Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (1) (2022), 27 – 47
DOI : 10.15672/hujms.885150

Research Article

High order approach for solving chaotic and hyperchaotic problems

Meltem Evrenosoglu Adiyaman

Department of Mathematics, Faculty of Sciences, Dokuz Eylül University, Tınaztepe, Buca, 35160 Izmir, Turkey

Abstract

In this work, the method of Taylor's decomposition on two points is suggested in order to find approximate solutions of chaotic and hyperchaotic initial value problems and to analyze the behaviors of these solutions. Unlike to the classical Taylor's method, the proposed numerical scheme is based on the application of the Taylor's decomposition on two points to the system of nonlinear initial value problems, and as a result an implicit method is obtained. Stability and error analysis of the method are presented, and its high-order accuracy and A-stability are proven. One of the advantages of the proposed method is that it is a stable and very efficient method for chaotic problems as it is an implicit one-step method. The most important advantage of the Taylor's decomposition method is that it has high order accuracy for large step sizes with a simple algorithm compared to other methods. The applicability of the proposed method has been examined in some famous chaotic systems; the Lorenz and Chen systems, and hyperchaotic systems; the Chua and Rabinovich-Fabrikant systems, to emphasize both its accuracy and effectiveness. The accuracy of the proposed method is checked by comparing the calculated results with semi-explicit Adams-Bashforth-Moulton method and ninth order Runge-Kutta method. The calculated results are also compared with multi-stage spectral relaxation method and multi-domain compact finite difference relaxation method. Comparisons have shown that the method is more accurate and efficient than the other mentioned methods for large step sizes. The obtained results are also compared with the theoretical findings and it is shown that the theoretical and numerical results are consistent.

Mathematics Subject Classification (2020). 65L05, 65L70, 65P20, 37M05

Keywords. system of nonlinear initial value problems, Taylor's decomposition method on two points, chaotic and hyperchaotic systems, Lorenz system, Chen system, Chua system, Rabinovich-Fabrikant system

Email address: meltem.evrenosoglu@deu.edu.tr
Received: 22.03.2021; Accepted: 05.08.2021

1. Introduction

The theory of chaos was first introduced by Lorenz [20] in order to demonstrate the chaotic behaviour in a system of ordinary differential equations modelling the atmospheric convection. This was the first paper to illustrate the concept of chaos by showing that some systems are capable of producing outputs that seem random, yet are ordered. The discovery of chaos in physical and natural systems has initiated interdisciplinary analysis of nonlinear system of equations. This epochal discovery has a tremendous potentials for applications such as electrical circuits, lasers, fluid dynamics, mechanical devices, population growth, and many other scientific applications. We refer [28] for the history of chaos theory.

Rössler [29] noticed hyperchaotic behaviors in the chemical reaction models. The main difference between chaotic and hyperchaotic systems can be underlined as the chaotic system consists of only one Lyapunov exponent while the hyperchaotic system is determined by at least two positive Lyapunov exponents. In addition, the hyperchaotic systems generally have more complex dynamical behaviors compared to the behavior of ordinary chaotic systems.

Since chaotic systems are expressed by complex dynamical systems which are characterized by rapidly changing solutions, chaotic systems have high sensitivity to small perturbations of the initial data. Computing solutions of the chaotic systems is an active research area over the last few years. Finding accurate solutions for such problems is an effective tool for researchers to show the accuracy and performance of their methods. It is commonly concluded that accurate solutions for chaotic systems can only be determined over short time intervals. The main reason of this conclusion is the sensitivity of numerical solutions to numerical errors and the size of the time step. In addition, in [35] it is shown that some of the numerical methods may fail to converge even if the size of the time step is decreased. Some of the numerical methods used to find solutions of chaotic and hyperchaotic problems are the piecewise-spectral parametric iteration method [16], the piecewise successive linearization method (PSLM) [25, 26], the multistage Adomian decomposition method [1, 19, 23], multistage homotopy analysis method [4, 5], multistage differential transformation method [14, 15, 27], multistage variational iteration method [7, 17, 18], multistage homotopy perturbation methods [11–13, 34], multistage spectral relaxation method [24], semi-implicit multistep extrapolation [10], semi-explicit composition method [9], and composition semi-implicit methods [8]. One can find other techniques for different types of nonlinear systems of differential equations in [30–32].

In this work, we focus on Taylor's decomposition method on two points (simply Taylor's decomposition method) in order to approximate solutions of the chaotic systems such as the Lorenz and Chen systems, and hyperchaotic systems such as the Chua and RabinovichFabrikant systems. The reason for this preference is that the proposed method is implicit, A-stable, and has highly accurate solutions. This is the most important difference of Taylor's decomposition method compared with the classical Taylor's method. Taylor's decomposition technique [6] is efficiently manipulated to approximate the solutions of these nonlinear system of initial value problems as in [2], [3]. The most important advantage of the proposed method is that it has a high order of accuracy for large step sizes, since it is A-stable. The error analysis was given for special problems in [2] and [3]. In this work, we additionally present the error analysis and stability analysis for general n-dimensional initial value problems. The accuracy of the computed numerical graphs and results are checked by comparing results of ninth order Runge-Kutta method and semi-explicit Adams-Bashforth-Moulton method [33]. They are also compared with multi-stage spectral relaxation method (MSRM) [24] and multi-domain compact finite difference relaxation method [22]. Observations show that proposed method is more accurate than the other methods for large step sizes. Therefore we conclude that the method is accurate, efficient, and very easy to apply compared to other methods.

In Section 2, Taylor's decomposition on two points is described and application of the method to chaotic systems is given. Stability and convergence of the method are analyzed in Section 3. In Section 4, numerical solutions of chaotic systems are given. In the Conclusion, the study is summarized and the numerical observations are dicussed.

2. Application of Taylor's decomposition method to chaotic and hyperchaotic initial value problems

Consider the initial value problem (IVP) of the form

$$y'(t) + a(t)y(t) = f(t), \qquad 0 < t \le T, \qquad y(0) = y_0. \tag{2.1}$$

In order to find approximate solution of (2.1) with a high order accuracy, Taylor's decomposition method on two points, which is given in [6], is used. Following theorem is taken from [6] directly to describe this method.

Theorem 2.1. *Let the function $v(t)$ $(0 \le t \le T)$ have $(p+q+1)$-th continuous derivative and t_{k-1}, $t_k \in [0,T]_h$;*

$$[0,T]_h = \{t_k = kh,\ k = 0,1,\ldots,N,\ Nh = T\}.$$

Then the following relation holds:

$$\begin{aligned} v(t_k) - v(t_{k-1}) + \sum_{j=1}^{p} \alpha_j v^{(j)}(t_k)\, h^j - \sum_{j=1}^{q} \beta_j v^{(j)}(t_{k-1})\, h^j \\ = \frac{(-1)^p}{(p+q)!} \int_{t_{k-1}}^{t_k} (t_k - s)^q (s - t_{k-1})^p\, v^{(p+q+1)}(s)\, ds, \end{aligned} \tag{2.2}$$

where

$$\alpha_j = \frac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!}, \qquad 1 \le j \le p,$$

$$\beta_j = \frac{(p+q-j)!q!}{(p+q)!j!(p-j)!}, \qquad 1 \le j \le q.$$

The proof is given in [6].

Chaotic and hyperchaotic problems are of the form

$$Y'(t) = F(t, Y(t)), \quad Y(t_0) = Y_0, \tag{2.3}$$

where Y, $Y_0 \in \mathbb{R}^n$, n is number of the differential equations in (2.3). In this paper, Taylor's decomposition method is suggested to approximate solutions of chaotic and hyperchaotic problems. For this, F must have $p+q$-th continuous derivative with respect to x on the given interval. In order to find approximate solutions of (2.3), Theorem (2.1) is modified as

$$\begin{aligned} Y(t_k) - Y(t_{k-1}) + \sum_{j=1}^{p} \alpha_j Y^{(j)}(t_k)\, h^j - \sum_{j=1}^{q} \beta_j Y^{(j)}(t_{k-1})\, h^j \\ = \frac{(-1)^p}{(p+q)!} \int_{t_{k-1}}^{t_k} (t_k - s)^q (s - t_{k-1})^p\, Y^{(p+q+1)}(s)\, ds, \end{aligned} \tag{2.4}$$

on the uniform grid

$$[t_0, t_N]_h = \{t_k = t_0 + kh,\ k = 0,1,\ldots,N,\ Nh = t_N - t_0\},$$

where

$$\alpha_j = \frac{(p+q-j)!p!(-1)^j}{(p+q)!j!(p-j)!} \qquad 1 \le j \le p,$$

$$\beta_j = \frac{(p+q-j)!q!}{(p+q)!j!(p-j)!} \qquad 1 \le j \le q.$$

By neglecting the last term of the formula (2.4), using $Y'(t) = F(t, Y(t))$ and assuming $p = q$; for the approximate solution of problem (2.3) the following one step difference scheme with $2p$-order accuracy (will be proved in Theorem 3.3) is obtained as

