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On $|V,\lambda|$ Summability of A Factored Fourier Series

by

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On $|V,\lambda|$ Summability of A Factored Fourier Series

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ABSTRACT

Let f (t) be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let

$$f(x) \sim \sum_{1}^{\infty} A_n(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We write

$$\varphi(t) = \frac{1}{2} [f(x+t) - f(x-t) - 2f(x)]$$

and

$$\Phi(t) = \int_0^t | \varphi(x) | dx.$$

In this paper the following theorem has been proved which generalizes certain results due to Verma [Abstract, Proc. Indian Sci. Congress [1972]; Liu [Proc. Japan Acad, 41 (1965), 757–757–762].

Theorem. Let $\{\mu_n\}$ be a positive sequence such that $\{\mu_n/(\log n)^\alpha\}$ is monotonic non increasing and $\Sigma (n\mu_n/\lambda^2_n)(\log n)^{1-\alpha} < \infty$, $(0 \leq \alpha < 1)$, then $\Sigma\mu_n A_n(t)$ is summable $| V, \lambda |$ at every point x satisfying

$$\text{(i)} \quad \Sigma \frac{(n \log n)^{1-\alpha}}{\lambda_n} \mu_n < \infty$$

$$\text{(ii)} \quad \int_t^\pi \frac{| \varphi(u) |}{u} du = 0 ((\log 1/t)^{1-\alpha}) \text{ as } t \to 0.$$

1. Let Σa_n be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\lambda = \{\lambda_n\}$ be a monotone non-decreasing

sequence of natural numbers with $\lambda_{n+1} \mp \lambda_n \leq 1$ and $\lambda_1 = 1$. The sequence to sequence transformation:

$$V_n(\lambda) = \lambda_n^{-1} \sum_{r=n-\lambda_n+1}^{n} s_r ,$$

defines generalized de-la Vallée Poussin means of the sequence $\{s_n\}$ generated by the sequence $\{\lambda_n\}$. The series Σa_n is said to be $|V,\lambda|$, if the series ([1], [2])

$$\sum_{n=1}^{\infty} | V_{n+1}(\lambda) - V_n(\lambda) | < \infty .$$

Let f (t) be a periodic function with period 2π and integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$. Let

$$\Sigma A_n(t) = 1/2\, a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

be the Fourier series of f (t).

We write

$$\varphi(t) = 1/2 \{ f(x+t) + f(x-t) - 2f(x) \};$$

and

$$\Phi(t) = \int_0^t | \varphi(u) | du.$$

2. In this note, we prove the following:

Theorem 1. If $\{\mu_n\}$ is a positive sequence such that $\{\mu_n/(\log n)^{\alpha}\}$ is monotonic non-increasing $\Sigma \frac{n\mu_n}{\lambda_n^2} (\log n)^{1-\alpha} < \infty$, $(o \leq \alpha < 1)$,

then $\Sigma \mu_n A_n(t)$ is summable $|V,\lambda|$ at every point x satisfying

(i) $\Sigma \dfrac{n (\log n)^{1-\alpha}}{\lambda_n} \triangle \mu_n$

(ii) $\displaystyle\int_t^{\pi} \frac{|\varphi(u)|}{u} du = 0 ((\log 1/t)^{1-\alpha})$ as $t \longrightarrow 0$.

(*) Taking $\lambda_n = n$, this reduces to the following known result for absolute Cesaro summability obtained by Verma [3].

Theorem : If $\{\lambda_n\}$ is a positive sequence such that $\{\lambda_n/(\log n)^{\alpha}\}$

is monotinic non-increasing and $\sum_{n=1}^{\infty} n^{-}\lambda_n (\log n)^{1-\alpha} < \infty$, $(0 \leq \alpha < 1)$,

then $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t = x$ at which

$$\int_t^{\pi} \frac{|\varphi(u)|}{u} du = 0 ((\log 1/t)^{1-\alpha}), \text{ as } t \to 0.$$

It may be remarked here that Verma's result under the above hypotheses is an extension of the result of Liu [5] which is further extension of Pati [4]. It may also be noted that our result also generalizes the result of Sharma and Jain [6].

