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Research Article

Some Properties of Amply Weak Essential Supplemented Modules

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Abstract

An investigation on the features of amply weak essential supplemented modules are presented in this paper. Assume that *K* is a weakly essential supplemented and projective *R*-module and *M* is a finitely *K*-generated *R*-module. Then *M* gets amply weak essential supplemented. Assume that *R* is a ring. In this case each finitely generated *R*-module is amply weak essential supplemented iff $_RR$ is weakly essential supplemented. If the module *K* is amply weak essential supplemented, then each e-supplement submodule in *K* is amply weak essential supplemented.

Keywords: Small Submodules , Essential Submodules, Supplemented Modules, Essential Supplemented Modules.

Bol Zayıf Büyük Tümlenmiş Modüllerin Birtakım Özellikleri

Öz

Bu çalışmada bol zayıf büyük tümlenmiş modüllerle ilgili birtakım özellikler araştırıldı. Eğer *K* projektif ve zayıf büyük tümlenmiş bir *R*-modül ve *M* sonlu *K*-üretilmiş bir *R*-modül olsun. Bu durumda *M* bol zayıf büyük tümlenmiştir. Kabul edelim ki *R* bir halka olsun. Bu durumda $_RR$ modülü zayıf büyük tümlenmiştir gerek ve yeter şart her sonlu üretilmiş *R*-modülü bol zayıf büyük tümlenmiştir. Eğer *K* bir bol zayıf büyük tümlenmiş modül ise *K* modülünde her e-tümleyen alt modül bol zayıf büyük tümlenmiştir.

Anahtar Kelimeler: Büyük Tümlenmiş Modüller, Tümlenmiş Modüller, Küçük Alt Modüller, Büyük Alt Modüller.

1. Introduction

All rings are associative with an identity element, in this study. Unless otherwise specified, *R* represents an arbitrary ring and every module will be a left unitary *R*-module. We refer to a submodule *L* of *K* as $L \leq K$ in this section. A submodule *N* of *K* is said to be a *small* (or *superfluous*) in *K*, *if* $T=K$ for every submodule *T* of *K* such that $K=N+T$. This submodule of *K* is indicated by the symbol $N \ll K$. $T \leq K$ is referred to as an *essential* (or *large*), and it is indicated by $T \trianglelefteq K$, in this case $M \cap T \neq 0$ for every submodule $M \neq 0$, or equivalently, $T \cap X=0$ for $X \leq K$ implies that $X=0$. Let $T,L \leq K$. If $K=T+L$ and *L* is minimal with respect to this property, or alternatively, if $K=T+L$ and $T \cap L \ll L$, then *L* is called a *supplement* of *T* in *K*. If each submodule of *K* has a supplement in *K*, then *K* is referred to as *supplemented*. From now on, we will use the *sM* notation instead of the supplemented module. *K* is called *essential*

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supplemented (or, shortly, *esM*) if it has a supplement for each of its essential submodules. Let $U \leq K$. If for each $U' \leq K$ with $K=U+U'$, there exists $X \leq U'$ which X is a supplement of U in K, it is said that U has *ample supplements* in K. K is referred to as *amply supplemented* if each submodule of K has ample supplements in K. From now on, we will use the *asM* notation instead of the amply supplemented module. K is termed *amply essential supplemented* (or shortly, *ae-sM*) if each essential submodule of K gets ample supplements in K. Let $T,L \leq K$. If $K=T+L$ and $T \cap L \ll K$, then L is referred to as a *weak supplement* of T in K. K is referred to as *weakly supplemented* provided that each submodule of K has a weak supplement in K. From now on, we will use the *wsM* notation instead of the weakly supplemented module. K is said to be *weakly essential supplemented* (or shortly, *we-sM*) provided that each essential submodule of K has a weak supplement in K. The *radical* of K, represented by $RadK$, is the intersection of all maximal submodules of K. We indicate $RadK=K$ if K have no maximal submodules. We define as β^* relation on the set of submodules of K by $T\beta^*L$ with $T \leq K$ and $L \leq K$ if and only if for each $B \leq K$ with $T+B=K$ then $L+B=K$ and for each $M \leq K$ with $L+M=K$ then $T+M=K$. Let $M \leq L \leq K$, if $L/M \ll K/M$, then it is called *L lies above M* in K. An R-module K is said to be *π-projective* if for every $T,L \leq K$ with $K=T+L$ there exists an R-module homomorphism $f: K \to K$ such that $f(K) \leq T$ and $(1-f)(K) \leq L$. Assume K and N be R-modules. If there exists an R-module epimorphism $f: K^{(\Lambda)} \to N$ with the index set Λ, then N is referred to as a *K-generated* module. If there is an R-module epimorphism $f: K^{(\Lambda)} \to N$ with a finite index set Λ, then N is said to be *finitely K-generated*.