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} \alpha_j F^{(j-1)}(t_k, Y_k)\, h^j - \sum_{j=1}^{p} (-1)^j \alpha_j F^{(j-1)}(t_{k-1}, Y_{k-1})\, h^j = 0, \quad k = 1, \ldots, N, \tag{2.5}$$

where Y_k is the approximate vector of $Y(t_k)$, Y_k and $F^{(j)}(t_k, Y_k)$ are in $\mathbb{R}^n$ for $j = 0, \ldots, 2p$, n is the number of differential equations in chaotic or hyperchaotic system, $F^{(j)}(t, Y(t)) = \dfrac{\partial^j}{\partial t^j} F(t, Y(t))$ and

$$\begin{aligned} \alpha_j &= \frac{(2p-j)!p!(-1)^j}{(2p)!j!(p-j)!} \\ \beta_j &= \frac{(2p-j)!p!}{(2p)!j!(p-j)!} = (-1)^j \alpha_j. \end{aligned} \tag{2.6}$$

Since Taylor's decomposition method is an implicit method, a predictor method or a root finding technique is needed. In this work, Newton's method is preferred to find Y_k by choosing Y_{k-1} as initial guess in each step.

One can use the following algorithm to apply Taylor's decomposition method to (2.3):
1. **Set** the values p and N;
2. **Input** the values t_0, t_N, Y_0 and F;
3. **Calculate** $h = (t_N - t_0)/N$;
4. **Input** $k = 0$; $t_1 = t_0$ and $L = \{Y_0\}$;
5. **Calculate** the values α_j and β_j by the formulas (2.6);
6. **Calculate** the derivatives of F with respect to t up to $(p-1)$-th order
7. $t := t_1$; $t_1 := t + h$; $k = k + 1$
8. **Solve** the value Y_k from the nonlinear equation

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} \alpha_j F^{(j-1)}(t_1, Y_k)\, h^j - \sum_{j=1}^{p} (-1)^j \alpha_j F^{(j-1)}(t, Y_{k-1}) = 0;$$

9. **Insert** Y_k to L;
10. **If** $t_1 < t_N$ then **Go to** Step 7;
11. **Print** L.

3. Error analysis

Error analysis of the proposed method for special cases of the function F is given in [2] and [3]. In this section, stability analysis and error analysis of Taylor's decomposition method for n-dimensional initial value problems of the form

$$\begin{aligned} Y'(t) &= F(t, Y(t)), \\ Y(t_0) &= Y_0 \end{aligned} \tag{3.1}$$

are examined.

Lemma 3.1. *Let $Y(t)$ have $(2p+1)$ continuous derivatives on $[t_0, t_N]$, then the truncation error τ_k at t_k for the Taylor's decomposition method (2.5) satisfies*

$$\|\tau_k\| \leq \frac{M\, h^{2p+1}}{(2p)!}, \tag{3.2}$$

where $M = \max_{t\in[t_0,t_N]} \left\|F^{(2p)}(t, Y(t))\right\|$, $F^{(j)}(t, Y(t)) = \dfrac{\partial^j}{\partial t^j} F(t, Y(t))$ *for* $j = 0, \ldots, 2p$, $Y'(t) = F(t, Y(t))$, *and* $\| \bullet \|$ *denotes* $\| \bullet \|_\infty$.

Proof. Using (2.4) and $Y'(t) = F(t, Y(t))$, the lemma is proved as follows:

$$\begin{aligned}
\|\tau_k\| &\leq \frac{1}{(2p)!} \int_{t_{k-1}}^{t_k} \|(t_k - s)^p\, (s - t_{k-1})^p\, Y^{(2p+1)}(s)\|\, ds \\
&\leq \frac{1}{(2p)!}\, h^{2p} \int_{t_{k-1}}^{t_k} \|F^{(2p)}(s, Y(s))\|\, ds \\
&\leq \frac{M}{(2p)!}\, h^{2p} \int_{t_{k-1}}^{t_k} ds \\
&\leq \frac{M}{(2p)!}\, h^{2p+1}.
\end{aligned}$$

□

Lemma 3.2. *Taylor's decomposition method on two points is A-stable.*

Proof. A numerical method is A-stable if when it is applied to

$$y' = \lambda y \quad y(0) = y_0, \tag{3.3}$$

where λ is any complex number with $\mathrm{Re}(\lambda) < 0$, the numerical solution $y_k \to 0$ as $k \to \infty$, for any step size $h > 0$. So, Taylor's decomposition method is applied to (3.3) to show that it is A-stable.

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} \alpha_j F^{(j-1)}(t_k, Y_k)\, h^j - \sum_{j=1}^{p} (-1)^j \alpha_j F^{(j-1)}(t_{k-1}, Y_{k-1})\, h^j = 0,$$

where $F^{(j)}(t_k, Y_k) = \lambda^{j+1} Y_k$. Then

$$Y_k - Y_{k-1} + \sum_{j=1}^{p} \alpha_j \lambda^j Y_k\, h^j - \sum_{j=1}^{p} (-1)^j \alpha_j \lambda^j Y_{k-1}\, h^j = 0,$$

which yields

$$\left(1 + \sum_{j=1}^{p} \alpha_j \lambda^j\, h^j\right) Y_k - \left(1 + \sum_{j=1}^{p} (-1)^j \alpha_j \lambda^j\, h^j\right) Y_{k-1} = 0.$$

Rewriting last equation gives

$$Y_k = \frac{1 + \sum_{j=1}^{p} \alpha_j \lambda^j\, h^j}{1 + \sum_{j=1}^{p} (-1)^j \alpha_j \lambda^j\, h^j} Y_{k-1}.$$

Backward process of the above recurrence relation gives

$$Y_k = \left(\frac{1 + \sum_{j=1}^{p} \alpha_j \lambda^j \, h^j}{1 + \sum_{j=1}^{p} (-1)^j \alpha_j \lambda^j \, h^j} \right)^k Y_0.$$

Taking limit of Y_k as $k \to \infty$ gives the exact solution

$$\begin{aligned} \lim_{k\to\infty} Y_k \quad &= e^{\lambda\, h\, k}\, Y_0 \\ &= 0, \end{aligned}$$

since $\mathrm{Re}(\lambda) < 0$. Hence Taylor's decomposition method is A-stable. □

Being A-stable means that the results of the Taylor's decomposition method are unaffected by the growth of round-off errors and the perturbations in initial data which might cause a large deviation of final result from the exact solution for any step size $h > 0$. Therefore, for any step size $0 < h < 1$, Taylor's decomposition method gives accurate solutions.

Theorem 3.3. *If $F^{(j)}$ is Lipschitz in Y with constant L_j, $j = 0, \ldots, p$, $L = \max_{0 \le j \le p-1} L_j$ and if the local truncation error at each step satisfies Lemma 3.1, then the global error for (2.4) is bounded by*

$$\|Y(t_k) - Y_k\| \le C_0 \, \|Y(0) - Y_0\| + C_1 \frac{M\, h^{2p}}{(2p)!},$$

where $C_0 = e^{\bar{t} \frac{L\, p}{1 - h\, L\, p\, \beta_1}}$ and $C_1 = \frac{C_0}{L}$ for some $\bar{t} > t_0$.

Proof. Subtracting equation (2.5) from (2.4) for $p = q$ and taking the norms of both sides yield

$$\begin{aligned} \|Y(t_k) - Y_k\| \quad &\le \|Y(t_{k-1}) - Y_{k-1}\| + \sum_{j=1}^{p} \beta_j \|Y^{(j)}(t_k) - F^{(j-1)}(t_k, Y_k)\| h^j \\ &+ \sum_{j=1}^{p} \beta_j \|Y^{(j)}(t_{k-1}) - F^{(j-1)}(t_{k-1}, Y_{k-1})\| h^j + \|\tau_k\|. \end{aligned}$$

By defining $E_k = \|Y(t_k) - Y_k\|$, Lipschitz property of $F^{(j)}(t, Y)$ with respect to Y and $Y^{(j)}(t) = F^{(j-1)}(t, Y(t))$ give

$$\begin{aligned} E_k \quad &\le E_{k-1} + \sum_{j=1}^{p} \beta_j h^j \|F^{(j-1)}(t_k, Y(t_k)) - F^{(j-1)}(t_k, Y_k)\| \\ &+ \sum_{j=1}^{p} \beta_j h^j \|F^{(j-1)}(t_{k-1}, Y(t_{k-1})) - F^{(j-1)}(t_{k-1}, Y_{k-1})\| + \|\tau_k\| \\ &\le E_{k-1} + \sum_{j-1}^{p} \beta_j h^j L_{j-1} \|Y(t_k) - Y_k\| + \sum_{j=1}^{p} \beta_j h^j L_{j-1} \|Y(t_{k-1}) - Y_{k-1}\| + \|\tau_k\| \\ &\le E_{k-1} + \sum_{j=1}^{p} \beta_j h^j L\, E_k + \sum_{j=1}^{p} \beta_j h^j L\, E_{k-1} + \|\tau_k\|. \end{aligned}$$

