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Family of Analytic Functions with Negative Coefficients Involving $q-$Analogue of Multiplier Transformation Operator

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Abstract
We introduce a new class of analytic functions with negative coefficients by using the $q-$analogue of multiplier transformation operator. Coefficient inequalities, distortion theorems, closure theorems, and some properties involving the modified Hadamard products, radii of close-to-convexity, starlikeness, and convexity, and integral operators associated with functions belonging to this class are obtained.

Keywords: *Univalent function; convolution; $q-$convex, $q-$starlike, $q-$analogue of multiplier transformation operator.*

AMS Subject Classification (2020): *30C45*

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1. Introduction

Let $\mathcal{A}(j)$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}(1) = \mathcal{A}$. For functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{k=j+1}^{\infty} b_k z^k \quad (j \in \mathbb{N}), \tag{1.2}$$

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the Hadamard product or convolution of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z)=z+\sum_{k=j+1}^{\infty}a_kb_kz^k=(g*f)(z). \tag{1.3}$$

Quantum calculus or $q-$calculus is an ordinary calculus without limit. In recent years, the study of $q-$theory attracted the researches due to its applications in various branches of mathematics and physics, for example, in the areas of special functions, ordinary fractional calculus, $q-$difference, $q-$integral equations and in $q-$transform analysis (see, for instance, [1], [2], [3], [4], [5], [6], [7], [8], [9] and [10]).

For $f\in\mathcal{A}(j)$ given by (1.1) and $0<q<1$, the $q-$derivative of f is defined by (see [11], [12], [13], [14], [15] and [16])

$$D_{q,j}f(z)=\begin{cases} f'(0) & \text{if } z=0,\\ \dfrac{f(z)-f(qz)}{(1-q)z} & \text{if } z\neq 0,\end{cases} \tag{1.4}$$

and $D^2_{q,j}f(z)=D_{q,j}\left(D_{q,j}f(z)\right)$. From (1.1) and (1.4), we deduce that

$$D_{q,j}f(z)=1+\sum_{k=j+1}^{\infty}[k]_q\,a_kz^{k-1}\quad (j\in\mathbb{N};z\neq 0), \tag{1.5}$$

where $[k]_q$ is $q-$integer number k defined by

$$[k]_q=\frac{1-q^k}{1-q}=1+q+q^2+...+q^{k-1}\quad (0<q<1)\,. \tag{1.6}$$

We note that $D_{q,1}f(z)=D_qf(z)$ and

$$\lim_{q\to 1^-}D_{q,j}f(z)=\lim_{q\to 1^-}\frac{f(z)-f(qz)}{(1-q)z}=f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. As a right inverse, the $q-$integral of f is introduced by

$$\int_0^z f(t)\,d_qt=z(1-q)\sum_{k=0}^{\infty}q^kf\left(zq^k\right),$$

provided that the series converges (see [17] and [18]). For a function f given by (1.1), we observe that

$$\int_0^z f(t)\,d_qt=\frac{z^2}{[2]_q}+\sum_{k=j+1}^{\infty}\frac{a_kz^{k+1}}{[k+1]_q}$$

and

$$\lim_{q\to 1^-}\int_0^z f(t)\,d_qt=\frac{z^2}{2}+\sum_{k=j+1}^{\infty}\frac{a_kz^{k+1}}{k+1}=\int_0^z f(t)\,dt,$$

where $\int_0^z f(t)\,dt$ is the ordinary integral.

Making use of the $q-$derivative $D_{q,j}f(z)$, we introduce the subclasses $\mathcal{S}_{q,j}(\alpha)$ and $\mathcal{C}_{q,j}(\alpha)$ of the class $\mathcal{A}(j)$ for $0<q<1, j\in\mathbb{N}$ and $0\leq\alpha<1$ as follows:

$$\mathcal{S}_{q,j}(\alpha)=\left\{f\in\mathcal{A}(j):\Re\frac{zD_{q,j}f(z)}{f(z)}>\alpha, z\in\mathbb{U}\right\}, \tag{1.7}$$

$$\mathcal{C}_{q,j}(\alpha)=\left\{f\in\mathcal{A}(j):\Re\frac{D_{q,j}(zD_{q,j}f(z))}{D_{q,j}f(z)}>\alpha, z\in\mathbb{U}\right\}, \tag{1.8}$$