Additional details regarding (amply) supplemented modules are in Clark et al. (2006), Nebiyev and Pancar (2003b), Nebiyev and Pancar (2013), Nebiyev and Sökmez (2010) and (Wisbauer, 1991). The statement of β^* relation and some features of this relation are in Birkenmeier et al. (2010) and (Sökmez et al., 2008). Additional details regarding π-projective modules are in Nebiyev and Pancar (2003a) and (Wisbauer, 1991). More details about weakly supplemented modules are in Lomp (1999) and (Nebiyev, 2005). More informations about (amply) *e-sM* are in Nebiyev (2016a), Nebiyev (2017) and (Nebiyev et al., 2018a and b). The definition of *we-sM* are in Nebiyev (2016b) and (Nebiyev and Koşar, 2018).

In this part, we will provide the definitions and propositions required for our work.

Lemma 1.1. Assume that K is an R-module.

(1) If $M \leq L \leq K$, then $M \trianglelefteq K$ if and only if $M \trianglelefteq L \trianglelefteq K$.

(2) For $M \leq L \leq K$, if $L/M \trianglelefteq K/M$, then $L \trianglelefteq K$.

(3) Assume that N is an R-module and $h: K \to N$ be an R-module homomorphism. If $M \trianglelefteq N$, then $h^{-1}(M) \trianglelefteq K$.

(4) If $M_1 \trianglelefteq L_1 \leq K$ and $M_2 \trianglelefteq L_2 \leq K$, then $M_1 \cap M_2 \trianglelefteq L_1 \cap L_2$.

(5) If $M_1 \trianglelefteq K$ and $M_2 \trianglelefteq K$, then $M_1 \cap M_2 \trianglelefteq K$.

Proof. See Wisbauer (1991), 17.3.

Definition 1.2. Assume that K is an R-module and $T \leq K$. T is referred to as an *e-supplement* (shortly, *e-s*) submodule in K provided that there exists $L \trianglelefteq K$ which T is a supplement of L in K (See (Nebiyev et al., 2018a)).

Definition 1.3. Assume that K is an R-module and $T \leq K$. T is referred to as a *weak e-supplement* (shortly, *we-s*) submodule in K if there exists $L \trianglelefteq K$ which T is a weak supplement of L in K (See (Nebiyev and Koşar, 2018)).

Lemma 1.4. Suppose K is an R-module, L is a supplement of T in K and $S,F \leq L$. If F is a weak supplement of $T+S$ in K, then F is a weak supplement of S in L.

Proof. Due to the fact that F is a weak supplement of $T+S$ in K, $K=T+S+F$ and $(T+S)\cap F<<K$. Because $K=T+S+F$ and $S+F \leq L$ and L is a supplement of T in K, $L=S+F$. Since $S\cap F \leq (T+S)\cap F<<K$ and L is a supplement of T in K, $S\cap F<<L$. Thus F is a weak supplement of S in L, as desired.

Lemma 1.5. Suppose that K is an R-module. K is *we-sM* provided that for each $T \trianglelefteq K$, T is β^* equivalent to a weak supplement submodule in K (See Nebiyev and Koşar (2018), Lemma 2.13).

2. Amply Weak Essential Supplemented Modules (*awe-sM*)

Definition 2.1. Assume that K is an R-module and $T \leq K$. If for each $L \leq K$ with $K=T+L$, T has a weak supplement X in K with $X \leq L$, we say T has *ample weak supplements* in K (See (Nebiyev, 2005). K is referred to as *amply weak essential supplemented* (or shortly, *awe-sM*) provided that for each $T \trianglelefteq K$, T has ample weak supplements in K (See also (Nebiyev and Ökten, 2017)).

Every *awe-sM* is clearly a *we-sM*, as can be shown.