Notice that

$$\frac{(2p - (j+1))!p!}{(2p)!(j+1)!(p - (j+1))!} = \frac{((2p - j) - 1)!(2p)!}{(2p)!(j+1)j!((p - j) - 1)!} = \frac{(2p - j)!p!(p - j)}{(2p - j)(j+1)j!(p - j)!}$$

which gives

$$\beta_{j+1} = \frac{(p-j)}{(2p-j)(j+1)}\beta_j.$$

Since $\beta_j < \beta_1 = \frac{1}{2},\ j = 2, \ldots, p$, it is obtained that

$$E_k \quad \leq E_{k-1} + L\,\beta_1 \sum_{j=1}^{p} h^j\, E_k + L\,\beta_1 \sum_{j=1}^{p} h^j\, E_{k-1} + \|\tau_k\|.$$

Rewriting the last inequality,

$$E_k \quad \leq \frac{1 + h\,L\,\beta_1 \frac{1-h^p}{1-h}}{1 - h\,L\,\beta_1 \frac{1-h^p}{1-h}} E_{k-1} + \frac{1}{1 - h\,L\,\beta_1 \frac{1-h^p}{1-h}} \|\tau_k\|$$

for sufficiently small h. Backward process of the above recurrence relation gives

$$E_k \leq \left(1 + \frac{2h\,L\,\beta_1 \frac{1-h^p}{1-h}}{1 - h\,L\,\beta_1 \frac{1-h^p}{1-h}}\right)^n E_0 + \frac{1}{2h\,L\,\beta_1 \frac{1-h^p}{1-h}} \left(\frac{2h\,L\,\beta_1 \frac{1-h^p}{1-h}}{1 - h\,L\,\beta_1 \frac{1-h^p}{1-h}}\right)^n \|\tau_k\|. \tag{3.4}$$

Since values of $\frac{1-h^p}{1-h}$ increases on $[0, 1]$ and $\frac{1-h^p}{1-h} \to p$ as $h \to 1$, the proof is completed as follows:

$$\begin{aligned} E_k \quad &\leq \left(1 + \frac{h\,L\,p}{1 - h\,L\,p\,\beta_1}\right)^n E_0 + \frac{1}{h\,L}\left(\frac{h\,L\,p}{1 - h\,L\,p\,\beta_1}\right)^n \|\tau_k\| \\ &\leq e^{t_k \frac{L\,p}{1-h\,L\,p\,\beta_1}}\, E_0 + \frac{1}{h\,L}\|\tau_k\| \left(e^{t_k \frac{L\,p}{1-h\,L\,p\,\beta_1}}\right) \\ &\leq C_0\, E_0 + C_1 \frac{M\,h^{2p}}{(2p)!}. \end{aligned}$$

□

From the above theorem it is concluded that

$$E_k = O(h^{2p})$$

for $E_0 = 0$. In other words, proposed method has $2p$-order of accuracy.

All analyzes and calculations in this article are made for equal step sizes. Variable step size can be used if one wants to make more precise calculations and keep the error in each step less than a certain tolerance value. Suppose that one wants the local truncation error to be less than $\epsilon\, h^{2p}$ at all points t_k, that is,

$$\frac{M_k\, h_k^{2p+1}}{(2p)!} \leq \epsilon\, h^{2p},$$

where $M_k = \max_{t \in [t_{k-1}, t_k]} \left\| F^{(2p)}(t, Y(t)) \right\|$. To achieve this goal, the step size h_k can be chosen as

$$h_k = \min\left(\frac{(2p)!}{M_k}\epsilon, h\right).$$

Then, it can be easily proved that $\|\tau_k\| = O(h^{2p})$ and $E_k = O(h^{2p-1})$ by using the above procedures. A similar application can be found in [21].

4. Numerical results and discussion

In this part, first, in order to compare the theoretical findings and numerical results, Taylor's decomposition method (TDM) is applied to a test problem in Example 4.1. To emphasis that TDM is A-stable, Table 1, which shows that TDM results for large step sizes are more accurate than semi-explicit Adams-Bashforth-Moulton method (SE-ABM) [33] results, is also given in Example 4.1. Then proposed method is applied to the following chaotic and hyperchaotic systems to demonstrate the accuracy, efficiency, and the power of the method. Obtained results are compared with ninth order Runge-Kutta method (RKM9) and semi-explicit Adams-Bashforth-Moulton method to check the accuracy. The obtained results are also compared with the multi-stage spectral relaxation method (MSRM) [24] and the multi-domain compact finite difference relaxation method (MD-CFDRM) [22]. In Tables 2, 4, 5, and 6, the results are obtained for TDM ($p = 3$), RKM9, SE-ABM using 10^3, 10^4, 10^5 subintervals, respectively. From these tables, it is concluded that TDM has the same accuracy as RKM9 and SE-ABM, more accurate than MSRM and MD-CFDRM, even at larger step sizes than others. Moreover the algorithm of proposed method is more simple than the others, especially Runge-Kutta method. So, Taylor's decomposition method is the most powerful method with the simple algorithm, high order accuracy, and being A-stable.

Example 4.1. Consider the following test problem:

$$\frac{dx_1}{dt} = x_1 + 2x_2 - x_3,$$

$$\frac{dx_2}{dt} = x_2 + x_3,$$

$$\frac{dx_3}{dt} = -x_1 + x_3,$$

with the initial conditions

$$x_1(0) = 0,\ x_2(0) = 0,\ x_3(0) = 1.$$

The exact solution of the above IVP is

$$\begin{aligned} x_1(t) &= 2e^t - 2e^t\cos t - e^t\sin t,\\ x_2(t) &= e^t\sin t,\\ x_3(t) &= 2e^t\cos t. \end{aligned} \tag{4.1}$$

In Table 1, the upper bounds of the errors, which are obtained from Equation (3.4) and numerical results obtained by TDM are compared for different p values and step sizes. In order to make comparison, SE-ABM results are given in Table 1, too. Observed and expected orders and CPU-time values (with Intel Core i7 Cpu 3.2 GHz, 16 GB RAM hardware and Mathematica 12 software) are also given in Table 1. From the table it can be seen that theoretical findings and obtained numerical results are in good agreement with each other and TDM results (especially for p=3) are much more accurate than SE-ABM results.

In Table 1, the observed orders of Taylor's decomposition method is calculated by using the formula

$$\text{order}(h) = \frac{1}{\ln 2}\ln\left(\frac{\max\limits_{0\le i\le n}|x(t_i) - x_{i,h}|}{\max\limits_{0\le i\le 2n}|x(t_i) - x_{i,h/2}|}\right),$$

where $x_{i,j}$ is the approximate value of $x(t_i)$, $t_i = i\,h,\ i = 0,\dots,j$ and $h = 1/j$.

Table 1. Maximum errors of numerical results of TDM ($p = 2,\ 3$) and SE-ABM, upper bounds of the errors, obtained and expected orders, and CPU-times

	h	Obtained maximum errors using TDM $p=2$	Obtained maximum errors using TDM $p=3$	Max Errors for SE-ABM	Upper bounds of the errors for TDM using Equation 3.4 $p=2$	Upper bounds of the errors for TDM using Equation 3.4 $p=3$
$x_1(t)$	1/8	4.698×10^{-6}	2.317×10^{-9}	2.025×10^{-6}	5.447×10^{-2}	3.615×10^{-4}
	1/16	2.979×10^{-7}	3.619×10^{-11}	2.519×10^{-7}	4.306×10^{-3}	3.774×10^{-6}
	1/32	1.863×10^{-8}	5.662×10^{-13}	3.167×10^{-8}	4.560×10^{-4}	8.600×10^{-8}
	1/64	1.164×10^{-9}	9.825×10^{-15}	2.533×10^{-9}	5.305×10^{-5}	2.378×10^{-9}
$x_2(t)$	1/8	5.091×10^{-6}	2.492×10^{-10}	4.854×10^{-6}	5.447×10^{-2}	3.615×10^{-4}
	1/16	3.183×10^{-7}	3.876×10^{-12}	8.602×10^{-7}	4.306×10^{-3}	3.774×10^{-6}
	1/32	1.989×10^{-8}	5.861×10^{-14}	7.997×10^{-8}	4.560×10^{-4}	8.600×10^{-8}
	1/64	1.243×10^{-9}	8.881×10^{-16}	5.886×10^{-9}	5.305×10^{-5}	2.378×10^{-9}
$x_3(t)$	1/8	1.119×10^{-6}	1.136×10^{-9}	4.698×10^{-6}	5.447×10^{-2}	3.615×10^{-4}
	1/16	6.954×10^{-8}	1.776×10^{-11}	3.390×10^{-7}	4.306×10^{-3}	3.774×10^{-6}
	1/32	4.339×10^{-9}	2.775×10^{-13}	3.120×10^{-8}	4.560×10^{-4}	8.600×10^{-8}
	1/64	2.711×10^{-10}	4.884×10^{-15}	2.253×10^{-9}	5.305×10^{-5}	2.378×10^{-9}
		Observed orders			Expected orders	
$x_1(t)$	1/8	3.97924	6.00027	3.00705	4	6
	1/16	3.99905	5.99840	2.99202	4	6
	1/32	3.99976	5.84868	3.64420	4	6
$x_2(t)$	1/8	3.99956	6.00645	2.49644	4	6
	1/16	3.99989	6.04737	3.42716	4	6
	1/32	3.99999	6.04439	3.76391	4	6
$x_3(t)$	1/8	4.00919	5.99970	3.79282	4	6
	1/16	4.00231	6.00029	3.44178	4	6
	1/32	4.00058	5.82828	3.79119	4	6
		CPU-time				
	1/8	0.0156	0.0156	0.0156		
	1/16	0.0312	0.0156	0.0156		
	1/32	0.0312	0.0625	0.0156		
	1/64	0.0625	0.0781	0.0312		