3. We require the following lemmas for the proof of our theorem.

Lemma 1. [7]. If condition (ii) of Theorem 1 holds then

$$\int_0^t |\varphi(u)| du = 0 (t. (\log 1/t)^{1-\alpha}). \tag{3.1}$$

Lemma 2. If condition (ii) of Theorem 1 holds and $S_n(x)$ is the nth partial sum of the Fourier series $\Sigma A_n(x)$, then

$$\sum_{k=0}^{n} |S_k(x) - f(x)| = 0 (n. (\log n)^{(1-\alpha)}), n \to \infty, 0 \leq \alpha < 1.$$

Proof. First we shall estimate the order of the integral

$$\int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t^2} dt.$$ By integration by parts

$$\text{(3.2)} \int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t^2} dt = \left[\frac{\Phi(t)}{t^2}\right]_{\pi/n}^{\pi} + 2 \int_{\pi/n}^{\pi} \frac{\Phi(t)}{t^3} dt$$

(*) When $\lambda_n = n$, (i) follows from hypothesis $\Sigma \frac{n\mu_n}{\lambda_n^2} (\log n)^{1-\alpha} < \infty$ by a lemma due to Pati [4].

$$= 0\,(1) + 0\,(n.\ (\log n)^{1-\alpha}) + 0\left(\int_{\pi/n}^{\pi} \frac{(\log 1/t)^{1-\alpha}}{t^2}\,dt\right)$$

$$= 0\ (n\ (\log n)^{1-\alpha}).$$

by using Lemma 1.

Now consider

$$\sum_{v=1}^{n} (s_v(x)-f(x))^2 = \sum_{v=1}^{n} (2/\pi)\,(2/\pi \int_0^{\pi} \varphi(t)\frac{\sin vt}{t}\,dt + 0\,(1))^2$$

$$= \sum_{v=1}^{n} \Big\{ 4/\pi^2 \int_0^{\pi} \varphi(t)\frac{\sin vt}{t}\,dt \int_0^{\pi} \varphi(u)\frac{\sin vu}{u}\,du$$

$$+ 0\left(\int_0^{\pi} \varphi(t)\frac{\sin vt}{t}\,dt\right) + 0\ (1).$$

$$= 4/\pi^2 \int_0^{\pi} \frac{\varphi(t)}{t}\,dt \int_0^{\pi} \frac{\varphi(u)}{u}\left(\sum_{v=1}^{n} \sin v\,t \sin vu\right) du$$

$$+ \left(\sum_{v=1}^{n} \int_0^{\pi} \varphi(t)\frac{\sin vt}{t}\,dt\right) + 0\ (n)$$

$$= I_1 + 0\,(\sqrt{I_1}) + 0\ (n)$$

where

$$I_1 = 4/\pi^2 \int_0^{\pi} \frac{\varphi(t)}{t}\,dt \int_0^{\pi} \frac{\varphi(u)}{u}\left(\sum_{v=1}^{n} \Sigma \sin v\,t \sin vu\right) du$$

We shall devide I_1 into four parts

$$I_1 = \frac{4}{\pi^2}\left(\int_0^{\pi/n}\int_0^{\pi/n} + \int_0^{\pi/n}\int_{\pi/n}^{\pi/n} + \int_{\pi/n}^{\pi}\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\int_{\pi/n}^{\pi}\right)$$

$$= J_1 + J_2 + J_3 + J_4 -$$

By condition (Lemma 1.), we get

$$|J_1| \leq \frac{4}{\pi^2}\int_0^{\pi/n} |\varphi(t)|\,dt \int_0^{\pi/n} |\varphi(u)| \left(\sum_{v=1}^{n} v^2\right) du$$

$$= 0\ (n\ (\log n)^{2(1-\alpha)})$$

By Lemma 1 and (3.1) we get

$$|J_2| \leq \frac{4}{\pi^2} \int_0^{\pi/n} |\varphi(t)| \, dt \int_{\pi/n}^{\pi} \frac{|\varphi(u)|}{u} \left(\sum_{r=1}^{n} v \right) du$$

