Ikonion Journal of Mathematics 3(1) (2021) 20-34

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

ISSN: 2687-6531

A New Moving Frame For Trajectories on Regular Surfaces

Kahraman Esen Özen[1], **Murat Tosun**[2]

Keywords
Angular momentum, Kinematics of a particle, Moving frame, Regular surfaces, Smarandache curves

Abstract — In this study, we introduce a new moving frame on regular surfaces for trajectories with non-vanishing angular momentum and give the angular velocity vector for this frame. Then, we consider the special trajectories generated by Smarandache curves according to this frame in three-dimensional Euclidean space and investigate the Serret-Frenet apparatus of them. Moreover, we provide an illustrative example explaining how this frame is constructed and how the aforementioned special trajectories are generated. This moving frame is a new contribution to the field and we expect that it will be useful in some specific applications of differential geometry and kinematics in the future.

Subject Classification (2020): 70B99, 70B05, 57R25, 53A05, 53A04.

1. Introduction

In differential geometry, the theory of surfaces in 3-dimensional Euclidean space has an important place. Although the theory of surfaces in 3-dimensional Euclidean space had already been developed widely when the Serret-Frenet frame was introduced by Serret and Frenet, Serret-Frenet frame helped developing this theory further by researchers. This theory is still an issue of interest despite its long history. The approaches followed by Serret and Frenet led to the success of adapting the method of moving frames to the surface curves. This was carried out by Jean Gaston Darboux. He introduced a moving frame which is constructed on a surface. It is called as Darboux frame. At all non-umbilic points of a surface, Darboux frame exists. Thus, it exists at all the points of a curve on a regular surface [9, 12]. Darboux frame is a useful tool for investigating the theory of surfaces. From the discovery of this frame until now, many researchers have carried out lots of interesting studies on this theory by using this frame. Some of these studies can be found in [2, 7, 8, 10, 14, 17].

In Euclidean 3-space, a point particle of constant mass moving on a regular surface curve has a position vector according to Darboux frame of this curve. So, an arbitrary point of the trajectory can be represented by the aforesaid particle. As a result of this case, there is a very close relationship between the differential geometry of the trajectory, the differential geometry of the surface and the kinematics of the moving particle. This relationship has motivated us to prepare this study. In this study, a new moving frame on regular

[1]kahraman.ozen1@ogr.sakarya.edu.tr (Corresponding Author); [2]tosun@sakarya.edu.tr
[1,2]Department of Mathematics, Faculty of Arts and Sciences, Sakarya University, Sakarya, TURKEY
Article History: Received: 20.02.2021 - Accepted: 05.04.2021 - Published: 02.06.2021

surfaces for trajectories with non-vanishing angular momentum has been constructed by considering the Darboux frame of the trajectory. It is expected that this moving frame will enable more convenient observation environment for the researchers studying on modern robotics. Note that we carried out a similar study [11] for trajectories, not necessarily lying on a surface, by considering Serret-Frenet frame. The present study includes similar techniques and approaches given in [11].

Let E^3 be endowed with the standard inner product $\langle \mathbf{D}, \mathbf{E} \rangle = d_1 e_1 + d_2 e_2 + d_3 e_3$ where $\mathbf{D} = (d_1, d_2, d_3)$, $\mathbf{E} = (e_1, e_2, e_3)$ are arbitrary vectors in this space. The norm of the vector $\mathbf{D}$ is stated as $\|\mathbf{D}\| = \sqrt{\langle \mathbf{D}, \mathbf{D} \rangle}$. If a differentiable curve $\chi = \chi(s) : I \subset \mathbb{R} \to E^3$ satisfies the equality $\left\| \frac{d\chi}{ds} \right\| = 1$ for all $s \in I$, this curve is called a unit speed curve. In this case, s is said to be arc-length parameter of χ. A differentiable curve is called regular curve if its derivative is nonzero along the curve. Regular curves can be reparameterized by the arc-length [13]. In the rest of the paper, the differentiation with respect to the arc-length parameter s will be shown with a dash.

The Serret-Frenet frame of the curve $\chi = \chi(s)$ is denoted by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$. The unit vectors $\mathbf{T}(s), \mathbf{N}(s)$ and $\mathbf{B}(s)$ are called the unit tangent, unit principal normal and unit binormal vectors, respectively. On the other hand, the Serret-Frenet formulas are given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix} \tag{1.1}$$

where $\kappa(s) = \|\mathbf{T}'(s)\|$ is the curvature function and $\tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle$ is the torsion function [13].

Suppose that $\chi : I \subset R \to M \subset E^3$ is a unit speed curve which lies on a regular surface M. In that case, there exists Darboux frame denoted by $\{\mathbf{T}, \mathbf{Y}, \mathbf{U}\}$ along the curve χ. $\mathbf{T}$ is the unit tangent vector of χ, $\mathbf{U}$ is the unit normal vector of M restricted to χ and $\mathbf{Y}$ is the unit vector given by $\mathbf{Y} = \mathbf{U} \times \mathbf{T}$. The derivative formulas of Darboux frame are as follows:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{Y}' \\ \mathbf{U}' \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{Y} \\ \mathbf{U} \end{pmatrix}. \tag{1.2}$$

Here, the functions k_g, k_n and τ_g are called geodesic curvature, normal curvature and geodesic torsion of the curve χ, respectively [6, 9].

This study is organized as follows. In Section 2, we explain how our frame is constructed and give the relation matrix between this frame and Darboux frame. Afterwards, we obtain derivative formulas and complete the set of apparatus of this frame. Also, angular velocity vector is obtained for this frame. In Section 3, we study the special trajectories generated by Smarandache curves according to this frame in three-dimensional Euclidean space.