From (1.7) and (1.8), we have

$$f\in\mathcal{C}_{q,j}(\alpha)\Leftrightarrow zD_{q,j}f\in\mathcal{S}_{q,j}(\alpha)\,.$$

We note that $\mathcal{S}_{q,1}(\alpha) = \mathcal{S}_q(\alpha)$ and $\mathcal{C}_{q,1}(\alpha) = \mathcal{C}_q(\alpha)$ (see [16]) and

$$\lim_{q\to 1^-} \mathcal{S}_{q,1}(\alpha) = \mathcal{S}(\alpha) \quad and \quad \lim_{q\to 1^-} \mathcal{C}_{q,1}(\alpha) = \mathcal{C}(\alpha),$$

where $\mathcal{S}(\alpha)$ and $\mathcal{C}(\alpha)$ are, respectively, the classes of starlike of order α and convex of order α in $\mathbb{U}$.

Now, we define the $q-$analogue of multiplier transformation operator

$$\mathcal{J}_{q,j}^{m}(l) : \mathcal{A}(j) \to \mathcal{A}(j) \quad (l > -1; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; j \in \mathbb{N}),$$

as follows:

$$\begin{aligned}
\mathcal{J}_{q,j}^{-m}(l) f(z) &= \frac{[l+1]_q}{z^l} \int_0^z t^{l-1} \mathcal{J}_{q,j}^{-(m-1)}(l) f(t)\, d_q t \quad (z \in \mathbb{U}), \\
&\vdots \\
\mathcal{J}_{q,j}^{-2}(l) f(z) &= \frac{[l+1]_q}{z^l} \int_0^z t^{l-1} \mathcal{J}_{q,j}^{-1}(l) f(t)\, d_q t \quad (z \in \mathbb{U}), \\
\mathcal{J}_{q,j}^{-1}(l) f(z) &= \frac{[l+1]_q}{z^l} \int_0^z t^{l-1} f(t)\, d_q t \quad (z \in \mathbb{U}), \\
\mathcal{J}_{q,j}^{0}(l) f(z) &= f(z) \quad (z \in \mathbb{U}), \\
\mathcal{J}_{q,j}^{1}(l) f(z) &= \frac{z^{1-l}}{[l+1]_q} D_{q,j}\left(z^l f(z)\right) \quad (z \in \mathbb{U}), \\
\mathcal{J}_{q,j}^{2}(l) f(z) &= \frac{z^{1-l}}{[l+1]_q} D_{q,j}\left(z^l \mathcal{J}_{q,j}^{1}(l) f(z)\right) \quad (z \in \mathbb{U}). \\
&\vdots \\
\mathcal{J}_{q,j}^{m}(l) f(z) &= \frac{z^{1-l}}{[l+1]_q} D_{q,j}\left(z^l \mathcal{J}_{q,j}^{m-1}(l) f(z)\right) \quad (z \in \mathbb{U}).
\end{aligned}$$

We see that for $f \in \mathcal{A}(j)$, we have

$$\mathcal{J}_{q,j}^{m}(l) f(z) = z + \sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m a_k z^k \tag{1.9}$$

$$(0 < q < 1; l > -1; m \in \mathbb{Z} = \{0, \pm 1, \pm 2, ...\}; j \in \mathbb{N}).$$

It is readily verified from (1.9) that

$$q^l\, z D_{q,j}\left(\mathcal{J}_{q,j}^{m}(l) f(z)\right) = [l+1]_q\ \mathcal{J}_{q,j}^{m+1}(l) f(z) - [l]_q\, \mathcal{J}_{q,j}^{m}(l) f(z) \quad (m \in \mathbb{Z}). \tag{1.10}$$

We observe that the operator $\mathcal{J}_{q,j}^{m}(l)$ generalize several previously familiar operators, and we will show some of the interesting particular cases as follows:

(i) $\mathcal{J}_{q,j}^{m}(0) f(z) = \mathcal{S}_{q,j}^{m} f(z)$ and $\mathcal{J}_{q,1}^{m}(0) f(z) = \mathcal{S}_q^m f(z)\, (m \in \mathbb{N}_0)$ (see [19]);

(ii) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{m}(0) f(z) = \mathcal{D}^m f(z)\, (m \in \mathbb{N}_0)$ (see [20], [21], [22] and [23]);

(iii) $\lim_{q\to 1^-} \mathcal{J}_{q,j}^{m}(l) f(z) = \mathcal{I}_{l,j}^{m} f(z)$ and $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{m}(l) f(z) = \mathcal{I}_l^m f(z)\, (l \geq 0; m \in \mathbb{N}_0)$ (see [24] and [25]);

(iv) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{m}(1) f(z) = D^m f(z)\, (m \in \mathbb{N}_0)$ (see [26]);

(v) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{-m}(1) f(z) = I^m f(z)\, (m \in \mathbb{N}_0)$ (see [27]);

(vi) $\lim_{q\to 1^-} \mathcal{J}_{q,1}^{-1}(c) f(z) = F_c f(z) = \frac{1+c}{z^c}\int_0^z t^{c-1} f(t)\,dt\ \ (c > -1)$ is the well-known Bernardi integral operator [28].

With the help of the operator $\mathcal{J}_{q,j}^m(l)$, we say that a function f belonging to the class $\mathcal{A}(j)$ is in the class $\mathcal{L}_q^m(l,\lambda,\alpha;j)$ if and only if

$$\Re\left\{\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^m(l) f(z)\right) + \lambda q z^2 D_{q,j}^2\left(\mathcal{J}_{q,j}^m(l) f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^m(l) f(z) + \lambda z D_{q,j}\left(\mathcal{J}_{q,j}^m(l) f(z)\right)}\right\} > \alpha \tag{1.11}$$

$$(z\in\mathbb{U}; m\in\mathbb{Z}; 0<q<1; l>-1; 0\le\lambda\le 1; 0\le\alpha<1).$$

Let $\mathcal{T}(j)$ denote the subclass of $\mathcal{A}(j)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k > 0; j\in\mathbb{N}) \tag{1.12}$$

Further, we define the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$ by

$$\mathcal{H}_q^m(l,\lambda,\alpha;j) = \mathcal{L}_q^m(l,\lambda,\alpha;j)\cap\mathcal{T}(j).$$

We note that

(i) $\lim_{q\to 1^-} \mathcal{H}_q^m(0,\lambda,\alpha;j) = \mathcal{P}(j;\lambda,\alpha,m)\ (m\in\mathbb{N})$ (Aouf and Srivastava [29]);

(ii) $\lim_{q\to 1^-} \mathcal{H}_q^0(0,0,\alpha;1) = \mathcal{S}(\alpha)$ and $\lim_{q\to 1^-} \mathcal{H}_q^0(0,1,\alpha;1) = \mathcal{C}(\alpha)$ (Silverman [30]);

(iii) $\lim_{q\to 1^-} \mathcal{H}_q^0(0,0,\alpha;j) = \mathcal{S}(\alpha;j)$ and $\lim_{q\to 1^-} \mathcal{H}_q^0(0,1,\alpha;j) = \mathcal{C}(\alpha;j)$ (Chatterjea [31] and Srivastava et al. [32]);

(iv) $\mathcal{H}_q^m(0,\lambda,\alpha;j) = \mathcal{H}_q^m(\lambda,\alpha;j)$

$$= \left\{ f\in\mathcal{T}(j) : \Re\left\{\frac{zD_{q,j}\left(\mathcal{S}_{q,j}^m f(z)\right) + \lambda q z^2 D_{q,j}^2\left(\mathcal{S}_{q,j}^m f(z)\right)}{(1-\lambda)\mathcal{S}_{q,j}^m f(z) + \lambda z D_{q,j}\left(\mathcal{S}_{q,j}^m f(z)\right)}\right\} > \alpha\right\};$$

(v) $\lim_{q\to 1^-} \mathcal{H}_q^m(l,\lambda,\alpha;j) = \mathcal{H}^m(l,\lambda,\alpha;j)$

$$= \left\{ f\in\mathcal{T}(j) : \Re\left\{\frac{z\left(\mathcal{I}_{l,j}^m f(z)\right)' + \lambda z^2\left(\mathcal{I}_{l,j}^m f(z)\right)''}{(1-\lambda)\mathcal{I}_{l,j}^m f(z) + \lambda z\left(\mathcal{I}_{l,j}^m f(z)\right)'}\right\} > \alpha\right\}.$$

The present paper aims at providing a systematic investigation of the various interesting properties and characteristics of the general class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$.