Example 4.2. Consider the Lorenz system

$$\frac{d\,x_1}{d\,t} = a(x_2 - x_1),$$

$$\frac{d\,x_2}{d\,t} = -x_1\,x_3 + b\,x_1 - x_2,$$

$$\frac{d\,x_3}{d\,t} = x_1\,x_2 - c\,x_3,$$

where a, b, and c are positive constants. These equations were derived by Lorenz [20] in the modelling of two dimensional fluid cell between two parallel plates at different temperatures. The parameters and initial conditions are choosen as

$$a = 10,\ b = 28,\ c = 8/3,$$

$$x_1(0) = 1,\ x_2(0) = 5,\ x_3(0) = 10.$$

Graphs of the approximate solutions obtained by TDM and phase portraits of the Lorenz system are given in Figure 1 and Figure 2, respectively. In Table 2, approximate results of the Lorenz system obtained by TDM for $p = 2$ and $p = 3$ are given by comparing with the approximate results of the methods RKM9, SE-ABM, MSRM, and MD-CFDRM.

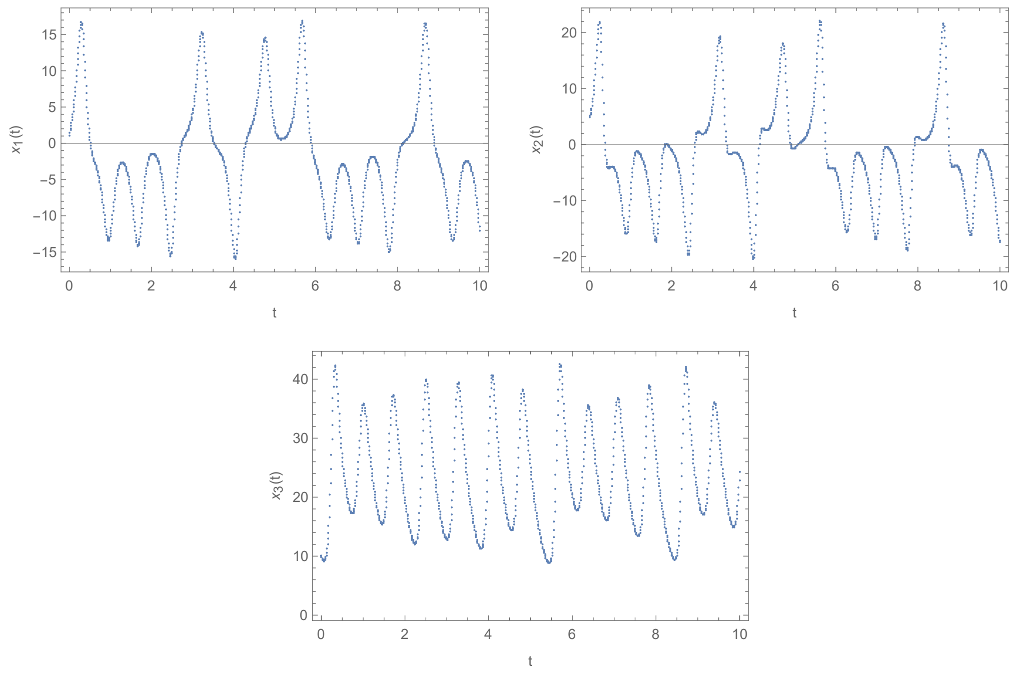

Figure 1. Graphs of approximate solutions of the Lorenz system obtained by using TDM

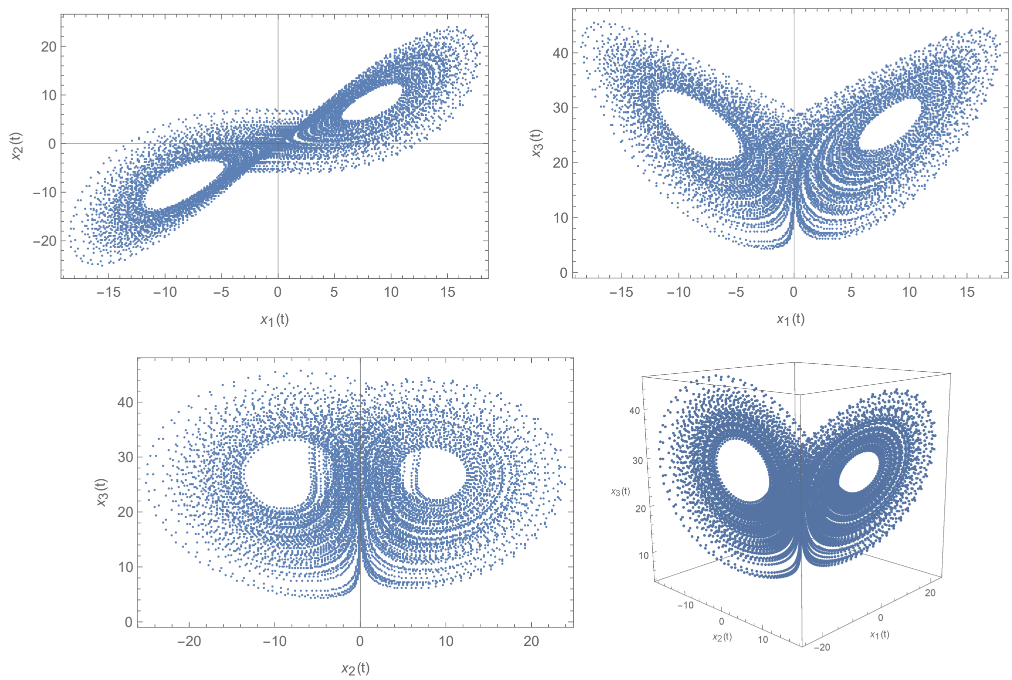

Figure 2. Phase portraits of the solutions obtained by TDM for the Lorenz system

Table 2. Approximate results of the Lorenz system obtained by using TDM, RKM9, SE-ABM, MSRM, and MD-CFDRM

t	TDM for p=2	TDM for p=3	RKM9	SE-ABM	MSRM	MD-CFDRM
$x_1(t)$						
2	-1.4443646255	-1.4443586788	-1.4442408018	-1.4443586723	-1.444359	-1.444359
4	-14.6751264379	-14.6750799809	-14.6749457046	-14.6750801064	-14.675080	-14.675080
6	-2.8828766366	-2.8830279688	-2.8831825022	-2.8830280947	-2.883028	-2.883028
8	-2.6781172029	-2.6792512793	-2.6797833651	-2.6792514161	-2.679253	-2.679253
10	-12.036449759	-12.0266562959	-12.0266549650	-12.0266535306	-12.026645	-12.026645
$x_2(t)$						
2	-1.0749787470	-1.0749770845	-1.0748932312	-1.0749770999	-1.074977	-1.074977
4	-20.1889615745	-20.1891066633	-20.1876761823	-20.1891067052	-20.189107	-20.189107
6	-4.7635416251	-4.7635574036	-4.7635587657	-4.7635574818	-4.763557	-4.763557
8	1.4279932894	1.4294747341	1.4315756838	1.4294751774	1.429476	1.429476
10	-17.5154943794	-17.5202758914	-17.5202766042	-17.5202775518	-17.520281	-17.520281
$x_3(t)$						
2	19.5170798208	19.5170571929	19.5170241404	19.5170571141	19.517057	19.517057
4	29.0636782394	29.0633619918	29.0637283497	29.0633623795	29.063361	29.063361
6	20.3569024033	20.3555595015	20.3553911934	20.3555589103	20.355558	20.355558
8	27.1018558173	27.1056547421	27.1060015188	27.1056555304	27.105659	27.105659
10	24.3415431129	24.3002008944	24.3001951698	24.3001887180	24.300153	24.300154
CPU time	2.7187	4.10938	3.46875	3.875		

In Table 3, the observed orders of TDM are calculated by using the formula

$$\text{order}(h) = \frac{1}{\ln 2} \ln \left(\frac{x_{n,2n} - x_{n/2,n}}{x_{2n,4n} - x_{n,2n}} \right),$$

where $x_{i,j}$ is the approximate value of $x(t_i)$, $t_i = i\frac{10}{j}$, that is, $x_{i,2i}$ is the approximate value of $x(5)$.