2. Positional Adapted Frame on Regular Surfaces

In E^3, let a point particle of constant mass m move on a curve which lies on a regular surface M. Denote by $\mathbf{x}$ the position vector of this particle relative to fixed origin O at time t. Let the curve $\chi = \chi(s)$ be the unit speed parametrization of the trajectory of the particle where the arc-length s of χ corresponds to time t. In that case, the unit tangent vector, velocity vector and linear momentum vector at the point $\chi(s)$ (at time t) are given by

$$\begin{aligned} \mathbf{T}(s) &= \frac{d\mathbf{x}}{ds} \\ \mathbf{v}(t) &= \frac{d\mathbf{x}}{dt} = \left(\frac{ds}{dt}\right)\mathbf{T}(s) \\ \mathbf{p}(t) &= m\mathbf{v}(t) = m\left(\frac{ds}{dt}\right)\mathbf{T}(s), \end{aligned} \tag{2.1}$$

respectively [4]. Also, we can write

$$\mathbf{x} = \langle \chi(s), \mathbf{T}(s)\rangle \mathbf{T}(s) + \langle \chi(s), \mathbf{Y}(s)\rangle \mathbf{Y}(s) + \langle \chi(s), \mathbf{U}(s)\rangle \mathbf{U}(s) \tag{2.2}$$

at the point $\chi(s)$ (at time t) with respect to Darboux frame. By vector product of $\mathbf{x}$ and $\mathbf{p}(t)$, the angular momentum vector (at time t) of the particle about O is found as:

$$\mathbf{H}^O = m\langle \chi(s), \mathbf{U}(s)\rangle \left(\frac{ds}{dt}\right)\mathbf{Y}(s) - m\langle \chi(s), \mathbf{Y}(s)\rangle \left(\frac{ds}{dt}\right)\mathbf{U}(s). \tag{2.3}$$

Throughout the paper, we suppose that angular momentum vector of the aforementioned particle never vanishes. In other words, we restrict ourselves to the trajectories having non-vanishing angular momentum. This assumption ensures that the coefficient functions $\langle \chi(s), \mathbf{Y}(s)\rangle$ and $\langle \chi(s), \mathbf{U}(s)\rangle$ of the position vector are not zero simultaneously. That is, we ensure that the tangent line never passes through the origin along the trajectory. Let us return to the position vector. The opposite of this vector is given as in the following:

$$-\mathbf{x} = \langle -\chi(s), \mathbf{T}(s)\rangle \mathbf{T}(s) + \langle -\chi(s), \mathbf{Y}(s)\rangle \mathbf{Y}(s) + \langle -\chi(s), \mathbf{U}(s)\rangle \mathbf{U}(s). \tag{2.4}$$

The projections of it on the instantaneous planes $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$ yield two vectors playing important roles to construct a new moving frame on M along χ. These roles are stated in detail below.

The vector, whose starting point is $\chi(s)$ and endpoint is the foot of perpendicular (from O to $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$), can be given by

$$\mathbf{r}(s) = \langle -\chi(s), \mathbf{T}(s)\rangle \mathbf{T}(s) + \langle -\chi(s), \mathbf{Y}(s)\rangle \mathbf{Y}(s) \tag{2.5}$$

and corresponds to the aforementioned projection on $Sp\{\mathbf{T}(s), \mathbf{Y}(s)\}$. On the other hand the vector, whose starting point is $\chi(s)$ and endpoint is the foot of the perpendicular (from origin to $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$), can be given by

$$\mathbf{r}^*(s) = \langle -\chi(s), \mathbf{T}(s)\rangle \mathbf{T}(s) + \langle -\chi(s), \mathbf{U}(s)\rangle \mathbf{U}(s) \tag{2.6}$$

and corresponds to the aforementioned projection on $Sp\{\mathbf{T}(s), \mathbf{U}(s)\}$. From the Equation 2.5 and Equation 2.6, we can get the vector

$$\mathbf{r}(s) - \mathbf{r}^*(s) = \langle -\chi(s), \mathbf{Y}(s)\rangle \mathbf{Y}(s) + \langle \chi(s), \mathbf{U}(s)\rangle \mathbf{U}(s) \tag{2.7}$$

whose starting point is $\chi(s)$ and which lies on the instantaneous plane $Sp\{\mathbf{Y}(s), \mathbf{U}(s)\}$. We must emphasize that the vector $\mathbf{r}(s)-\mathbf{r}^*(s)$ is equivalent to the vector whose starting point is the aforesaid foot on $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ and endpoint is the other aforesaid foot on $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ (see Figure 1).

Let us talk about the determination of unit vector in direction $\mathbf{r}(s)-\mathbf{r}^*(s)$. If both planes $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ do not contain the origin, the foots are distinct from each other and from the origin. Therefore, two distinct foots generate the non-zero vector $\mathbf{r}(s)-\mathbf{r}^*(s)$. In this case, the desired unit vector can be obtained. When only one of the planes $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ passes through the origin, the foot of the perpendicular on the plane, containing origin, is taken as the origin. Certainly, the other foot is distinct from the origin. In that case, the desired unit vector is determined similarly. The case both the planes $Sp\{\mathbf{T}(s),\mathbf{Y}(s)\}$ and $Sp\{\mathbf{T}(s),\mathbf{U}(s)\}$ include the origin simultaneously causes not to be determined of the desired unit vector because the both of the aforesaid foots correspond to the origin. This situation occurs only when the tangent line contains the origin. Fortunately, the assumption on the angular momentum vector averts this. Let the unit vector in direction $\mathbf{r}(s)-\mathbf{r}^*(s)$ be denoted by $\mathbf{H}(s)$. Namely,

$$\mathbf{H}(s)=\frac{\mathbf{r}(s)-\mathbf{r}^*(s)}{\|\mathbf{r}(s)-\mathbf{r}^*(s)\|}=\frac{\langle-\chi(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)+\frac{\langle\chi(s),\mathbf{U}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s). \tag{2.8}$$

By vector product $\mathbf{H}(s)$ and $\mathbf{T}(s)$, we can get the another basis vector. We show it with $\mathbf{G}(s)$. Then we obtain

$$\mathbf{G}(s)=\mathbf{H}(s)\wedge\mathbf{T}(s)=\frac{\langle\chi(s),\mathbf{U}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{Y}(s)+\frac{\langle\chi(s),\mathbf{Y}(s)\rangle}{\sqrt{\langle\chi(s),\mathbf{Y}(s)\rangle^2+\langle\chi(s),\mathbf{U}(s)\rangle^2}}\mathbf{U}(s). \tag{2.9}$$

This completes the positively oriented orthonormal moving frame $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s)\}$.