2. Coefficient estimates

Unless otherwise mentioned, we assume throughout this section that $m\in\mathbb{Z}$, $j\in\mathbb{N}$, $0<q<1$, $l>-1$, $0\le\lambda\le 1$, $0\le\alpha<1$, $z\in\mathbb{U}$ and $[k]_q$ is given by (1.6).

Theorem 2.1. *Let the function f be defined by (1.12). Then $f\in\mathcal{H}_q^m(l,\lambda,\alpha;j)$ if and only if*

$$\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right) a_k \le 1-\alpha. \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\left| \frac{zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)+\lambda qz^{2}D_{q,j}^{2}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^{m}(l)f(z)+\lambda zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}-1\right|$$

$$\leq \frac{\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left([k]_q-1\right)\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}$$

$$\leq \frac{\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left([k]_q-1\right)\left\{1+\left([k]_q-1\right)\lambda\right\}a_k}{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}a_k}\leq 1-\alpha.$$

This shows that the values of the function

$$\phi(z)=\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)+\lambda qz^{2}D_{q,j}^{2}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^{m}(l)f(z)+\lambda zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)} \tag{2.2}$$

lie in a circle which is centered at $w=1$ and whose radius is $1-\alpha$. Hence f satisfies the condition (1.11).

Conversely, assume that the function f is in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then we have

$$\Re\left\{\frac{zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)+\lambda qz^{2}D_{q,j}^{2}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}{(1-\lambda)\mathcal{J}_{q,j}^{m}(l)f(z)+\lambda zD_{q,j}\left(\mathcal{J}_{q,j}^{m}(l)f(z)\right)}\right\}$$

$$=\Re\left\{\frac{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}a_k|z|^{k-1}}\right\}>\alpha, \tag{2.3}$$

for some $\alpha\,(0\leq\alpha<1)$, $m\in\mathbb{Z}$, $0<q<1$, $l>-1$, $0\leq\lambda\leq1$ and $z\in\mathbb{U}$. Choose values of z on the real axis so that ϕ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z\to1^-$ through real values, we can see that

$$1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}a_k$$

$$\geq\alpha\left(1-\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}a_k\right). \tag{2.4}$$

Thus we have the inequality (2.1). This completes the proof of Theorem 2.1. □

Corollary 2.1. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then*

$$a_k\leq\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}\;(k\geq j+1;j\in\mathbb{N}) \tag{2.5}$$

The equality in (2.5) is attained for the function f given by

$$f(z)=z-\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^{m}\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}z^{k}\quad(k\geq j+1;j\in\mathbb{N}). \tag{2.6}$$

Theorem 2.2. *If $0\leq\alpha_1<\alpha_2<1$, then*

$$\mathcal{H}_q^m(l,\lambda,\alpha_2;j)\subseteq\mathcal{H}_q^m(l,\lambda,\alpha_1;j). \tag{2.7}$$

Proof. Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha_2;j)$. Then, by Theorem 2.1, we have

$$\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_2\right)a_k\leq 1-\alpha_2 \tag{2.8}$$

and

$$\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k\leq \frac{1-\alpha_2}{[j+1]_q-\alpha_2}<1. \tag{2.9}$$

Consequently,

$$\begin{aligned}
&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_1\right)a_k\\
=&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_2\right)a_k\\
&+(\alpha_2-\alpha_1)\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}a_k\\
\leq&\ 1-\alpha_1.
\end{aligned} \tag{2.10}$$