$$= 0 \, (n \, (\log n)^{2(1-\alpha)}).$$

J_3 is equal to J_2. Hence it remains to estimate J_4:

$$J_4 = \frac{2}{\pi^2} \int_{\pi/n}^{\pi} \frac{\varphi(t)}{t} \, dt \int_{\pi/n}^{\pi} \frac{\varphi(u)}{u} \int \sum_{v=1}^{n} (\cos v \, (u-t)$$

$$- \cos v \, (u+t) \,) \,) \, du$$

$$= \frac{2}{\pi^2} \int_{\pi/n}^{\pi} \frac{\varphi(t)}{t} \, dt \int_{\pi/n}^{\pi} \frac{\varphi(u)}{u} \, (D_n \, (u-t) - Dn \, (u+t)) du$$

$$= 0 \left(\int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt \int_{\pi/n}^{\pi} \frac{|\varphi(u)|}{u} \, \frac{|\sin (n+1/2)(u-t)|}{|u-t|} du \right)$$

$$+ 0 \left(\int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt \int_{\pi/n}^{\pi} \frac{|\varphi(u)|}{u} \, \frac{|\sin (n+1/2) \, (u+t)|}{u+t} \, du \right)$$

$$= 0 \, (J'_4 + J''_4)$$

then

$$J'_4 = \int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt \left(\int_{|u-t| \leq \pi/2n} + \int_{|u-t| > \pi 2n)} \right) \frac{|\varphi(u)|}{u}$$

$$\times \, \frac{|\sin (n+1/2) \, (u-t)|}{|u-t|} \, du$$

$$= J'_{41} + J'_{42}$$

By integration by parts and Lemma 1 and (3.2) we get

$$J'_{41} \leq (n+1/2) \int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt \int_{t-\pi/n}^{t+\pi/n} \frac{|\varphi(u)|}{u} \, du$$

$$= (n+1/2) \int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} \, dt \left\{ \left[\frac{\varphi(t+\pi/2n)}{t+\pi/2n} - \frac{\varphi(t-\pi/2n)}{t-\pi/2n} \right] \right.$$

$$+ \int_{t-\pi/n}^{t+\pi/n} \frac{\varphi(u)}{u^2}\, du \Big\}$$

$$= 0\left(n \int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} \Big\{ (\log(t+\pi/2n))^{1-\alpha} - (\log(t-\pi/2n))\Big)^{1-\alpha} \Big\}\, dt\right.$$

$$\left. + 0\left(n \int_{\pi/n}^{\pi} \frac{|\pi(t)|}{t} \Big\{ (\log(t+\pi/2n))^{1-\alpha} - (\log(t-\pi/2n)^{1-\alpha}) \Big\} dt\right)\right.$$

$$= 0\,(n\,(\log n)^{1-\alpha}),$$

further,

$$J'_{42} \leq \left(\int_{\pi/n}^{\pi-\pi/2n} \int_{t+\pi/2n}^{\pi} + \right.$$

$$\left. + \int_{\pi/n+\pi/2n}^{\pi} \int_{\pi/n}^{t-\pi/n} \right) \frac{|\varphi(u)|}{t} \frac{|\varphi(u)|}{u\,|u-t|}\, dt\, du$$

$$= J'_{421} + J'_{422}.$$

By integration by parts and by Lemma 1 and (3.1) we get

$$J'_{421} = \int_{\pi/n}^{\pi-\pi/2n} \frac{|\varphi(t)|}{t}\, dt \Big\{ \left[\frac{\varphi(u)}{u\,|u-t|}\right]_{t+\pi/2n}^{\pi}$$

$$- \int_{t+\pi/2n}^{\pi} \frac{\varphi(u)\,(2u-t)}{u^2\,(u-t)^2}\, du$$

$$= \frac{\varphi(\pi)}{\pi} \int_{\pi/n}^{\pi-\pi/2n} \frac{|\varphi(t)|}{t\,(\pi-t)}\, dt$$