Since the vectors $\mathbf{Y}(s),\mathbf{U}(s),\mathbf{G}(s)$ and $\mathbf{H}(s)$ lie on the plane $\{\mathbf{T}(s)\}^{\perp}$, there is a relation between the Darboux frame and this frame as follows:

$$\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix}=\begin{pmatrix}1&0&0\\ 0&\cos\Omega(s)&-\sin\Omega(s)\\ 0&\sin\Omega(s)&\cos\Omega(s)\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{Y}(s)\\ \mathbf{U}(s)\end{pmatrix} \tag{2.10}$$

where $\Omega(s)$ is the angle between the vectors $\mathbf{U}(s)$ and $\mathbf{H}(s)$ which is positively oriented from $\mathbf{U}(s)$ to $\mathbf{H}(s)$ (see Figure 1). By using the Equation 1.2 and Equation 2.10, we can write

$$\begin{aligned}\mathbf{G}'(s) &= (\cos\Omega(s)\mathbf{Y}(s)-\sin\Omega(s)\mathbf{U}(s))'\\ &= -\Omega'(s)\sin\Omega(s)\mathbf{Y}(s)+\cos\Omega(s)\left(-k_g(s)\mathbf{T}(s)+\tau_g(s)\mathbf{U}(s)\right)\\ &\quad -\Omega'(s)\cos\Omega(s)\mathbf{U}(s)+\sin\Omega(s)\left(k_n(s)\mathbf{T}(s)+\tau_g(s)\mathbf{Y}(s)\right)\\ &= \left(-k_g(s)\cos\Omega(s)+k_n(s)\sin\Omega(s)\right)\mathbf{T}(s)+\left(\tau_g(s)-\Omega'(s)\right)[\sin\Omega(s)\mathbf{Y}(s)+\cos\Omega(s)\mathbf{U}(s)]\\ &= \left(-k_g(s)\cos\Omega(s)+k_n(s)\sin\Omega(s)\right)\mathbf{T}(s)+\left(\tau_g(s)-\Omega'(s)\right)\mathbf{H}(s)\end{aligned}$$

and

$$\begin{aligned}
\mathbf{H}'(s) &= (\sin\Omega(s)\mathbf{Y}(s)+\cos\Omega(s)\mathbf{U}(s))' \\
&= \Omega'(s)\cos\Omega(s)\mathbf{Y}(s)+\sin\Omega(s)\left(-k_g(s)\mathbf{T}(s)+\tau_g(s)\mathbf{U}(s)\right) \\
&\quad -\Omega'(s)\sin\Omega(s)\mathbf{U}(s)-\cos\Omega(s)\left(k_n(s)\mathbf{T}(s)+\tau_g(s)\mathbf{Y}(s)\right) \\
&= \left(-k_g(s)\sin\Omega(s)-k_n(s)\cos\Omega(s)\right)\mathbf{T}(s)+\left(\Omega'(s)-\tau_g(s)\right)[\cos\Omega(s)\mathbf{Y}(s)-\sin\Omega(s)\mathbf{U}(s)] \\
&= \left(-k_g(s)\sin\Omega(s)-k_n(s)\cos\Omega(s)\right)\mathbf{T}(s)+\left(\Omega'(s)-\tau_g(s)\right)\mathbf{G}(s).
\end{aligned}$$

In that case, differentiating the vector $\mathbf{G}(s)\wedge\mathbf{H}(s)$ gives us the following:

$$\begin{aligned}
\mathbf{T}'(s) &= (\mathbf{G}(s)\wedge\mathbf{H}(s))' \\
&= \mathbf{G}'(s)\wedge\mathbf{H}(s)+\mathbf{G}(s)\wedge\mathbf{H}'(s) \\
&= \left[\left(-k_g(s)\cos\Omega(s)+k_n(s)\sin\Omega(s)\right)\mathbf{T}(s)+\left(\tau_g(s)-\Omega'(s)\right)\mathbf{H}(s)\right]\wedge\mathbf{H}(s) \\
&\quad +\mathbf{G}(s)\wedge\left[\left(-k_g(s)\sin\Omega(s)-k_n(s)\cos\Omega(s)\right)\mathbf{T}(s)+\left(\Omega'(s)-\tau_g(s)\right)\mathbf{G}(s)\right] \\
&= \left(k_g(s)\cos\Omega(s)-k_n(s)\sin\Omega(s)\right)\mathbf{G}(s)+\left(k_g(s)\sin\Omega(s)+k_n(s)\cos\Omega(s)\right)\mathbf{H}(s).
\end{aligned}$$

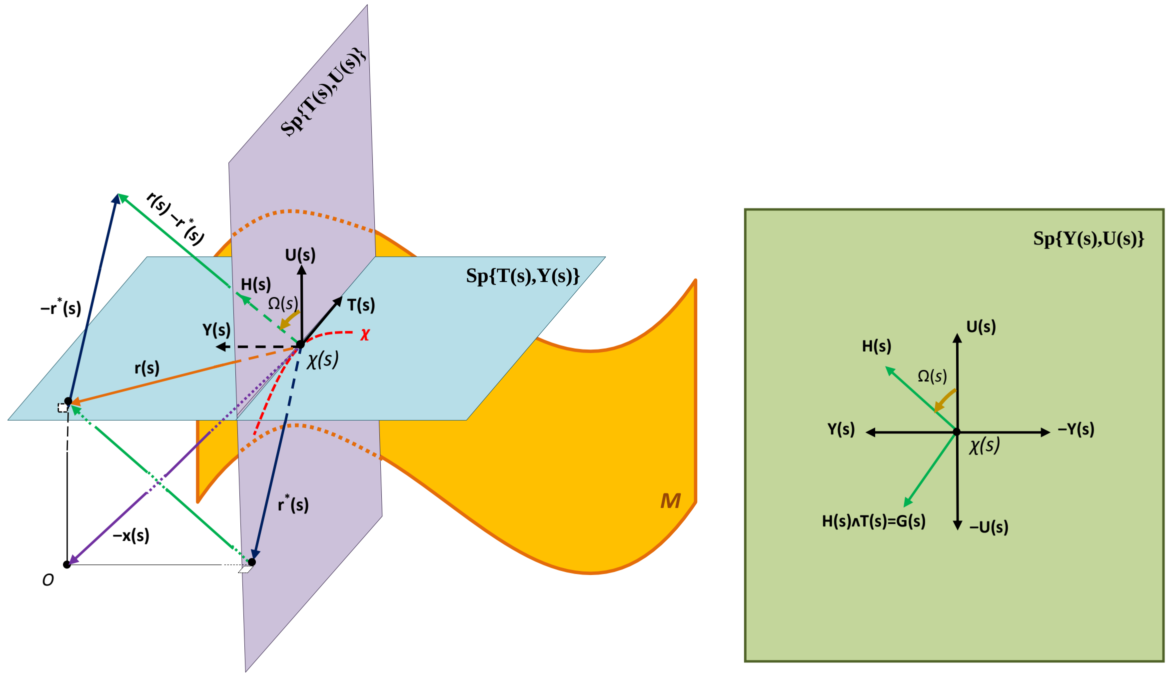
Therefore, the derivative formulas are given by