Table 3. Observed orders of TDM for $p = 2,\ 3,\ 4$ obtained by using approximate results of the Lorenz system

	h	$p=2$	$p=3$	$p=4$
$x_1(5)$	1/20	3.41463	5.95654	8.07042
	1/40	3.91562	5.98123	7.96719
	1/80	3.98215	5.99563	8.63308
	1/160	3.99575	6.01454	-
$x_2(5)$	1/20	3.90324	5.95708	8.05699
	1/40	3.976	5.98247	7.97199
	1/80	3.99406	5.99584	8.1702
	1/160	3.99852	6.01154	-
$x_3(5)$	1/20	3.96008	5.95894	8.03799
	1/40	3.98511	5.98443	7.97686
	1/80	3.99594	5.99621	7.85537
	1/160	3.99897	6.00752	-

In Theorem 3.3, it was concluded that proposed method is of order $2p$. From Table 3, it is observed that calculated orders are well confirmed with theoretical aspects.

Example 4.3. Consider the Chen dynamical system

$$\frac{dx_1}{dt} = a(x_2 - x_1),$$

$$\frac{dx_2}{dt} = (c - a)x_1 - x_1 x_3 + c x_2,$$

$$\frac{dx_3}{dt} = x_1 x_2 - b x_3.$$

This system is solved for the parameters

$$a = 35,\ b = 3,\ c = 28,$$

and the initial conditions

$$x_1(0) = -10,\ x_2(0) = 0,\ x_3(0) = 37.$$

Graphs of the approximate solutions obtained by using TDM and phase portraits of the Chen system are given in Figure 3 and Figure 4, respectively. In Table 4, approximate results of the Chen system obtained by using TDM for $p = 3$ are given by comparing with the approximate results of the methods RKM9, SE-ABM, MSRM, and MD-CFDRM.

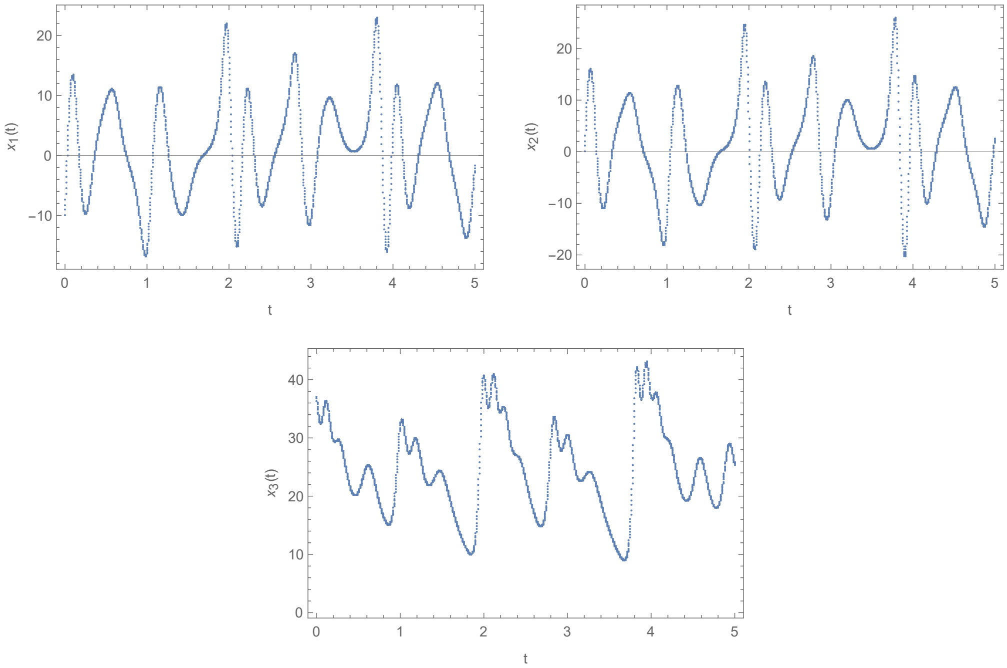

Figure 3. Graphs of approximate solutions of the Chen system obtained by using TDM

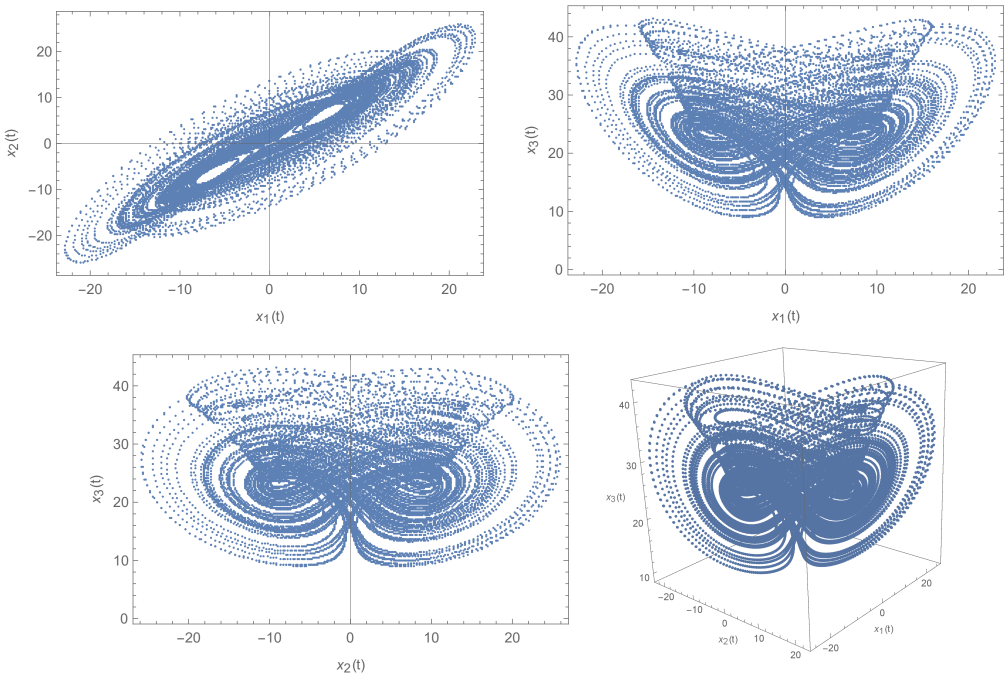

Figure 4. Phase portraits of the solutions obtained by TDM for the Chen system

Example 4.4. Consider the Rabinovich-Fabrikant dynamical system which is a hyper-chaotic system

$$\frac{d\,x_1}{d\,t} = x_2\left(x_3 - 1 + x_1^2\right) + a\,x_1,$$

$$\frac{d\,x_2}{d\,t} = x_1\left(3x_3 + 1 - x_1^2\right) + a\,x_2,$$

$$\frac{d\,x_3}{d\,t} = -2x_3\left(b + x_1\,x_2\right).$$

Table 4. Approximate results of the Chen system obtained by using TDM, RKM9, SE-ABM, MSRM, and MD-CFDRM

t	TDM Results for p=3	RKM9	SE-ABM	MSRM	MD-CFDRM
$x_1(t)$					
1	-15.9049235755	-15.9049219635	-15.9049228567	-15.904923	-15.904923
2	16.3181688833	16.3159039457	16.3181198452	16.318160	16.318160
3	-10.7052451851	-10.7028677928	-10.7052684669	-10.705249	-10.705249
4	3.7711541919	3.7694543689	3.7731990657	3.771509	3.771507
5	-1.7514486088	-1.7515327933	-1.7490522600	-1.751032	-1.751032
$x_2(t)$					
1	-13.1622222778	-13.1622216884	-13.1622209599	-13.162222	-13.162222
2	6.3694451074	6.3717554354	6.3693670977	6.369432	6.369432
3	-8.4740537505	-8.4736733546	-8.4740717331	-8.474057	-8.474057
4	12.5241109861	12.5238794950	12.5251562259	12.524292	12.524291
5	2.6181188848	2.6181139950	2.6182771002	2.618147	2.618147
$x_3(t)$					
1	32.0900475842	32.0900446617	32.0900477024	32.090048	32.090048
2	40.6559495468	40.6504438689	40.6559403140	40.655948	40.655948
3	30.5025814895	30.4975124986	30.5026425038	30.502592	30.502592
4	37.0324379667	37.0369199903	37.0316502489	37.032301	37.032302
5	25.3296425751	25.3297960909	25.3253034668	25.328889	25.328889
CPU time	4.09	3.50313	3.73125		

This system is solved for the parameters

$$a = 0.1,\ b = 0.98,$$

and the initial conditions

$$x_1(0) = -0.5,\ x_2(0) = 6,\ x_3(0) = 1.1.$$

Graphs of the approximate solutions obtained by TDM and phase portraits of the Rabinovich-Fabrikant system are given in Figure 5 and Figure 6, respectively. In Table 5 approximate results of the Rabinovich-Fabrikant system obtained by using TDM for $p = 3$ are given by comparing with the approximate results of the methods RKM9, SE-ABM, MSRM, and MD-CFDRM.