$$+ 0\,\left(n. \int_{\pi/n}^{\pi-\pi/2n} \frac{|\varphi(t)| \log \dfrac{1}{(t+\pi/2n)}}{t}\, dt\right) +$$

$$+ 0\left(\left(\int_{\pi/n}^{\pi-\pi/2n} \frac{|\varphi(t)|}{t}\, dt \left(\int_{t+\pi/2n}^{\pi} \frac{(2u-t)\,(\log 1/u)^{\alpha}}{u\,(u-t)^2}\, du\right)\right)\right.$$

$$= 0\,(n\,(\log n)^{1-\alpha}.\ (\log n)^{1-\alpha} = (n.\ (\log n)^{2(1-\alpha)})$$

J'_{422} is equal to J'_{421}. By (3.2) and (3.1)

$$J''_4 \leq \int_{\pi/n}^{\pi} \frac{|\varphi(t)|}{t} dt \int_{\pi/n}^{\pi} \frac{|\varphi(u)|}{u^2} du$$

$$= 0 \left((\log n)^{1-\alpha}. n (\log n)^{1-\alpha} = 0 \left((\log n)^{2(1-\alpha)}.n \right) \right.$$

Thus we get the conclusion

$$\sum_{v-1}^{n} (s_v(x)-f(x))^2 = 0 \left(n (\log n)^{2(1-\alpha)}\right).$$

Now by Cauchy's inequality we get

$$\sum_{v-1}^{n} |s_v(x)-f(x)| = 0 \left(n. (\log n)^{(1-\alpha)}\right)$$

Lemma 3. If $\int_t^{\pi} \frac{|\varphi(u)|}{u} du = 0 (\log 1/t)^{1-\alpha}$ as $t \longrightarrow 0$,

$0 \leq \alpha < 1$, and $T_n(x) = 1/n+1 \sum_{k-1}^{n} k A_k(x)$, then

$$\sum_{k-1}^{n} |T_k(x)| = 0 \left(n (\log n)^{1-\alpha}\right), n \longrightarrow \infty, 0 \leq \alpha < 1.$$

Proof. Let

$$p_n(x) = \frac{1}{n+1} \sum_{k=o}^{n} s_k(x).$$

Then by Lemma 1,

$$|p_n(x) - f(x)| \leq \frac{1}{n+1} \sum_{k=1}^{n} |s_k(x) - f(x)|$$

$$= 0 \left(\frac{1}{n+1} .n. (\log n)^{1-\alpha} \right)$$

$$= 0 \left((\log n^{1-\alpha}), n \longrightarrow \infty, 0 \leq \alpha < 1, \right.$$

so that

$$\sum_{k=1}^{n} | p_k(x) - f(x) | = \sum_{k=1}^{n} 0 \left((\log k)^{(1-\alpha)} \right),$$

$$= 0 \left(n (\log n)^{1-\alpha}\right).$$

Since $T_n(x) = S_n(x) - p_n(x)$, we have

$$\sum_{k=1}^{n} | T_n(x) | \triangle \sum_{k=1}^{n} | s_k(x) - f(x) | + \sum_{k=1}^{n} | p_k(x) - f(x) |$$

$$= 0 \,(n. (\log n)^{1-\alpha}).$$

Lemma 4. If $\{\mu_n\}$ is positive sequence such that $\{\mu_n / (\log n)^{\alpha}\}$ is monotonic non increasing and $\Sigma \frac{n\,\mu_n}{\lambda^2_n} (\log n)^{1-\alpha} < \infty$, then

(i) $\sum_{n=1}^{m} \log(n+1). \triangle\mu_n = 0\,(1)$, as $m \longrightarrow \infty$.

(ii) $\sum_{n=1}^{m} n. \lg(n+1). \triangle^2\mu_n = 0\,(1)$, as $m \longrightarrow \infty$.

Proof. The convergence of $\Sigma \frac{n\,\mu_n}{\lambda^2_n} (\log n)^{1-\alpha} \Rightarrow$ the convergence of $\Sigma \frac{\mu_n}{n}$. For the remainder of the proof, see [8] and [4] respectively.