This completes the proof of Theorem 2.2 with the aid of Theorem 2.1. □

Theorem 2.3. *If* $0\leq\lambda_1\leq\lambda_2\leq 1$, *then*

$$\mathcal{H}_q^m(l,\lambda_2,\alpha;j)\subseteq\mathcal{H}_q^m(l,\lambda_1,\alpha;j). \tag{2.11}$$

Proof. It follows from Theorem 2.1 that

$$\begin{aligned}
&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda_1\right\}\left([k]_q-\alpha\right)a_k\\
\leq&\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda_2\right\}\left([k]_q-\alpha\right)a_k\\
\leq&\ 1-\alpha.
\end{aligned}$$

for $f\in\mathcal{H}_q^m(l,\lambda_2,\alpha;j)$. This completes the proof of Theorem 2.3 □

Similarly we can prove

Theorem 2.4. *If* $m\in\mathbb{Z}$, *then*

$$\mathcal{H}_q^{m+1}(l,\lambda,\alpha;j)\subseteq\mathcal{H}_q^m(l,\lambda,\alpha;j).$$

3. Distortion theorems and convex linear combinations

Theorem 3.1. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then, for $|z|<r<1$,*

$$\begin{aligned}
&r-\frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}r^{j+1}\leq|f(z)|\\
&\leq r+\frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}r^{j+1}.
\end{aligned} \tag{3.1}$$

The equality in (3.1) is attained for the function f given by

$$f(z)=z-\frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)}z^{j+1}. \tag{3.2}$$

Proof. It is easy to see from Theorem 2.1 that

$$\begin{aligned} &\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q\ -\alpha\right)\sum_{k=j+1}^{\infty}a_k \\ \le\ &\sum_{k=j+1}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([j+1]_q\ -\alpha\right)a_k \le 1-\alpha, \end{aligned}$$

so that

$$\sum_{k=j+1}^{\infty}a_k \le \frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q\ -\alpha\right)}. \tag{3.3}$$

Making use of (3.3), we have

$$\begin{aligned} |f(z)| &\ge r-\sum_{k=j+1}^{\infty}a_k\, r^k \le r-r^{j+1}\sum_{k=j+1}^{\infty}a_k \\ &\ge r-\frac{(1-\alpha)}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q\ -\alpha\right)}r^{j+1} \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\le r+\sum_{k=j+1}^{\infty}a_k\, r^k \le r+r^{j+1}\sum_{k=j+1}^{\infty}a_k \\ &\le r+\frac{(1-\alpha)}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m \left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q\ -\alpha\right)}r^{j+1} \end{aligned}$$

which prove the assertion (3.1). Finally, we note that the equality in (3.1) is attained for the function f defined by (3.2). This completes the proof of Theorem 3.1. □

Now, we shall prove that the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$ is closed under convex linear combinations.

Theorem 3.2. *$\mathcal{H}_q^m(l,\lambda,\alpha;j)$ is a convex set.*

Proof. Let the functions

$$f_v(z)=z-\sum_{k=j+1}^{\infty}a_{v.k}z^k\ (a_{v,k}>0;v=1,2;j\in\mathbb{N}) \tag{3.4}$$

be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=(1-\gamma)f_1(z)+\gamma f_2(z)\ (0\le\gamma\le 1) \tag{3.5}$$

is also in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Since, for $0\le\gamma\le 1$,

$$h(z)=z-\sum_{k=j+1}^{\infty}\left\{(1-\gamma)a_{1,k}+\gamma a_{2,k}\right\}z^k, \tag{3.6}$$

with the aid of Theorem 2.1, we have

$$\sum_{k=2}^{\infty}\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)\left\{(1-\gamma)a_{1,k}+\gamma a_{2,k}\right\}\le 1-\alpha, \tag{3.7}$$

which implies that $h\in\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Hence $\mathcal{H}_q^m(l,\lambda,\alpha;j)$ is a convex set. □

Theorem 3.3. *Let* $f_j(z) = z$ *and*

$$f_k(z) = z - \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} z^k \, (k \geq j+1; j \in \mathbb{N}). \tag{3.8}$$

Then f *is in the class* $\mathcal{H}_q^m(l, \lambda, \alpha; j)$ *if and only if it can be expressed in the form:*

$$f(z) = \sum_{k=j}^{\infty} \mu_k z^k \left(\mu_k \geq 0, k \geq j; \sum_{k=j}^{\infty} \mu_k = 1\right). \tag{3.9}$$

Proof. Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k z^k = z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} \mu_k z^k. \tag{3.10}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{1-\alpha} \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} \mu_k$$

$$= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \leq 1$$

So, by Theorem 2.1, $f \in \mathcal{H}_q^m(l, \lambda, \alpha; j)$.