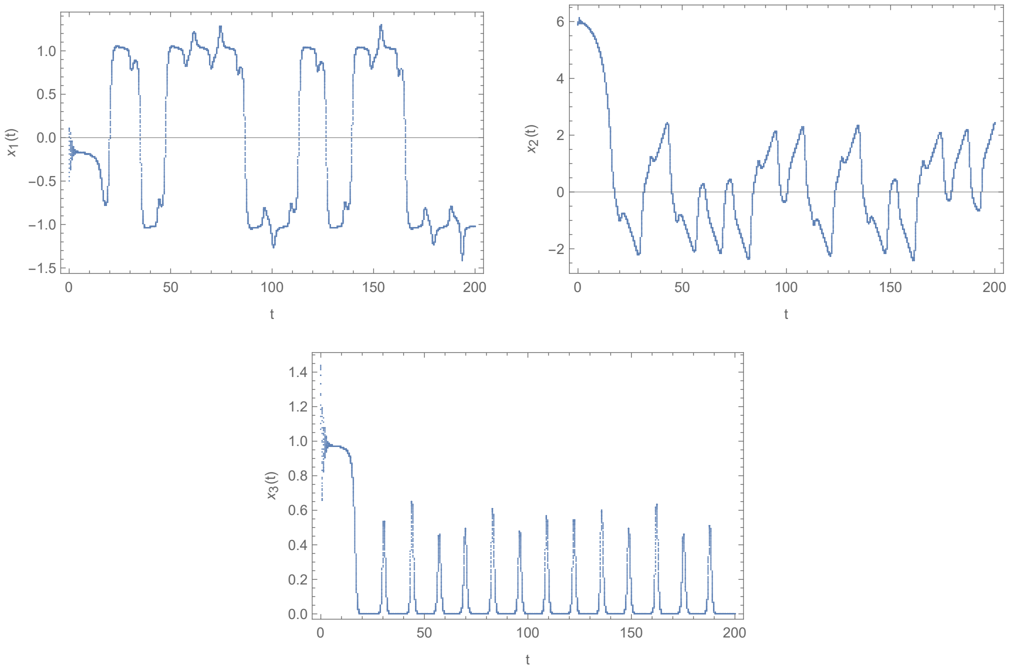

Figure 5. Graphs of approximate solutions of the Rabinovich-Fabrikant system obtained by using TDM

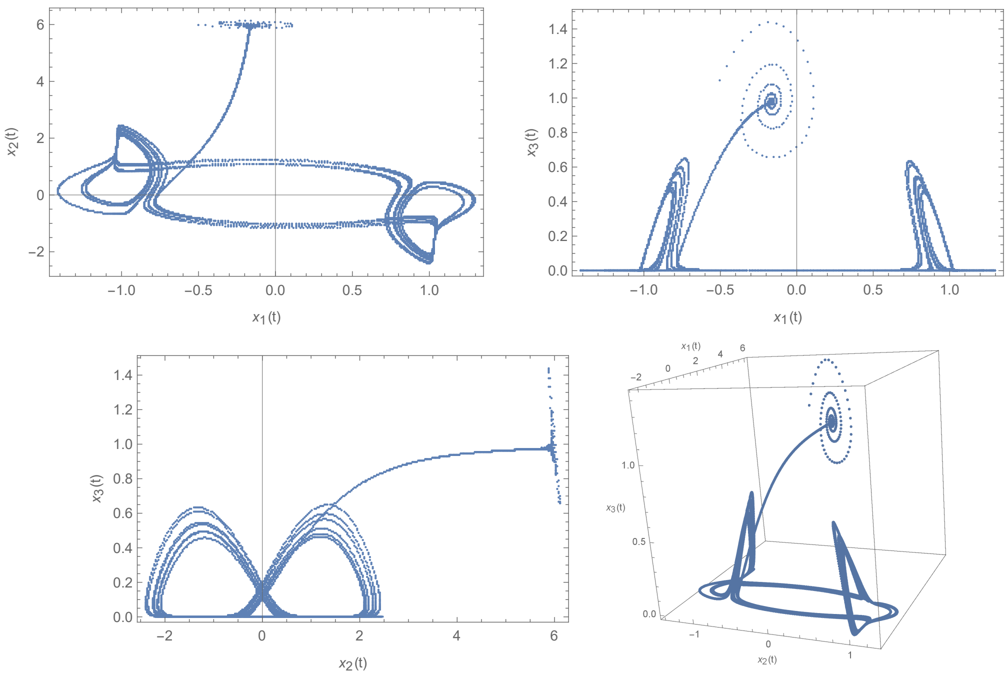

Figure 6. Phase portraits of the solutions obtained by TDM for the Rabinovich-Fabrikant system

Table 5. Approximate results of the Rabinovich-Fabrikant system obtained by using TDM, RKM9, SE-ABM, MSRM, and MD-CFDRM

t	TDM Results for p=3	RKM9	SE-ABM	MSRM	MD-CFDRM
$x_1(t)$					
10	-0.1937695597	-0.1937696325	-0.1937697037	-0.193770	-0.193770
20	0.0301646717	0.0301777927	0.0301805675	0.030179	0.030179
30	0.8787299632	0.8787274040	0.8787264034	0.878727	0.878727
40	-1.0298239954	-1.0298239610	-1.0298239266	-1.029824	-1.029824
50	1.0375504336	1.0375508411	1.0375510002	1.037551	1.037551
60	1.0340037515	1.0340050797	1.0340056666	1.034005	1.034005
70	0.7985405088	0.7985394581	0.7985404438	0.798540	0.798540
80	1.0261912353	1.0261912785	1.0261911707	1.026191	1.026191
$x_2(t)$					
10	5.0469589272	5.0469558001	5.0469551555	5.046955	5.046955
20	-1.0201349684	-1.0201358401	-1.0201360883	-1.020136	-1.020136
30	-1.8454031979	-1.8453922322	-1.8453865806	-1.845389	-1.845389
40	1.7715210364	1.7715232715	1.7715247555	1.771524	1.771524
50	-0.8786317495	-0.8786360485	-0.8786343489	-0.878634	-0.878634
60	0.2771270069	0.2771291779	0.2771264729	0.277127	0.277127
70	-0.8761434596	-0.8761345056	-0.8761233162	-0.876128	-0.876128
80	-1.9798538870	-1.9798545817	-1.9798577565	-1.979857	-1.979857
$x_3(t)$					
10	0.9644535495	0.9644535192	0.9644534897	0.964453	0.964454
20	0.0001915919	0.0001915877	0.0001915863	0.000192	0.000192
30	0.3995160522	0.3995218156	0.3995240246	0.399523	0.399523
40	0.0000928562	0.0000928614	0.0000928586	0.000093	0.000093
50	0.0000912749	0.0000912740	0.0000912747	0.000091	0.000091
60	0.0028618398	0.0028616944	0.0028617270	0.002862	0.002862
70	0.4579521073	0.4579519045	0.4579481304	0.457949	0.457949
80	0.0008755513	0.0008754129	0.0008755793	0.000876	0.000876
CPU time	7.9375	6.34375	6.76563		