$$\begin{pmatrix}\mathbf{T}'(s)\\ \mathbf{G}'(s)\\ \mathbf{H}'(s)\end{pmatrix}=\begin{pmatrix}0 & k_1(s) & k_2(s)\\ -k_1(s) & 0 & k_3(s)\\ -k_2(s) & -k_3(s) & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}(s)\\ \mathbf{G}(s)\\ \mathbf{H}(s)\end{pmatrix} \tag{2.11}$$

where

$$\begin{aligned}
k_1(s) &= k_g(s)\cos\Omega(s)-k_n(s)\sin\Omega(s) \\
k_2(s) &= k_g(s)\sin\Omega(s)+k_n(s)\cos\Omega(s) \\
k_3(s) &= \tau_g(s)-\Omega'(s).
\end{aligned} \tag{2.12}$$

Based on the relationship of the frame $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s)\}$ to the position vector, we call it as "Positional Adapted Frame on Regular Surface". We will use the abbreviation PAFORS for it in the rest of the study. Also, we call the set $\{\mathbf{T}(s),\mathbf{G}(s),\mathbf{H}(s),k_1(s),k_2(s),k_3(s)\}$ as PAFORS apparatus of the regular surface curve $\chi=\chi(s)$.



(a) **(b)**

Figure 1. An illustration for explaining the construction of PAFORS

From the Equation 2.8, Equation 2.9 and Equation 2.10, the followings can be written easily:

$$\sin\Omega(s) = \frac{-\left\langle \chi(s), \mathbf{Y}(s)\right\rangle}{\sqrt{\left\langle \chi(s), \mathbf{Y}(s)\right\rangle^2 + \left\langle \chi(s), \mathbf{U}(s)\right\rangle^2}} \tag{2.13}$$

$$\cos\Omega(s) = \frac{\left\langle \chi(s), \mathbf{U}(s)\right\rangle}{\sqrt{\left\langle \chi(s), \mathbf{Y}(s)\right\rangle^2 + \left\langle \chi(s), \mathbf{U}(s)\right\rangle^2}}. \tag{2.14}$$

Then we obtain

$$\tan\Omega(s) = -\frac{\left\langle \chi(s), \mathbf{Y}(s)\right\rangle}{\left\langle \chi(s), \mathbf{U}(s)\right\rangle}. \tag{2.15}$$

Taking into consideration Figure 1 and Equations 2.13, 2.14, 2.15, the rotation angle $\Omega(s)$ is determined as

$$\Omega(s) = \begin{cases} \arctan\left(-\frac{\langle \chi(s), \mathbf{Y}(s)\rangle}{\langle \chi(s), \mathbf{U}(s)\rangle}\right) \ if \ \left\langle \chi(s), \mathbf{U}(s)\right\rangle > 0 \\ \arctan\left(-\frac{\langle \chi(s), \mathbf{Y}(s)\rangle}{\langle \chi(s), \mathbf{U}(s)\rangle}\right) + \pi \ if \ \left\langle \chi(s), \mathbf{U}(s)\right\rangle < 0 \\ -\frac{\pi}{2} \ if \ \left\langle \chi(s), \mathbf{U}(s)\right\rangle = 0\,, \ \left\langle \chi(s), \mathbf{Y}(s)\right\rangle > 0 \\ \frac{\pi}{2} \ if \ \left\langle \chi(s), \mathbf{U}(s)\right\rangle = 0\,, \ \left\langle \chi(s), \mathbf{Y}(s)\right\rangle < 0. \end{cases} \tag{2.16}$$

When $\left\langle \chi(s), \mathbf{U}(s)\right\rangle = 0, \left\langle \chi(s), \mathbf{Y}(s)\right\rangle > 0$, PAFORS apparatus $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s), k_1(s), k_2(s), k_3(s)\}$ correspond to $\left\{\mathbf{T}(s), \mathbf{U}(s), -\mathbf{Y}(s), k_n(s), -k_g(s), \tau_g(s)\right\}$. Similar to above, in the case $\left\langle \chi(s), \mathbf{U}(s)\right\rangle = 0$, $\left\langle \chi(s), \mathbf{Y}(s)\right\rangle < 0$, $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s), k_1(s), k_2(s), k_3(s)\}$ correspond to the apparatus $\left\{\mathbf{T}(s), -\mathbf{U}(s), \mathbf{Y}(s), -k_n(s), k_g(s), \tau_g(s)\right\}$.

Now, we will get the angular velocity vector for PAFORS. A better insight into the structure of the derivative formulas, given in (2.11), is presented by the help of the angular velocity vector $\omega(s)$. The evolution of PAFORS $\{\mathbf{T}(s), \mathbf{G}(s), \mathbf{H}(s)\}$ is specified by its angular velocity via

$$\begin{aligned} \mathbf{T}'(s) &= \omega(s) \wedge \mathbf{T}(s) \\ \mathbf{G}'(s) &= \omega(s) \wedge \mathbf{G}(s) \\ \mathbf{H}'(s) &= \omega(s) \wedge \mathbf{H}(s). \end{aligned} \tag{2.17}$$

Let us obtain the vector $\omega(s)$. Assume that it is written with respect to PAFORS as follows:

$$\omega(s) = \lambda_1(s)\mathbf{T}(s) + \lambda_2(s)\mathbf{G}(s) + \lambda_3(s)\mathbf{H}(s)$$

where $\lambda_1(s)$, $\lambda_2(s)$ and $\lambda_3(s)$ are real-valued functions of s. In this case, (2.17) becomes

$$\begin{aligned} \mathbf{T}'(s) &= -\lambda_2(s)\mathbf{H}(s) + \lambda_3(s)\mathbf{G}(s) \\ \mathbf{G}'(s) &= \lambda_1(s)\mathbf{H}(s) - \lambda_3(s)\mathbf{T}(s) \\ \mathbf{H}'(s) &= -\lambda_1(s)\mathbf{G}(s) + \lambda_2(s)\mathbf{T}(s). \end{aligned} \tag{2.18}$$

By comparing (2.11) with (2.18) we find

$$\begin{aligned} \lambda_1(s) &= k_3(s) \\ \lambda_2(s) &= -k_2(s) \\ \lambda_3(s) &= k_1(s). \end{aligned}$$

Consequentially, the angular velocity vector is given as

$$\omega(s) = \left[\tau_g(s) - \Omega'(s)\right]\mathbf{T}(s) - \left[k_g(s)\sin\Omega(s) + k_n(s)\cos\Omega(s)\right]\mathbf{G}(s) + \left[k_g(s)\cos\Omega(s) - k_n(s)\sin\Omega(s)\right]\mathbf{H}(s)$$

for PAFORS.