Conversely, assume that the function f defined by (1.12) belongs to the class $\mathcal{H}_q^m(l, \lambda, \alpha; j)$. Then

$$a_k \leq \frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)} \, (k \geq j+1; j \in \mathbb{N}) \tag{3.11}$$

Setting

$$\mu_k = \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{1-\alpha} \, (k \geq j+1; j \in \mathbb{N}) \tag{3.12}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k,$$

we can see that f can be expressed in the form (3.9). This completes the proof of Theorem 3.3. □

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 4.1. *Let the function* f *defined by (1.12) be in the class* $\mathcal{H}_q^m(l, \lambda, \alpha; j)$. *Then* f *is close-to-convex of order* $\rho\,(0 \leq \rho < 1)$ *in* $|z| < r_1$, *where*

$$r_1 = \inf_k \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1+\left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} (k \geq j+1). \tag{4.1}$$

The result is sharp, the extremal function f *being given by (2.6).*

Proof. We must show that

$$\left|f'(z) - 1\right| \leq 1 - \rho \quad \text{for} \quad |z| < r_1,$$

where r_1 is given by (4.1). Indeed we find from the definition (1.12) that

$$\left|f'(z) - 1\right| = \sum_{k=j+1}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$\left| f^{'}(z) - 1 \right| \leq 1 - \rho,$$

if

$$\sum_{k=j+1}^{\infty} \frac{k}{1-\rho} a_k |z|^{k-1} \leq 1. \tag{4.2}$$

But, by Theorem 2.1, (4.2) will be true if

$$\frac{k}{1-\rho} a_k |z|^{k-1} \leq \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{k(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{4.3}$$

Theorem 4.1 follows easily from (4.3). □

Theorem 4.2. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l, \lambda, \alpha; j)$. Then f is starlike of order $\rho\,(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_k \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{4.4}$$

The result is sharp, the extremal function f being given by (2.6).

Proof. It is sufficient to show that

$$\left| \frac{z f^{'}(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad \text{for} \quad |z| < r_2,$$

where r_2 is given by (4.4). Indeed we find, again from the definition (1.12), that

$$\left| \frac{z f^{'}(z)}{f(z)} - 1 \right| = \frac{\sum\limits_{k=j+1}^{\infty} (k-1) a_k |z|^{k-1}.}{1 - \sum\limits_{k=j+1}^{\infty} a_k |z|^{k-1}}$$

Thus

$$\left| \frac{z f^{'}(z)}{f(z)} - 1 \right| \leq 1 - \rho,$$

if

$$\sum_{k=j+1}^{\infty} \frac{k-\rho}{1-\rho} a_k |z|^{k-1} \leq 1. \tag{4.5}$$

But, by Theorem 2.1, (4.5) will be true if

$$\frac{k-\rho}{1-\rho} a_k |z|^{k-1} \leq \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{4.6}$$

Theorem 4.2 follows easily from (4.6). □

Similarly, we can prove the following theorem.

Theorem 4.3. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then f is convex of order $\rho\,(0\leq\rho<1)$ in $|z|<r_3$, where*

$$r_3=\inf_k\left[\frac{(1-\rho)\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}}\quad(k\geq j+1)\,. \tag{4.7}$$

The result is sharp, the extremal function f being given by (2.6).

