b}' &= -\frac{k_2}{k_1}\mathbb{m} + \dot{\rho}\mathbb{b} + \frac{k_3}{k_1}\mathbb{e},\\
\mathbb{e}' &= -\frac{k_3}{k_1}\mathbb{b} + \dot{\rho}\mathbb{e}.
\end{aligned}$$

**Definition 3.1.** The function $\mathbb{k}_m : I \to R$, $m = 1,2,3$, is depicted as

$$\mathbb{k}_1 = \dot{\rho},\ \ \mathbb{k}_2 = \frac{k_2}{k_1}, \mathbb{k}_3 = \frac{k_3}{k_1} \tag{3.7}$$

is named the $m^{th}$ equiform curvature of the curve. In addition, the formula in Equiform geometry of $G_4$ with similar logic, it is expressed as follows

$$\begin{aligned}
\mathbb{t}' &= \mathbb{k}_1\mathbb{t} + \mathbb{m},\\
\mathbb{m}' &= \mathbb{k}_1\mathbb{m} + \mathbb{k}_2\mathbb{b}\\
\mathbb{b}' &= -\mathbb{k}_2\mathbb{m} + \mathbb{k}_1\mathbb{b} + \mathbb{k}_3\mathbb{e},\\
\mathbb{e}' &= -\mathbb{k}_3\mathbb{b} + \mathbb{k}_1\mathbb{e}.
\end{aligned} \tag{3.8}$$

[24].

## 4. Generalizated Mannheim curves in Equiform Geometry of Galilean 4-Space

Current porsion, we designate the generalizated Mannheim curve with respect to Equiform differential geometry of $G_4$.

**Definition 4.1.** Let's take as a special Frenet curve $C$ in Equiform differential geometry of $G_4$ . In the Equform differential geometry of $G_4$, there is a special Frenet curve $\hat{C}$ such that the first normal line at each point of $C$ is included in the plane generated by the second and third normal lines of $\hat{C}$ at corresponding point under $\emptyset$. Where $\emptyset : C \longmapsto \hat{C}$ is a bijection. As a consequence, the curve $\hat{C}$ is named the generalized Mannheim mate curve of $C$ under this circumstance.

Hereafter, a privative Frenet curve $C$ in Equiform differential geometry of $G_4$ is parametrized by parameter $\varrho$, that is, $C$ is dedicated by $x : L \ni \varrho \longmapsto x(\varrho) \in G_4$. When C is a generalized Mannheim curve in $G_4$ and , because of descript, a generalized Mannheim mate curve $\hat{C}$ is denoted as the map $\hat{x} : L \to G_4$ such that

$$\hat{x} = x(\varrho) + \alpha(\varrho)e_2(\varrho),\ \ \varrho \in L, \tag{4.1}$$

is written. Where $\alpha$ is a smooth function on $L$. We remark that the parameter $\varrho$ generally is not an arc-length parameter of $\hat{C}$. Let $\hat{\varrho}$ be the arc-length of $\hat{C}$ defined by

$$\hat{\varrho} = \int_0^s \left\| \frac{d\hat{x}(\varrho)}{d\varrho} \right\| d\varrho.$$

We can count a smooth function $\mathrm{F} : L \to \hat{L}$ given by $\mathrm{F}(\varrho) = \hat{\varrho}$. We remark that $\hat{s}$ is the parameter of $\hat{C}$, and the bijection $\emptyset : C \to \hat{C}$ is defined by $\emptyset(x(\varrho)) = \hat{x}(\mathrm{F}(\varrho))$. From the definition of the Mannheim curve, corresponding point under a bijection $\emptyset$ for each $\varrho \in L$ the vector $e_2(\varrho)$ is grant by linear combination of $\hat{e}_3(\mathrm{F}(\varrho))$ and $\hat{e}_4(\mathrm{F}(\varrho))$, that is, we can set $\boldsymbol{e_2}(\varrho) = g(\varrho)\,\hat{e}_3(\mathrm{F}(\varrho)) + h(\varrho)\,\hat{e}_4(\mathrm{F}(\varrho))$ for some smooth functions $g$ and $h$ on $L$. According to this definition, Differentiating (4.1) according to equiform invariant parameter $\varrho$ and using the equations (3.8), we have