Lemma 2.2. Assume that K is an *awe-sM* and L be an *e-s* submodule in K. Then L is also an *awe-sM*.

Proof. Let $T \trianglelefteq K$ and L be a supplement of T in K. Let $L=S+F$ with $S \trianglelefteq L$ and $F \leq L$. Then $K=T+L=T+S+F$. Due to fact that K is *awe-sM* and $T+S \trianglelefteq K$, $T+S$ has a weak supplement Z in K with $Z \leq F$. By Lemma 1.4, Z is a weak supplement of S in L. Moreover, $Z \leq F$. Hence S has ample weak supplements in L and L is an *awe-sM*.

Corollary 2.3. Assume that K is an *awe-sM* and V be an *e-s* submodule in K. Then V is a *we-sM*.

Proof. From Lemma 2.2, it's clear.

Lemma 2.4. Every factor module of an *awe-sM* is an *awe-sM* (See also (Nebiyev and Ökten, 2017)).

Proof. Assume that K is an *awe-sM* and K/Z is a factor module of K. Let $T/Z \trianglelefteq K/Z$ and $K/Z=T/Z+L/Z$ with $L/Z \leq K/Z$. Due to the fact that $T/Z \trianglelefteq K/Z$, by Lemma 1.1, $T \trianglelefteq K$. Due to the fact that $K/Z=T/Z+L/Z=(T+L)/Z$, $K=T+L$ and since K is an *awe-sM* and $T \trianglelefteq K$, T has a weak supplement N in K with $N \leq L$. Since N is a weak supplement of T in K, $(N+Z)/Z$ is a weak supplement of T/Z in K/Z. Moreover, $(N+Z)/Z \leq L/Z$. As a result, K/Z is an *awe-sM*.

Corollary 2.5. Each homomorphic image of an *awe-sM* is an *awe-sM* (See also (Nebiyev and Ökten, 2017)).

Proof. It is clear from Lemma 2.4.

Proposition 2.6. Assume that K is a *we-sM* and $T,L \trianglelefteq K$ with $K=T+L$. Then T has a weak supplement N in K with $N \leq L$.

Proof. Since $T,L \trianglelefteq K$, by Lemma 1.1, $T \cap L \trianglelefteq K$. Due to the fact that K is a *we-sM*, $T \cap L$ has a weak supplement N in K. Here $K=T \cap L+N$ and $T \cap L \cap N << K$. Due to the fact that $K=T \cap L+N$, $L=T \cap L+L \cap N$. Let $Z=L \cap N$. Thus, $K=T+L=T+T \cap L+L \cap N=T+Z$ and $T \cap Z=T \cap L \cap Z << K$. As a result, Z is a weak supplement of T in K with $Z \leq L$.

Lemma 2.7. Assume that K is a module. K is an *awe-sM* provided that every submodule of K is a *we-sM* (See also (Nebiyev and Ökten, 2017)).

Proof. Let $U \trianglelefteq K$ and $K=U+V$ with $V \leq K$. Since $U \trianglelefteq K$, $U \cap V \trianglelefteq V$. By hypothesis, V is a *we-sM*. Hence $U \cap V$ has a weak supplement X in V. Here $V=U \cap V+X$ and $U \cap V \cap X << V$. Then $K=U+V=U+U \cap V+X=U+X$ and $U \cap X=U \cap V \cap X << K$. Therefore, X is a weak supplement of U in K. Moreover, $X \leq V$. As a result, K is an *awe-sM*.

Lemma 2.8. Assume that R is any ring. Then each R-module is a *we-sM* iff each R-module is an *awe-sM*.

Proof. ($\Rightarrow$) Assume that K is any R-module. Since each R-module is a *we-sM*, each submodule of K is a *we-sM*. Then from Lemma 2.7, K is an *awe-sM*, as desired.

($\Leftarrow$) It is clear, since each *awe-sM* is a *we-sM*.

Lemma 2.9. Assume that K is a π-projective and *we-sM*. Then K is an *awe-sM*.