5. Modified Hadamard products and integral operator

Let the functions $f_v\,(v=1,2)$ be defined by (3.4). The modified Hadamard product of f_1 and f_2 is defined by

$$(f_1*f_2)(z)=z-\sum_{k=j+1}^{\infty}a_{1,k}a_{2,k}z^k. \tag{5.1}$$

Theorem 5.1. *Let each of the functions $f_v(z)\,(v=1,2)$ defined by (3.4) be in the class $\mathcal{H}_q^m(l,\lambda,\alpha;j)$. Then*

$$(f_1*f_2)(z)\in\mathcal{H}_q^m(l,\lambda,\beta;j)\,,$$

where

$$\beta=\frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-(1-\alpha)^2}. \tag{5.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [33], we need to find the largest β such that

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\beta\right)}{1-\beta}a_{1,k}a_{2,k}\leq 1. \tag{5.3}$$

Since

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{1,k}\leq 1 \tag{5.4}$$

and

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{2,k}\leq 1, \tag{5.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\sqrt{a_{1,k}a_{2,k}}\leq 1. \tag{5.6}$$

Thus it is sufficient to show that

$$\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\beta\right)}{1-\beta}a_{1,k}a_{2,k}\leq\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\sqrt{a_{1,k}a_{2,k}} \tag{5.7}$$

that is, that

$$\sqrt{a_{1,k}a_{2,k}}\leq\frac{(1-\beta)\left([k]_q-\alpha\right)}{(1-\alpha)\left([k]_q-\beta\right)}\quad(k>j+1)\,. \tag{5.8}$$

Note that

$$\sqrt{a_{1,k}a_{2,k}}\leq\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}\qquad(k\geq j+1)\,. \tag{5.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)} \leq \frac{(1-\beta)\left([k]_q-\alpha\right)}{(1-\alpha)\left([k]_q-\beta\right)} \quad (k \geq j+1), \tag{5.10}$$

or, equivalently, that

$$\beta \leq \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-(1-\alpha)^2} \quad (k \geq j+1). \tag{5.11}$$

Since

$$\Psi_q(k) = \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-(1-\alpha)^2} \quad (k \geq j+1) \tag{5.12}$$

is an increasing function of k $(k \geq j+1)$, letting $k = j+1$ in (5.12). we obtain

$$\beta \leq \Psi_q(j+1) = \frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-(1-\alpha)^2} \tag{5.13}$$

which proves the main assertion of Theorem 5.1. Finally, by taking the functions

$$f_i(z) = z - \frac{1-\alpha}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)} z^{j+1} \quad (i = 1, 2), \tag{5.14}$$

we can see that the result is sharp. □

Theorem 5.2. *Let $f_i \in \mathcal{H}_q^m(l, \lambda, \alpha_i; j)$ $(i = 1, 2)$. Then $(f_1 * f_2) \in \mathcal{H}_q^m(l, \lambda, \delta; j)$, where*

$$\delta = \frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha_1\right)\left([j+1]_q-\alpha_2\right)-[j+1]_q(1-\alpha_1)(1-\alpha_2)}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha_1\right)\left([j+1]_q-\alpha_2\right)-(1-\alpha_1)(1-\alpha_2)}. \tag{5.15}$$

The result is the best possible for the functions

$$f_i(z) = z - \frac{1-\alpha_i}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha_i\right)} z^{j+1} \ (i = 1, 2). \tag{5.16}$$

Proof. Proceeding as in the proof of Theorem 5.1, we get

$$\delta \leq \frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_1\right)\left([k]_q-\alpha_2\right)-[k]_q(1-\alpha_1)(1-\alpha_2)}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha_1\right)\left([k]_q-\alpha_2\right)-(1-\alpha_1)(1-\alpha_2)} \ (k \geq j+1). \tag{5.17}$$

Since the right-hand side of (5.17) is an increasing function of k, setting $k = j+1$ in (5.17), we obtain (5.15). This completes the proof of Theorem 5.2. □

Theorem 5.3. *Let each of the functions f_i $(i = 1, 2)$ defined by (3.4) be in the class $\mathcal{H}_q^m(l, \lambda, \alpha; j)$. Then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty}\left(a_{1,k}^2 + a_{2,k}^2\right) z^k \tag{5.18}$$

belongs to the class $\mathcal{H}_q^m(l, \lambda, \zeta; j)$, where

$$\zeta = \frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2(1-\alpha)^2}. \tag{5.19}$$

The result is sharp for the functions f_i $(i = 1, 2)$ defined by (5.14).