3. Some Special Trajectories Generated by Smarandache Curves According to PAFORS

In the study [1], author defined special Smarandache curves in the Euclidean space. Author considered a unit speed regular curve $\gamma = \gamma(s)$ with its Serret-Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and defined **TN**, **NB**, **TNB**$-$ Smarandache curves as follows:

$$\begin{aligned} \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{N}) \\ \beta(s^*) &= \frac{1}{\sqrt{2}}(\mathbf{N} + \mathbf{B}) \\ \beta(s^*) &= \frac{1}{\sqrt{3}}(\mathbf{T} + \mathbf{N} + \mathbf{B}), \end{aligned}$$

respectively. There can be found some studies [1, 3, 5, 15, 16, 18] on Smarandache curves in the literature.

In this section, we investigate special trajectories generated by Smarandache curves according to PAFORS in 3-dimensional Euclidean space.

3.1. Special Trajectories Generated by TG Smarandache Curves

Definition 3.1. In E^3, assume that a point particle P of constant mass moves on the regular surface M along the trajectory $\chi = \chi(s)$ which is a unit speed curve. Let PAFORS be shown with $\{\mathbf{T}_\chi, \mathbf{G}_\chi, \mathbf{H}_\chi\}$ for $\chi = \chi(s)$. Then, special trajectories generated by $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache curves may be defined as follows:

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\chi + \mathbf{G}_\chi\right). \tag{3.1}$$

For convenience, we call these trajectories as $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache trajectories.

Note that PAFORS apparatus of $\chi = \chi(s)$ will be denoted by $\{\mathbf{T}_\chi(s), \mathbf{G}_\chi(s), \mathbf{H}_\chi(s), k_1(s), k_2(s), k_3(s)\}$ in the rest of the paper.

Now, we will discuss Serret-Frenet apparatus of $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache trajectories. Differentiating the Equation 3.1 with respect to the arc-length parameter s of $\chi = \chi(s)$, we obtain

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\,\mathbf{H}_\chi\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\,\mathbf{H}_\chi\right). \tag{3.2}$$

From the Equation 3.2, one can easily find

$$\frac{ds^*}{ds} = \sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}}. \tag{3.3}$$

Thus, we can rewrite the Equation 3.2 as

$$\mathbf{T}_\sigma\sqrt{{k_1}^2 + \frac{(k_2+k_3)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\,\mathbf{H}_\chi\right). \tag{3.4}$$

The Equation 3.4 gives us the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_1}^2 + (k_2+k_3)^2}}\left(-k_1\mathbf{T}_\chi + k_1\mathbf{G}_\chi + (k_2+k_3)\,\mathbf{H}_\chi\right). \tag{3.5}$$

Differentiating the last equation with respect to the arc-length parameter s of $\chi = \chi(s)$, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_1}^2 + (k_2+k_3)^2\right)^{-3/2}\left(\xi_1\mathbf{T}_\chi + \xi_2\mathbf{G}_\chi + \xi_3\mathbf{H}_\chi\right) \tag{3.6}$$

where

$$\begin{aligned}
\xi_1 &= (k_2+k_3)\left[k_1k'_2 + k_1k'_3 - {k_1}^2k_2 - {k_1}^2k_3 - k'_1(k_2+k_3) - k_2\left(2{k_1}^2 + (k_2+k_3)^2\right)\right] - 2{k_1}^4 \\
\xi_2 &= (k_2+k_3)\left[-k_1k'_2 - k_1k'_3 - {k_1}^2k_2 - {k_1}^2k_3 + k'_1(k_2+k_3) - k_3\left(2{k_1}^2 + (k_2+k_3)^2\right)\right] - 2{k_1}^4 \\
\xi_3 &= k_1(k_2+k_3)\left[-2k'_1 - {k_2}^2 + {k_3}^2\right] + 2{k_1}^2\left[k'_2 + k'_3 + k_1k_3 - k_1k_2\right].
\end{aligned}$$

Considering the Equation 3.3 in the Equation 3.6, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2{k_1}^2 + (k_2+k_3)^2\right)^{-2}\left(\xi_1\mathbf{T}_\chi + \xi_2\mathbf{G}_\chi + \xi_3\mathbf{H}_\chi\right).$$

In that case, the curvature and principal normal vector of σ are obtained as in the following:

$$\kappa_\sigma = \left\|\frac{d\mathbf{T}_\sigma}{ds^*}\right\| = \frac{\sqrt{2\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}{\left(2{k_1}^2+(k_2+k_3)^2\right)^2}$$

and

$$\mathbf{N}_\sigma = \frac{1}{\sqrt{{\xi_1}^2+{\xi_2}^2+{\xi_3}^2}}\left(\xi_1\mathbf{T}_\chi + \xi_2\mathbf{G}_\chi + \xi_3\mathbf{H}_\chi\right).$$

Where

$$\begin{aligned}
\zeta_1 &= k_1\xi_3 - k_2\xi_2 - k_3\xi_2 \\
\zeta_2 &= k_2\xi_1 + k_3\xi_1 + k_1\xi_3 \\
\zeta_3 &= -k_1\xi_2 - k_1\xi_1,
\end{aligned}$$

we can get the binormal vector of σ as

$$\begin{aligned}
\mathbf{B}_\sigma &= \frac{1}{\sqrt{\left(2{k_1}^2+(k_2+k_3)^2\right)\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}\begin{vmatrix} \mathbf{T}_\chi & \mathbf{G}_\chi & \mathbf{H}_\chi \\ -k_1 & k_1 & k_2+k_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix} \\
&= \frac{1}{\sqrt{\left(2{k_1}^2+(k_2+k_3)^2\right)\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)}}\left(\zeta_1\mathbf{T}_\chi + \zeta_2\mathbf{G}_\chi + \zeta_3\mathbf{H}_\chi\right)
\end{aligned}$$

by vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$.