Example 4.5. Consider the Chua hyperchaotic system

$$\frac{d\,x_1}{d\,t} = b(x_2 - a\,x_1^3 - (1+c)x_1),$$

$$\frac{d\,x_2}{d\,t} = x_1 - x_2 + x_3,$$

$$\frac{d\,x_3}{d\,t} = -\beta x_2 - \gamma\,x_3 + x_4,$$

$$\frac{d\,x_4}{d\,t} = -s x_4 + x_2\,x_3,$$

This system is solved for the parameters

$$a = 0.03,\; b = 30,\; c = -1.2,\; \beta = 50,\; \gamma = 0.32,\; s = 0.1060,$$

and the initial conditions

$$x_1(0) = 3,\; x_2(0) = 1,\; x_3(0) = 6,\; x_4(0) = 1.$$

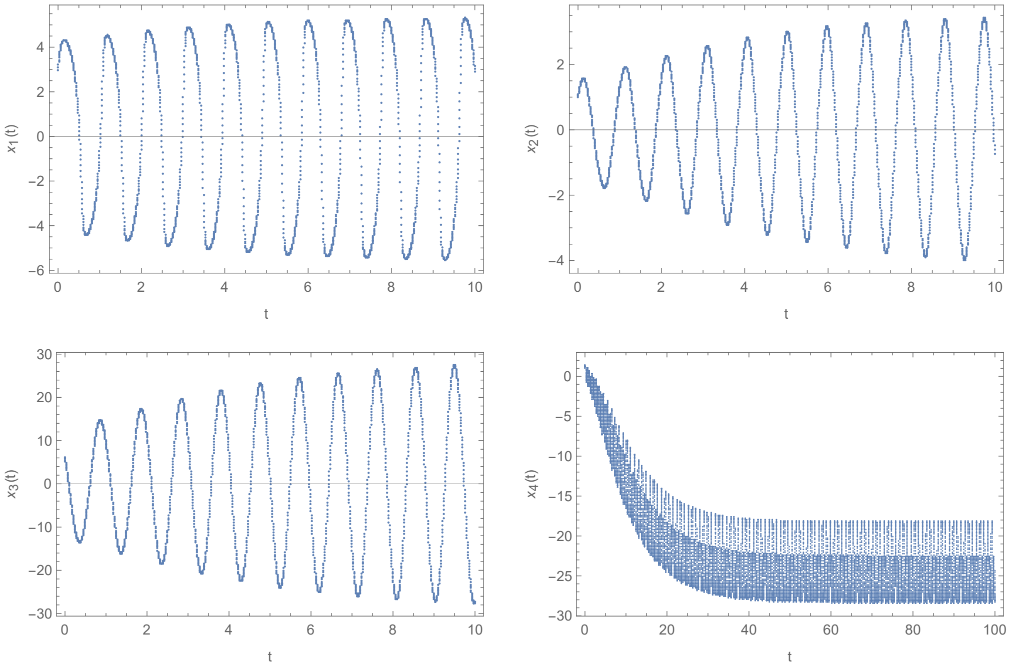

Figure 7. Graphs of approximate solutions of the Chua system obtained by using TDM

Graphs of the approximate solutions obtained by using TDM, phase portraits, and 3D phase portraits of the Chua system are given in Figure 7, Figure 8, and Figure 9, respectively. In Table 6 approximate results of the Chua system obtained by using TDM for $p = 3$ are given by comparing with the approximate results of the methods RKM9, SE-ABM, MSRM, and MD-CFDRM.

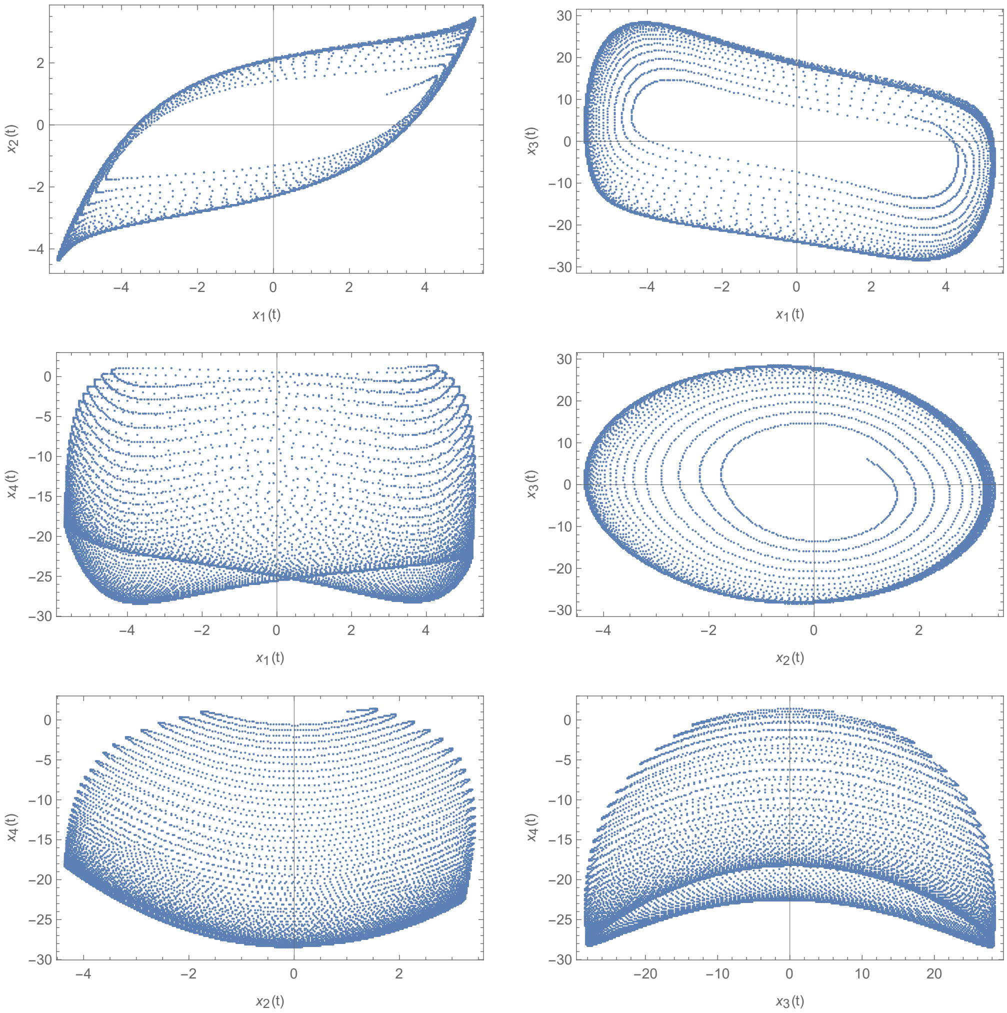

Figure 8. Phase portraits of the solutions obtained by TDM for the Chua system

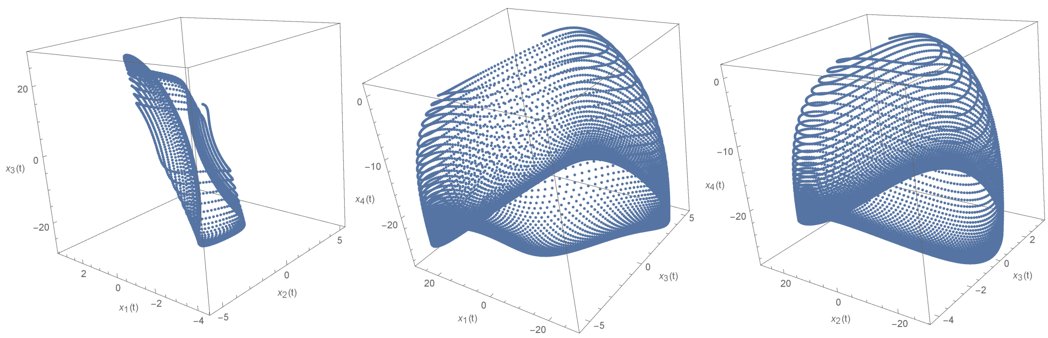

Figure 9. 3D phase portraits of the solutions obtained by TDM for the Chua system

Table 6. Approximate results of the Chua system obtained by TDM, RKM9, SE-ABM, MSRM, and MD-CFDRM

t	TDM Results for p=3	RKM9	SE-ABM	MSRM	MD-CFDRM
$x_1(t)$					
2	-0.1857682178	-0.1857684357	-0.1857682176	-0.170293	-0.170293
4	3.9637342754	3.9637338927	3.9637343418	3.990219	3.990219
6	5.1746286936	5.1746286655	5.1746286925	5.159513	5.159513
8	4.7199706841	4.7199707873	4.7199706975	4.648366	4.648366
10	2.8843671540	2.8843676014	2.8843672203	2.583704	2.583704
$x_2(t)$					
2	1.5451421862	1.5451421352	1.5451421838	1.551695	1.551695
4	2.5198497669	2.5198496851	2.5198497591	2.525985	2.525985
6	3.0972371198	3.0972371634	3.0972371159	3.058536	3.058536
8	1.9090383146	1.9090385989	1.9090383449	1.766664	1.76664
10	-0.7224935284	-0.7224931112	-0.7224934640	-0.980654	-0.980654
$x_3(t)$					
2	10.9965436146	10.9965439999	10.9965436135	11.014651	11.014651
4	7.1283745269	7.1283756748	7.1283746029	7.011997	7.011997
6	-6.3742256949	-6.3742237785	-6.3742254822	-6.752382	-6.752382
8	-21.8922265933	-21.8922250557	-21.8922263350	-22.334344	-22.334344
10	-27.2043968680	-27.2043973543	-27.2043968493	-27.076591	-27.076591
$x_4(t)$					
2	-1.3875689592	-1.3875690501	-1.3875689567	-1.493298	-1.493298
4	-2.8614383507	-2.8614385395	-2.8614383504	-3.597274	-3.597274
6	-5.2416766734	-5.2416765009	-5.2416766219	-7.371016	-7.371016
8	-11.9182359771	-11.9182354659	-11.9182358339	-16.055827	-16.055827
10	-16.1618725305	-16.1618730365	-16.1618724905	-22.167142	-22.167142
CPU time	6.48438	5.25313	5.82813		