Proof. By virtue of Theorem 2.1, we obtain

$$\sum_{k=j+1}^{\infty}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2 a_{1,k}^2$$

$$\leq\left[\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{1,k}\right]^2\leq 1 \tag{5.20}$$

and

$$\sum_{k=j+1}^{\infty}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2 a_{2,k}^2$$

$$\leq\left[\sum_{k=j+1}^{\infty}\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}a_{2,k}\right]^2\leq 1. \tag{5.21}$$

It follows from (5.20) and (5.21) that

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2\left(a_{1,k}^2+a_{2,k}^2\right)\leq 1 \tag{5.22}$$

Therefore, we need to find the largest ζ such that

$$\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\zeta\right)}{1-\zeta}\leq\frac{1}{2}\left[\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)}{1-\alpha}\right]^2 \tag{5.23}$$

that is,

$$\zeta\leq\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2(1-\alpha)^2}\quad (k\geq j+1)\,. \tag{5.24}$$

Since

$$\chi_q\left(k\right)=\frac{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2[k]_q(1-\alpha)^2}{\left(\frac{[l+k]_q}{[l+1]_q}\right)^m\left\{1+\left([k]_q-1\right)\lambda\right\}\left([k]_q-\alpha\right)^2-2(1-\alpha)^2} \tag{5.25}$$

is an increasing function of $k\,(k\geq j+1)$, we readily have

$$\zeta\leq\chi_q\left(j+1\right)=\frac{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2[j+1]_q(1-\alpha)^2}{\left(\frac{[l+j+1]_q}{[l+1]_q}\right)^m\left\{1+\left([j+1]_q-1\right)\lambda\right\}\left([j+1]_q-\alpha\right)^2-2(1-\alpha)^2} \tag{5.26}$$

and Theorem 5.3 follows at once. □

Theorem 5.4. *Let the function f defined by (1.12) be in the class $\mathcal{H}_q^m\left(l,\lambda,\alpha;j\right)$, and let c be a real number such that $c>-1$. Then the function*

$$\mathcal{J}_{q,j}^{-1}\left(c\right)f\left(z\right)=F_{c,q,j}\left(z\right)=\frac{[c+1]_q}{z^c}\int_0^z t^{c-1}f\left(t\right)d_qt\quad (c>-1) \tag{5.27}$$

also belongs to the class $\mathcal{H}_q^m\left(l,\lambda,\alpha;j\right)$.

Proof. From the representation (5.27) of $F_{c,q,j}(z)$, it follows that

$$F_{c,q,j}(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,$$

where $b_k = \frac{[c+1]_q}{[c+k]_q} a_k$ (see [34] and [35]). Therefore, we have

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right) b_k z^k \\
=& \sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right) \frac{[c+1]_q}{[c+k]_q} a_k z^k \\
\leq& \sum_{k=j+1}^{\infty} \left(\frac{[l+k]_q}{[l+1]_q}\right)^m \left\{1 + \left([k]_q - 1\right)\lambda\right\}\left([k]_q - \alpha\right) a_k z^k \\
\leq& \ 1 - \alpha,
\end{aligned}$$

since $f \in \mathcal{H}_q^m(l, \lambda, \alpha; j)$. Hence, by Theorem 2.1, $F_{c,q,j} \in \mathcal{H}_q^m(l, \lambda, \alpha; j)$. □

Remark 5.1. Taking $l = 0, m \in \mathbb{N}_0$ and $q \to 1^-$ in the above results, we obtain the results of Aouf and Srivastava [29] for the class $\mathcal{P}(j; \lambda, \alpha, m)$.

Remark 5.2. Putting $l = 0$ in the above results, we obtain the the corresponding results for the class $\mathcal{H}_q^m(\lambda, \alpha; j)$ involving an operator $\mathcal{S}_{q,j}^m$.

Remark 5.3. Putting $q \to 1^-$ in the above results, we obtain the corresponding results for the class $\mathcal{H}^m(l, \lambda, \alpha; j)$ involving multiplier transformation operator $\mathcal{I}_{l,j}^m$.

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