3.2. Special Trajectories Generated by TH Smarandache Curves

Definition 3.2. In E^3, suppose that a point particle P of constant mass moves on the regular surface M along the trajectory $\chi = \chi(s)$ which is a unit speed curve. Let PAFORS be denoted by $\{\mathbf{T}_\chi, \mathbf{G}_\chi, \mathbf{H}_\chi\}$ for $\chi = \chi(s)$. In this case, special trajectories generated by $\mathbf{T}_\chi\mathbf{H}_\chi$−Smarandache curves may be defined by

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{T}_\chi + \mathbf{H}_\chi\right). \tag{3.7}$$

For convenience, we call these trajectories as $\mathbf{T}_\chi\mathbf{H}_\chi$−Smarandache trajectories.

Now, we will investigate Serret-Frenet apparatus of $\mathbf{T}_\chi\mathbf{H}_\chi$−Smarandache trajectories. Differentiating the Equation 3.7 with respect to the arc-length parameter s of $\chi = \chi(s)$, we find

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\chi + (k_1-k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right)$$

and so

$$\mathbf{T}_\sigma \frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\chi + (k_1 - k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right). \tag{3.8}$$

From the Equation 3.8, one can easily obtain

$$\frac{ds^*}{ds} = \sqrt{{k_2}^2 + \frac{(k_1-k_3)^2}{2}}. \tag{3.9}$$

Therefore we can rewrite the Equation 3.8 as in the following:

$$\mathbf{T}_\sigma \sqrt{{k_2}^2 + \frac{(k_1-k_3)^2}{2}} = \frac{1}{\sqrt{2}}\left(-k_2\mathbf{T}_\chi + (k_1 - k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right). \tag{3.10}$$

The Equation 3.10 yields the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_2}^2 + (k_1-k_3)^2}}\left(-k_2\mathbf{T}_\chi + (k_1 - k_3)\,\mathbf{G}_\chi + k_2\mathbf{H}_\chi\right). \tag{3.11}$$

Differentiating the Equation 3.11 with respect to s, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_2}^2 + (k_1-k_3)^2\right)^{-3/2}\left(\mu_1\mathbf{T}_\chi + \mu_2\mathbf{G}_\chi + \mu_3\mathbf{H}_\chi\right) \tag{3.12}$$

where

$$\begin{aligned}
\mu_1 &= (k_3-k_1)\left[-k_2\left(k'_1 - k'_3\right) + 2k_1{k_2}^2 - k'_2\,(k_3-k_1) - {k_2}^2\,(k_3-k_1) + k_1(k_3-k_1)^2\right] - 2{k_2}^4\\
\mu_2 &= k_2\,(k_1-k_3)\left[-2k'_2 - {k_1}^2 + {k_3}^2 k_2\left(k'_1 - k'_3\right)\right] + 2{k_2}^2\left[k'_1 - k'_3 - k_1k_2 - k_2k_3\right]\\
\mu_3 &= (k_3-k_1)\left[k_2\left(k'_1 - k'_3\right) - 2k_3{k_2}^2 + k'_2\,(k_3-k_1) - {k_2}^2\,(k_3-k_1) - k_3(k_3-k_1)^2\right] - 2{k_2}^4.
\end{aligned}$$

Taking into consideration the Equation 3.9 in the Equation 3.12, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2{k_2}^2 + (k_1-k_3)^2\right)^{-2}\left(\mu_1\mathbf{T}_\chi + \mu_2\mathbf{G}_\chi + \mu_3\mathbf{H}_\chi\right).$$

In this case, the curvature and principal normal vector of σ are obtained as follows:

$$\kappa_\sigma = \left\| \frac{d\mathbf{T}_\sigma}{ds^*} \right\| = \frac{\sqrt{2\left({\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2\right)}}{\left(2{k_2}^2 + (k_1-k_3)^2\right)^2}$$

and

$$\mathbf{N}_\sigma = \frac{1}{\sqrt{{\mu_1}^2 + {\mu_2}^2 + {\mu_3}^2}}\left(\mu_1\mathbf{T}_\chi + \mu_2\mathbf{G}_\chi + \mu_3\mathbf{H}_\chi\right).$$

Where

$$\begin{aligned}\eta_1 &= k_1\mu_3 - k_3\mu_3 - k_2\mu_2\\ \eta_2 &= k_2\mu_1 + k_2\mu_3\\ \eta_3 &= -k_2\mu_2 + k_3\mu_1 - k_1\mu_1,\end{aligned}$$

we can immediately obtain the binormal vector of σ as

$$\begin{aligned}\mathbf{B}_\sigma &= \frac{1}{\sqrt{\left(2k_2{}^2 + (k_1-k_3)^2\right)\left(\mu_1{}^2+\mu_2{}^2+\mu_3{}^2\right)}}\begin{vmatrix}\mathbf{T}_\chi & \mathbf{G}_\chi & \mathbf{H}_\chi\\ -k_2 & k_1-k_3 & k_2\\ \mu_1 & \mu_2 & \mu_3\end{vmatrix}\\ &= \frac{1}{\sqrt{\left(2k_2{}^2 + (k_1-k_3)^2\right)\left(\mu_1{}^2+\mu_2{}^2+\mu_3{}^2\right)}}\left(\eta_1\mathbf{T}_\chi + \eta_2\mathbf{G}_\chi + \eta_3\mathbf{H}_\chi\right)\end{aligned}$$

by vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$.

3.3. Special Trajectories Generated by GH Smarandache Curves

Definition 3.3. In E^3, assume that a point particle P of constant mass moves on the regular surface M along the trajectory $\chi = \chi(s)$ which is a unit speed curve. Let $\{\mathbf{T}_\chi, \mathbf{G}_\chi, \mathbf{H}_\chi\}$ be PAFORS for $\chi = \chi(s)$. Then, special trajectories generated by $\mathbf{G}_\chi\mathbf{H}_\chi-$Smarandache curves can be defined as follows:

$$\sigma(s^*) = \frac{1}{\sqrt{2}}\left(\mathbf{G}_\chi + \mathbf{H}_\chi\right). \tag{3.13}$$

For convenience, we call these trajectories as $\mathbf{G}_\chi\mathbf{H}_\chi-$Smarandache trajectories.