$$\mathrm{F}'(\varrho)\hat{e}_1(\mathrm{F}(\varrho)) = e_1(\varrho) + \alpha'(\varrho)e_2(\varrho) + \alpha(\varrho)(\mathbb{k}_1(\varrho)e_2(\varrho) + \mathbb{k}_2(\varrho)e_3(\varrho))$$

$$\mathrm{F}'(\varrho)\hat{e}_1(\mathrm{F}(\varrho)) = e_1(\varrho) + (\alpha'(\varrho) + \alpha(\varrho)\mathbb{k}_1(\varrho))e_2(\varrho) + \alpha(\varrho)\mathbb{k}_2(\varrho)e_3(\varrho) \tag{4.2}$$

if we inner product both sides of this equation by $e_2(\varrho)$ and considering the following equality

$$< \hat{e}_1(\mathrm{F}(\varrho)), g(\varrho)\, \hat{e}_3(\mathrm{F}(\varrho)) + h(\varrho)\, \hat{e}_4(\mathrm{F}(\varrho)) >= 0,$$

we obtain

$$0 = (\alpha'(\varrho) + \alpha(\varrho)\mathbb{k}_1(\varrho))\rho^2, \rho \neq 0$$

from this equation we get

$$\alpha(\varrho) = e^{-\mathbb{k}_1\varrho}c \tag{4.3}$$

where $c \in R$. Thus we have

$$\mathrm{F}'(\varrho)\hat{e}_1(\mathrm{F}(\varrho)) = e_1(\varrho) + e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3(\varrho),$$

that is,

$$\hat{e}_1(\mathrm{F}(\varrho)) = \frac{e_1(\varrho)}{\mathrm{F}'(\varrho)} + \frac{e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3(\varrho)}{\mathrm{F}'(\varrho)},$$

where $\|\mathrm{F}'(\varrho)\| = \sqrt{1 + (e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho))^2}$ for $\varrho \in L$. If we take the differential of the above equality according to $\varrho$, we have

$$\begin{aligned}\mathrm{F}'(\varrho)\hat{e}_1{}'(\mathrm{F}(\varrho)) &= \left(\frac{1}{\mathrm{F}'(\varrho)}\right)' e_1(\varrho) + \frac{1}{\mathrm{F}'(\varrho)} e_1'(\varrho)\\ &+ \left(\frac{1}{\mathrm{F}'(\varrho)}\right)' e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3(\varrho)\\ &+ \frac{1}{\mathrm{F}'(\varrho)}\left(-\mathbb{k}_1 e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3(\varrho) + e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3'(\varrho)\right),\\ \mathrm{F}'(\varrho)\left(\widehat{\mathbb{k}}_1(\varrho)\hat{e}_1(\varrho) + \hat{e}_2(\varrho)\right) &= \left(\frac{1}{\mathrm{F}'(\varrho)}\right)' e_1(\varrho)\\ &+ \frac{1}{\mathrm{F}'(\varrho)}\left(\mathbb{k}_1(\varrho)e_1(\varrho) + e_2(\varrho)\right)\\ &+ \left(\frac{1}{\mathrm{F}'(\varrho)}\right)' e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3(\varrho)\\ &+ \frac{1}{\mathrm{F}'(\varrho)}\left(-\mathbb{k}_1 e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)e_3(\varrho)\right)\\ &+ \frac{1}{\mathrm{F}'(\varrho)} e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2(\varrho)\left(-\mathbb{k}_2(\varrho)e_2(\varrho) + \mathbb{k}_1(\varrho)e_3(\varrho) + \mathbb{k}_3(\varrho)e_4(\varrho)\right),\end{aligned}$$

If we inner product both sides of this equation by $e_2(\varrho)$ and using the $\boldsymbol{e_2}(\varrho) = g(\varrho)\, \hat{e}_3(\mathrm{F}(\varrho)) + h(\varrho)\, \hat{e}_4(\mathrm{F}(\varrho))$ equality

$$\frac{e^{\mathbb{k}_1\varrho}}{(\mathbb{k}_2)^2} = \text{const.}$$

In this way, we have following theorem from the explanations:

**Theorem 4.1.** Let's take a Frenet curve $C$ in equiform differential geometry of $G_4$. If the curve $\alpha$ is a generalized Mannheim curve, from here, the following equality is satisfied the relationship between the curve functions $\mathbb{k}_1$ and $\mathbb{k}_2$ of $C$.

$$\frac{e^{\mathbb{k}_1\varrho}}{(\mathbb{k}_2)^2} = \text{const.}, \varrho \in L. \tag{4.4}$$

Let $\hat{\varrho}$ be the arc-length of $\hat{C}$. Also, $\hat{\varrho}$ we know it is defined by

$$\hat{\varrho} = \int_0^s \left\| \frac{d\hat{x}(\varrho)}{d\varrho} \right\| d\varrho$$

for $\varrho \in L$. We can take into consideration a smooth function $\mathrm{F} : L \to \hat{L}$ given by $\mathrm{F}(\varrho) = \hat{\varrho}$. Considering the following equality

$$\|\mathrm{F}'(\varrho)\| = \sqrt{1 + (e^{-\Bbbk_1 \varrho} c \Bbbk_2)^2},$$

from (4.4) equality

$$\|\mathrm{F}'(\varrho)\| = \sqrt{1 + \frac{1}{(\Bbbk_2)^2}}$$

for $\varrho \in L$.

The description of $\hat{C}$ by arc-length parameter $\hat{\varrho}$ is denoted by $\hat{x}(\hat{\varrho})$, here we use the same letter "$\hat{x}$" for simplicity. Then we can simply write

$$\hat{x}(\hat{\varrho}) = \hat{x}(\mathrm{F}(\varrho)) = x(\varrho) + \alpha(\varrho) e_2(\varrho)$$

for curve $\hat{C}$. By receiving the derivative of this equation with respect to $\varrho$,

$$\frac{d\hat{x}(\hat{\varrho})}{d\varrho} = \frac{d\hat{x}(\hat{\varrho})}{d\hat{\varrho}} \mathrm{F}'(\varrho) = \mathrm{F}'(\varrho) \hat{e}_1(\mathrm{F}(\varrho))$$

and

$$\mathrm{F}'(\varrho) \hat{e}_1(\mathrm{F}(\varrho)) = e_1(\varrho) + e^{-\Bbbk_1 \varrho} c \Bbbk_2 e_3(\varrho)$$

Thus we have

$$\hat{e}_1(\mathrm{F}(\varrho)) = \frac{\mathbf{1}}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} e_1(\varrho) + \frac{e^{-\Bbbk_1 \varrho} c \Bbbk_2}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} e_3(\varrho)$$

for $\varrho \in L$. We differentiate of the above equality according to $\varrho$, then we have

$$\mathrm{F}'(\varrho) \left( \hat{\Bbbk}_1(\mathrm{F}(\hat{\varrho})) \hat{e}_1(\mathrm{F}(\varrho)) + \hat{e}_2(\mathrm{F}(\varrho)) \right) = \frac{\Bbbk_1}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} e_1(\varrho)$$

$$+ \left( \frac{1}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} - \frac{1}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} e^{-\Bbbk_1 \varrho} c \Bbbk_2^2 \right) e_2(\varrho)$$

$$+ \left( \frac{-\Bbbk_1 e^{-\Bbbk_1 \varrho} c \Bbbk_2}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} + \frac{\Bbbk_1 e^{-\Bbbk_1 \varrho} c \Bbbk_2}{\sqrt{1 + \frac{1}{(\Bbbk_2)^2}}} \right) e_3(\varrho)$$

$$+\frac{e^{-\mathbb{k}_1\varrho}c\mathbb{k}_2\mathbb{k}_3}{\sqrt{1+\frac{1}{(\mathbb{k}_2)^2}}}e_4(\varrho),$$

considering $1-e^{-\mathbb{k}_1\varrho}c\big(\mathbb{k}_2(\varrho)\big)^2=0$ equality, from the above equation, coefficient of $e_3(\varrho)$ is zero. Thus, for each $\varrho\in L$, the vector $\hat{e}_1(\mathrm{F}(\varrho))$ is granted by linear combination of $\boldsymbol{e_1}(\varrho)$ and $\boldsymbol{e_3}(\varrho)$. And, as above, the vector $\hat{e}_2(\mathrm{F}(\varrho))$ is given by linear combination of $\boldsymbol{e_1}(\varrho)$, $\boldsymbol{e_3}(\varrho)$ and $\boldsymbol{e_4}(\varrho)$.