5. Conclusion

In this paper, Taylor's decomposition method is applied to the chaotic Lorenz and Chen systems, and the hyperchaotic Rabinovich-Fabrikant and Chua systems. Numerical results are compared with theoretical aspects and comparisons show that numerical results are well-confirmed with the theoretical results. Furthermore, the power of accuracy and efficiency of the proposed method is tested against the other methods and it is concluded that proposed method is applicable, more accurate, and has great potential by having simple algorithm than the others. Another important feature of the method is that it is A-stable. This method can be extended to high dimensional chaotic and hyperchaotic systems and system of nonlinear boundary value problems with variable step sizes.

References

[1] O. Abdulaziz, N. Noor, I. Hashim, and M. Noorani, *Further accuracy tests on adomian decomposition method for chaotic systems*, Chaos, Solitons & Fractals **36**, 1405–1411, 2008.

[2] M.E. Adiyaman and S. Somali, *Taylor's decomposition on two points for one-dimensional Bratu problem*, Numer. Methods Partial Differential Equations **26**, 412–425, 2010.

[3] M.E. Adiyaman and S. Somali, *A new approach for linear eigenvalue problems and nonlinear Euler buckling problem*, Abstr. Appl. Anal. **2012**, Article ID 697013, 2012.

[4] A.K. Alomari, M.S.M. Noorani, and R. Nazar, *Homotopy approach for the hyperchaotic Chen system*, Phys. Scr. **8**, 045005, 2010.

[5] A.K. Alomari, M.S.M. Noorani and R. Nazar, *Adaptation of homotopy analysis method for the numeric analytic solution of Chen system*, Commun. Nonlinear Sci. Numer. Simul. **14**, 2336-2346, 2009.

[6] A. Ashyralyev and P.E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Operator Theory Advances and Applications, Birkhauser, Basel, 2004.

[7] B. Batiha, M.S.M. Noorani, I. Hashim, and E.S. Ismail, *The multistage variational iteration method for a class of nonlinear system of odes*, Phys. Scr. **76**, 388–392, 2007.

[8] D.N. Butusov, V.S. Andreev, and D.O. Pesterev, *Composition semi-implicit methods for chaotic problems simulation*, in: 2016 XIX IEEE International Conference on Soft Computing and Measurements (SCM), 107–110, 2016.

[9] D.N. Butusov, V.Y. Ostrovskii, A.I. Karimov, and V.S. Andreev, *Semi-explicit composition methods in memcapacitor circuit simulation*, Int. J. Embed. Real-Time Commun. Syst. **10**(2), 37–52, 2019.

[10] D.N. Butusov, A. Tutueva, P. Fedoseev, A. Terentev, and A. Karimov, *Semi-implicit multistep extrapolation ODE solvers*, Mathematics **8**(6), 943, 2020.

[11] M.S.H. Chowdhury, I. Hashim and S. Momani, *The multistage homotopy-perturbation method: a powerful scheme for handling the Lorenz system*, Chaos, Solitons & Fractals **40**, 1929–1937, 2009.

[12] M.S.H. Chowdhury, I. Hashim, S. Momani, and M.M. Rahman, *Application of multistage homotopy perturbation method to the chaotic Genesio system*, Abstr. Appl. Anal. **2012**, Article ID 974293, 2012.

[13] M.S.H. Chowdhury, I. Hashim, *Application of multistage homotopy-perturbation method for the solutions of the Chen system*, Nonlinear Anal.: Real World Appl. **10**, 381–391, 2009.

[14] Y.D.B. Jang, *Enhanced multistage differential transform method: application to the population models*, Abstr. Appl. Anal. **2012**, Article ID 253890, 2012.

[15] A. Freihat, S. Momani, *Adaptation of differential transform method for the numeric-analytic solution of fractional-order Rössler chaotic and hyperchaotic systems*, Abstr. Appl. Anal. **2012**, Article ID 934219, 2012.

[16] A. Ghorbani, J. Saberi-Nadja, *A piecewise-spectral parametric iteration method for solving the non-linear chaotic Genesio system*, Math. Comput. Model. **54**, 131–139, 2011.

[17] S.M. Goh, M.S.M. Noorani, and I. Hashim, *Efficacy of variational iteration method for chaotic Genesio systemclassical and multistage approach*, Chaos, Solitons & Fractals **40**, 2152–2159, 2009.

[18] S.M. Goh, M.S.M. Noorani, I. Hashim, and M.M. Al-Sawalha, *Variational iteration method as a reliable treatment for the hyperchaotic Rössler system*, Int. J. Nonlinear Sci. Numer. Simul. **10**, 363–371, 2009.

[19] G. González-Parra, A.J. Arenas, and L. Jódar,*Piecewise infinite series solutions of seasonal diseases models using multistage Adomian method*, Commun. Nonlinear Sci. Numer. Simul. **14**, 3967–3977, 2009.

[20] E. Lorenz, *Deterministic nonperiodic flow*, J. Atmos. Sci. **20**, 130–141, 1963.

[21] R. Lozi, V.A. Pogonin, and A.N. Pchelintsev, *A new accurate numerical method of approximation of chaotic solutions of dynamical model equations with quadratic nonlinearities*, Chaos, Solitons & Fractals **91**, 108–114, 2016.

[22] D. Mathale, P.G. Dlamini, and M. Khumalo, *Compact finite difference relaxation method for chaotic and hyperchaotic initial value systems*, Comp. Appl. Math. **37**, 5187–5202, 2018.

[23] M. Mossa Al-Sawalha, M.S.M. Noorani, and I.Hashim, *On accuracy of Adomian decomposition method for hyperchaotic Rössler system*, Chaos, Solitons & Fractals **40**, 1801–1807, 2009.

[24] S.S. Motsa, P. Dlamini, and M. Khumalo, *A new multistage spectral relaxation method for solving chaotic initial value systems*, Nonlinear Dyn. **72**, 265–283, 2013.

[25] S.S. Motsa, Y. Khan, and S. Shateyi, *Application of piecewise successive linearization method for the solutions of the chen chaotic system*, J. Appl. Math **2012**, Article ID 258948, 2012.

[26] S.S. Motsa, *A new piecewise-quasilinearization method for solving chaotic systems of initial value problems*, Cent. Eur. J. Phys. **10**, 936–946, 2012.

[27] Z.M. Odibat, C. Bertelle, M.A. Aziz-Alaoui, and G.H.E. Duchamp, *A multi-step differential transform method and application to non-chaotic or chaotic systems*, Comput. Math. Appl. **59**, 1462–1472, 2010.

[28] L.M. Resler, *Edward N. Lorenz's 1963 paper, "Deterministic nonperiodic flow", in Journal of the Atmospheric Sciences, Vol 20, pages 130-141: Its history and relevance to physical geography*, Prog Phys Geogr . Earth Environ. **40**, 175–180, 2015.

[29] O.E. Rössler, *An equation for continuous chaos*, Phys. Lett. A **57**, 397–398, 1976.

[30] M. Turkyilmazoglu, *A simple algorithm for high order Newton iteration formulae and some new variants*, Hacet. J. Math. Stat. **49**(1), 425–438, 2020.

[31] M. Turkyilmazoglu, *Is homotopy perturbation method the traditional Taylor series expansion*, Hacet. J. Math. Stat. **44**(3), 651–657, 2015.

[32] M. Turkyilmazoglu, *Purely analytic solutions of the compressible boundary layer flow due to a porous rotating disk with heat transfer*, Phys. Fluids **21**, 106104, 2009.

[33] A. Tutueva, T. Karimov, and D.N. Butusov, *Semi-implicit and semi-explicit Adams-Bashforth-Moulton methods*, Mathematics **8**(5), 780, 2020.

[34] S. Wang, Y. Yu, *Application of multistage homotopyperturbation method for the solutions of the chaotic fractional order systems*, Int. J. Nonlinear Sci. **13**, 3–14, 2012.

[35] L. Yao, *Computed chaos or numerical errors*, Nonlinear Anal. Model. Control **15**, 109–126, 2010.