Now, we will investigate Serret-Frenet apparatus of $\mathbf{G}_\chi\mathbf{H}_\chi-$Smarandache trajectories. Differentiating the Equation 3.13 with respect to the arc-length parameter s of $\chi = \chi(s)$, we get

$$\sigma' = \frac{d\sigma}{ds^*}\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right)$$

and so

$$\mathbf{T}_\sigma\frac{ds^*}{ds} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right). \tag{3.14}$$

From the Equation 3.14, one can easily obtain

$$\frac{ds^*}{ds} = \sqrt{k_3{}^2 + \frac{(k_1+k_2)^2}{2}}. \tag{3.15}$$

Therefore we can rewrite the Equation 3.14 as in the following:

$$\mathbf{T}_\sigma\sqrt{k_3{}^2 + \frac{(k_1+k_2)^2}{2}} = \frac{1}{\sqrt{2}}\left((-k_1-k_2)\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right). \tag{3.16}$$

The Equation 3.16 yields the tangent vector of σ:

$$\mathbf{T}_\sigma = \frac{1}{\sqrt{2{k_3}^2 + (k_1+k_2)^2}}\left((-k_1-k_2)\,\mathbf{T}_\chi - k_3\mathbf{G}_\chi + k_3\mathbf{H}_\chi\right). \tag{3.17}$$

Differentiating the Equation 3.17 with respect to s, we get

$$\frac{d\mathbf{T}_\sigma}{ds^*}\frac{ds^*}{ds} = \left(2{k_3}^2 + (k_1+k_2)^2\right)^{-3/2}\left(\upsilon_1\mathbf{T}_\chi + \upsilon_2\mathbf{G}_\chi + \upsilon_3\mathbf{H}_\chi\right) \tag{3.18}$$

where

$$\begin{aligned}
\upsilon_1 &= k_3\,(k_1+k_2)\left[2k'_3 + {k_1}^2 - {k_2}^2\right] + 2{k_3}^2\left[k_1k_3 - k_2k_3 - k'_1 - k'_2\right]\\
\upsilon_2 &= (k_1+k_2)\left[k_3\left(k'_1+k'_2\right) - 2k_1{k_3}^2 - k'_3\,(k_1+k_2) - {k_3}^2\,(k_1+k_2) - k_1(k_1+k_2)^2\right] - 2{k_3}^4\\
\upsilon_3 &= (k_1+k_2)\left[-k_3\left(k'_1+k'_2\right) - 2k_2{k_3}^2 + k'_3\,(k_1+k_2) - {k_3}^2\,(k_1+k_2) - k_2(k_1+k_2)^2\right] - 2{k_3}^4.
\end{aligned}$$

Taking into consideration the Equation 3.15 in the Equation 3.18, we find

$$\frac{d\mathbf{T}_\sigma}{ds^*} = \sqrt{2}\left(2{k_3}^2 + (k_1+k_2)^2\right)^{-2}\left(\upsilon_1\mathbf{T}_\chi + \upsilon_2\mathbf{G}_\chi + \upsilon_3\mathbf{H}_\chi\right).$$

In this case, the curvature and principal normal vector of σ are obtained as follows:

$$\begin{aligned}
\kappa_\sigma &= \frac{\sqrt{2\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}{\left(2{k_3}^2 + (k_1+k_2)^2\right)^2}\\
\mathbf{N}_\sigma &= \frac{1}{\sqrt{{\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2}}\left(\upsilon_1\mathbf{T}_\chi + \upsilon_2\mathbf{G}_\chi + \upsilon_3\mathbf{H}_\chi\right).
\end{aligned}$$

By vector product of $\mathbf{T}_\sigma$ and $\mathbf{N}_\sigma$, we can immediately obtain the binormal vector of σ as

$$\begin{aligned}
\mathbf{B}_\sigma &= \frac{1}{\sqrt{\left(2{k_3}^2 + (k_1+k_2)^2\right)\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}\begin{vmatrix} \mathbf{T}_\chi & \mathbf{G}_\chi & \mathbf{H}_\chi \\ -k_1-k_2 & -k_3 & k_3 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \end{vmatrix}\\
&= \frac{1}{\sqrt{\left(2{k_3}^2 + (k_1+k_2)^2\right)\left({\upsilon_1}^2 + {\upsilon_2}^2 + {\upsilon_3}^2\right)}}\left(\psi_1\mathbf{T}_\chi + \psi_2\mathbf{G}_\chi + \psi_3\mathbf{H}_\chi\right)
\end{aligned}$$

where

$$\begin{aligned}
\psi_1 &= -k_3\upsilon_3 - k_3\upsilon_2\\
\psi_2 &= k_3\upsilon_1 + k_2\upsilon_3 + k_1\upsilon_3\\
\psi_3 &= -k_1\upsilon_2 - k_2\upsilon_2 + k_3\upsilon_1.
\end{aligned}$$

Note that the torsions of $\mathbf{T}_\chi\mathbf{G}_\chi$, $\mathbf{T}_\chi\mathbf{H}_\chi$, $\mathbf{G}_\chi\mathbf{H}_\chi$-Smarandache trajectories can be obtained by following the similar steps given in this section. We leave this to the readers.