Since the curve $\hat{C}$ is a special Frenet curve in Equiform differential geometry of $G_4$, the vector $\boldsymbol{e_2}(\varrho)$ is grant by linear combination of $\hat{e}_3(\mathrm{F}(\varrho))$ and $\hat{e}_4(\mathrm{F}(\varrho))$.

With the above description, we have following theorem:

**Theorem 4.2.** Let C be a special Frenet curve in Equiform differential geometry of $G_4$ whose curvature functions $\mathbb{k}_1$ =constant and $\mathbb{k}_2$ =constant are constant functions and satisfy the equality: $1-e^{-\mathbb{k}_1\varrho}c\big(\mathbb{k}_2(\varrho)\big)^2=0$, $\varrho\in L$. If the curve $\hat{C}$ given by $\hat{x}(\hat{\varrho})=x(\varrho)+\alpha(\varrho)e_2(\varrho)$, $\varrho\in L$ is a private Frenet curve, then $C$ is a generalized Mannheim curve and $\hat{C}$ is the generalized Mannheim mate curve of $C$.

**Theorem 4.3.** Let's take the curve C defined by

$$x(\varrho)=\left(\varrho,\alpha\int\left(\int \hbar(\varrho)\sin\varrho\, d\varrho\right)d\varrho,\alpha\int\left(\int \hbar(\varrho)\cos\varrho\, d\varrho\right)d\varrho,\alpha\int\left(\int \hbar(\varrho)\, g(\varrho)d\varrho\right)d\varrho\right),\ s\in I,$$

where $\alpha$ is a positive constant number, $g$ and $\hbar$ are any smooth functions: $I\to R$, and F defined by

$$\hbar(\varrho)=\frac{\ln c(1+g(\varrho)^2)^{-3}(g(\varrho)(g(\varrho))')^2}{\alpha\sqrt{1+g(\varrho)^2}}$$

for $\varrho\in I$. Then the curvatures $\mathbb{k}_1$ and $\mathbb{k}_2$ of the curve C satisfy the equality

$$1-e^{-\mathbb{k}_1\varrho}c\big(\mathbb{k}_2(\varrho)\big)^2=0,$$

for each $\varrho\in I$.

**Proof.** First we have to find $\mathbb{k}_1$ and $\mathbb{k}_2$. Given the definitions of curvatures,

$$\mathbb{k}_1=\|e'_2(\varrho)\|=\alpha\hbar(\varrho)\sqrt{1+g^2(\varrho)}$$

is obtained. Similarly,

$$\mathbb{k}_2=\langle e'_2(\varrho),e_3(\varrho)\rangle=-(1+g(\varrho)^2)^{\frac{-3}{2}}g(\varrho)\big(g(\varrho)\big)'.$$

If we substitute these equations in the condition of Mannheim curve,

$$c=\frac{e^{-\alpha\hbar(\varrho)\sqrt{1+g^2(\varrho)}}}{-(1+g(\varrho)^2)^{\frac{-3}{2}}g(\varrho)(g(\varrho))'}$$

is obtained. If we leave $\hbar(\varrho)$ alone in the last eqation, it is

$$\hbar(\varrho)=\frac{\ln c\big(1+g(\varrho)^2\big)^{-3}\big(g(\varrho)(g(\varrho))'\big)^2}{\alpha\sqrt{1+g(\varrho)^2}}.$$

So briefly, considering the formula of curvature calculations in Equiform differential geometry of $G_4$ the proof is can be done easily.

## 5. Conclusion

In this study, we defined the Mannheim curve for the Equiform differential geometry. We have obtained a new characterizations between the curvatures of the Mnnheim curve. A new case is obtained if the curvatures are not zero. We gave the application of the Mannheim curve with a general example.

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