Proof. Let $K=T+L$ with $T \trianglelefteq K$ and $L \leq K$. Because K is a *we-sM*, T has a weak supplement N in K. Here $K=T+N$ and $T \cap N << K$. Since K is π-projective, there exists an R-module homomorphism $f: K \to K$ such that $f(K) \leq L$ and $(1-f)(K) \leq T$. We can see here that $f(T) \leq T$. Then $K=f(K)+(1-f)(K)=T+f(T+N)=T+f(T)+f(N)=T+f(N)$. Let $t \in T \cap f(N)$. Then $t \in T$ and $t \in f(N)$. Since $t \in f(N)$, there exists $z \in N$ with $t=f(z)$. Since $(1-f)(z) \in T$, $z=z-f(z)+f(z)=(1-f)(z)+t \in T$. Hence $z \in T \cap N$ and $t=f(z) \in f(T \cap N)$. Therefore, $T \cap f(N) \leq f(T \cap N)$. Because, $T \cap T << K$, $f(T \cap N) << K$. Hence $T \cap f(N) << K$ and $f(N)$ is a weak supplement of T in K. Furthermore, $f(N) \leq L$. Hence K is an *awe-sM*.

Corollary 2.10. Assume that K is a projective and *we-sM*. Then K is an *awe-sM*.

Proof. It is obvious from Lemma 2.9, since each projective module is π-projective.

Lemma 2.11. Assume that K is a π-projective module. In this case when for every $T \trianglelefteq K$ there exists a weak supplement submodule Z in K with $T\beta^{*}Z$, then K is an *awe-sM*.

Proof. Since for every $T \trianglelefteq K$ there exists a weak supplement submodule Z in K with $T\beta^{*}Z$, from Lemma 1.5, K is an *we-sM*. From Lemma 2.9, K is an *awe-sM*.

Corollary 2.12. Assume that K is a π-projective module. K is an *awe-sM* provided that each essential submodule of K is β^{*} equivalent to a *we-s* in K.

Proof. The proof is obvious from Lemma 2.11.

Corollary 2.13. Assume that K is a π-projective module. K is an *awe-sM* provided that each essential submodule of K lies above a weak supplement submodule in K.

Proof. The proof is obvious from Lemma 2.11.

Corollary 2.14. Assume that K is a π-projective module. K is an *awe-sM* provided that each essential submodule of K is a weak supplement submodule in K.

Proof. The proof is obvious from Lemma 2.11.

Lemma 2.15. Assume that K is a projective and *we-sM*. Then $K^{(\Lambda)}$ is an *awe-sM* for every finite index set Λ.

Proof. Since K is projective, $K^{(\Lambda)}$ is also projective. Since K is a *we-sM*, by Nebiyev and Koşar (2018) Corollary 2.8, $K^{(\Lambda)}$ is also a *we-sM*. Then, in accordance with Corollary 2.10, $K^{(\Lambda)}$ is an *awe-sM*.

Corollary 2.16. Assume that K is a projective and *we-sM*. Then every finitely K-generated module is an *awe-sM*.

Proof. Assume N is any finitely K-generated R-module. Then there exists an R-module epimorphism $f : K^{(\Lambda)} \to N$ with finite index set Λ. $K^{(\Lambda)}$ is an *awe-sM* according to Lemma 2.15. Then in accordance with Corollary 2.5, N is an *awe-sM*, as desired.

Proposition 2.17. Assume that R is a ring. The statements below are equivalent.

(i) ${}_RR$ is a *we-sM*.

(ii) ${}_RR$ is an *awe-sM*.

(iii) ${}_RR^{(\Lambda)}$ is a *we-sM*, for each finite index set Λ.

(iv) ${}_RR^{(\Lambda)}$ is an *awe-sM*, for each finite index set Λ.

(v) Each finitely generated R-module is a *we-sM*.

(vi) Each finitely generated R-module is an *awe-sM*.

Proof. (i)$\Rightarrow$(ii) It is obvious from Corollary 2.10, because ${}_RR$ is projective.

(ii)$\Rightarrow$(i) It is obvious from definitions.

(i)$\Rightarrow$(iii) It is obvious from Nebiyev and Koşar (2018), Corollary 2.8.

(iii)$\Rightarrow$(v) Clear from Nebiyev and Koşar (2018), Corollary 2.10.

(v)$\Rightarrow$(i) Clear.

(i)$\Rightarrow$(iv) Clear from Lemma 2.15, since $_RR$ is projective.

(iv)$\Rightarrow$(vi) Clear from Corollary 2.5.

(vi)$\Rightarrow$(ii) Clear.

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