Example 3.4. In E^3, assume that a point particle P of constant mass moves on the regular surface

$$M = \left\{(x,\, y,\, z) :\ x^2 + y^2 = 64,\ z \geq 0\right\}$$

along the trajectory

$$\delta:(0,255)\to M\subset E^3,\ \ \delta(t)=\left(8\cos\frac{t}{17},8\sin\frac{t}{17},\frac{t}{17}\right).$$

Reparameterization of $\delta=\delta(t)$ in terms of arc-length parameter is given as follows:

$$\chi(s)=\left(8\cos\frac{s}{\sqrt{65}},8\sin\frac{s}{\sqrt{65}},\frac{s}{\sqrt{65}}\right)$$

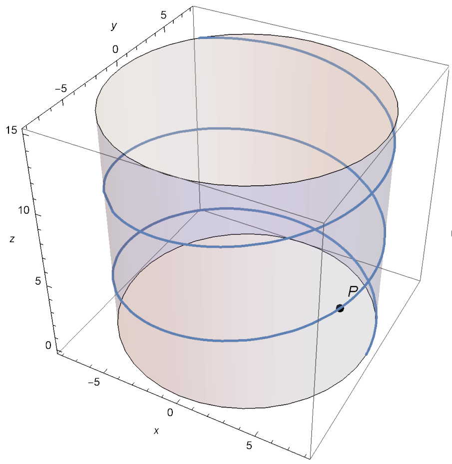
where $s=\frac{\sqrt{65}}{17}t$. One can easily calculate Darboux apparatus of this trajectory as in the following:

$$\begin{aligned}
\mathbf{T}(s)&=\left(\frac{-8}{\sqrt{65}}\sin\frac{s}{\sqrt{65}},\frac{8}{\sqrt{65}}\cos\frac{s}{\sqrt{65}},\frac{1}{\sqrt{65}}\right)\\
\mathbf{U}(s)&=\left(\cos\frac{s}{\sqrt{65}},\sin\frac{s}{\sqrt{65}},0\right)\\
\mathbf{Y}(s)&=\left(\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}},\frac{-1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}},\frac{8}{\sqrt{65}}\right)\\
k_g(s)&=0\\
k_n(s)&=\frac{-8}{65}\\
\tau_g(s)&=\frac{1}{65}.
\end{aligned}$$

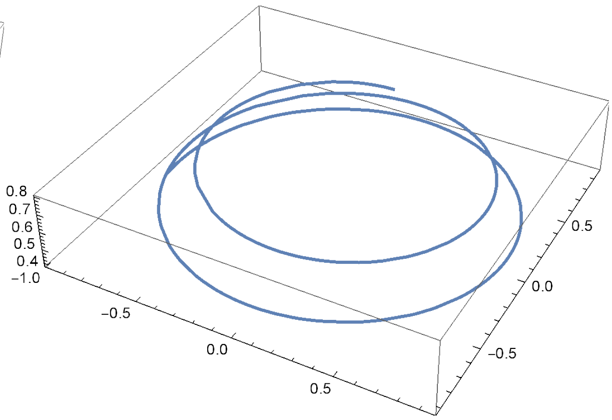
Then, $\langle\chi(s),\mathbf{Y}(s)\rangle=\frac{8}{65}s$ and $\langle\chi(s),\mathbf{U}(s)\rangle=8$ are obtained. Since $\langle\chi(s),\mathbf{U}(s)\rangle>0$ for all the values of parameter, we get $\Omega(s)=\arctan\left(-\frac{s}{65}\right)$. The above information yields the PAFORS apparatus as follows:

$$\begin{aligned}
\mathbf{T}(s)&=\left(\frac{-8}{\sqrt{65}}\sin\frac{s}{\sqrt{65}},\frac{8}{\sqrt{65}}\cos\frac{s}{\sqrt{65}},\frac{1}{\sqrt{65}}\right)\\
\mathbf{G}(s)&=\begin{pmatrix}\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)-\cos\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{-1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)-\sin\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{8}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)\end{pmatrix}\\
\mathbf{H}(s)&=\begin{pmatrix}\frac{1}{\sqrt{65}}\sin\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)+\cos\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{-1}{\sqrt{65}}\cos\frac{s}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)+\sin\frac{s}{\sqrt{65}}\cos\left(\arctan\left(\frac{-s}{65}\right)\right),\\ \frac{8}{\sqrt{65}}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)\end{pmatrix}\\
k_1(s)&=\frac{8}{65}\sin\left(\arctan\left(\frac{-s}{65}\right)\right)\\
k_2(s)&=-\frac{8}{65}\cos\left(\arctan\left(\frac{-s}{65}\right)\right)\\
k_3(s)&=\frac{1}{65}+\frac{65}{s^2+65^2}
\end{aligned}$$

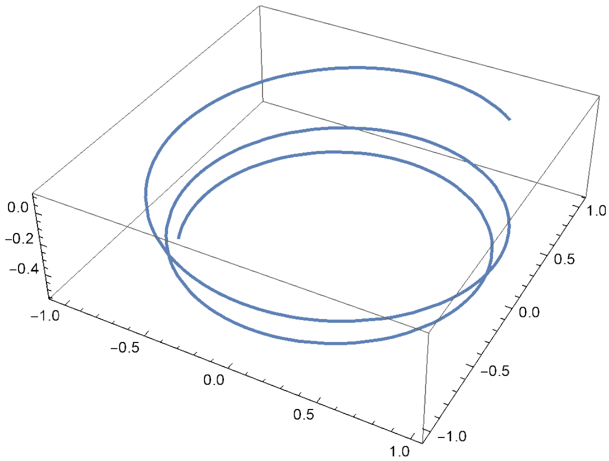
in the light of the Equation 2.10 and Equation 2.12. We can give the following figure for this example:



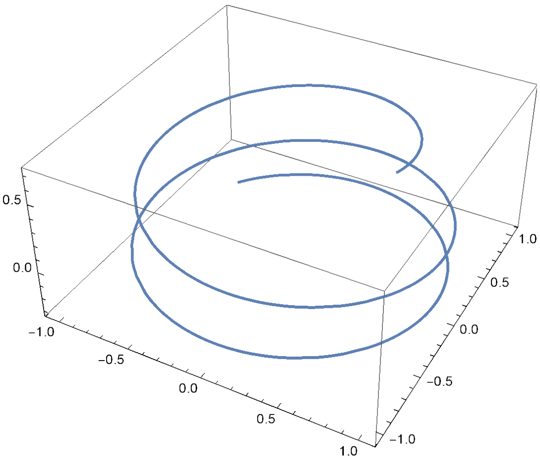
(a) The trajectory of the particle P



(b) $\mathbf{T}_\chi\mathbf{G}_\chi-$Smarandache trajectory



(c) $\mathbf{T}_\chi\mathbf{H}_\chi-$Smarandache trajectory



(d) $\mathbf{G}_\chi\mathbf{H}_\chi-$Smarandache trajectory

Figure 2. An illustration including special Smarandache trajectories

Acknowledgement

The authors thank Assistant Professor Furkan Semih Dündar for his help in drawing the Figure 2.

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