Proof of Theorem 1. It is easy to find

$$V_{n+1}(\lambda) - V_n(\lambda) = \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n \} a_k$$

Let $V_n(\lambda;x)$ denote the nth de la Vallee Poussin mean of the series $\Sigma\, \mu_n A_n(x)$. Then we have

$$\sum_{n=1}^{\infty} | V_{n+1}(\lambda;x) - V_n(\lambda;x) | = \sum_{n=1}^{\infty} \Big| \frac{1}{\lambda_n \lambda_{n+1}} \sum_{k=n-\lambda_n+2}^{n+1} \{ (\lambda_{n+1} - \lambda_n)(k-n-1) + \lambda_n \} \mu_k A_k(x) \Big|$$

Let Σ'_n be the summation over all a satisfying $\lambda_{n+1} = \lambda_n$ and Σ_n'' be the summation over all n where $\lambda_{n+1} > \lambda_n$. Then by Abel's transformation, we have

$$\Sigma'_n = \Sigma'_n \frac{1}{\lambda_{n+1}} \left| \sum_{k=n-\lambda_n+2}^{n+1} \frac{\mu_k}{k} kA_k(x) \right|$$

$$= \Sigma_n \frac{1}{\lambda_{n+1}} \left| \sum_{k=n-\lambda_n+2}^{n} \left(\sum_{v=1}^{k} v A_v(x) . \triangle \frac{\mu_k}{k} \right) - \right.$$

$$\left. - \frac{\mu_n-\lambda_n+2}{n-\lambda_n+2} \left(\sum_{v=1}^{n-\lambda_n+1} v A_v(x) \right) + \frac{\mu_n+1}{n+1} \left(\sum_{v=2}^{n+1} v A_v(x) \right) \right|$$

$$\leq \Sigma'_n \frac{1}{\lambda_{n+1}} \left\{ \sum_{k=n-\lambda_n+2}^{n} \left| \sum_{v=1}^{n} vA_v(x) \right| . \triangle \left(\frac{\mu_k}{k}\right) + \right.$$

$$\left. + \frac{\mu_{n-\lambda+2}}{n-\lambda_n+2} \left| \sum_{v=1}^{n-\lambda_n+1} v A_v(x) \right| + \frac{\mu_{n+1}}{n+1} \left| \sum_{v-1}^{n+1} v A_v(x) \right| \right\}$$

$$= \Sigma'_n (1) + \Sigma'_n (2) + \Sigma' (3), \text{ say.}$$

Since the inside lower indices $n- \lambda_n+2$ in $\Sigma'_n (1)$ are strictly increasing, we have

$$\Sigma'_n (1) = 0 \left\{ \sum_{k=1}^{\infty} k \left| T_k(x) \right| . \triangle \left(\frac{\mu_k}{k}\right) \sum_{n=k}^{k+\lambda_n-1} \frac{1}{\lambda_n} \right\}$$

$$= 0 \left\{ \sum_{k=1}^{\infty} k \left| T_k(x) \right| . \triangle \left(\frac{\mu_k}{k}\right) \right\} = M(x), \text{ say.}$$

Using Abel's transformation again, we get

$$M(x) = 0 \left\{ \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k \left| T_k(x) \right| \right) . \triangle^2 \left(\frac{\mu_n}{n}\right) \right\}$$

$$= 0 \left\{ \sum_{n=1}^{\infty} n^2 (\log n)^{1-\alpha} . \triangle^2 \left(\frac{\mu_n}{n}\right), \right\} \text{ by Lemma 3}$$

$$= 0 \left\{ \sum_{n=1}^{\infty} n (\log n)^{(1-\alpha)} . \triangle^2 \mu_n \right\}$$

$$+ 0 \left\{ \sum_{n=1}^{\infty} (\log n)^{(1-\alpha)} . \triangle \mu_n \right\}$$

$$+ 0 \left\{ \sum_{n=1}^{\infty} \frac{\mu_n}{n} (\log n)^{1-\alpha} \right\}$$

(4.1) $= 0$ (1). by Lemma 4 (i), 4 (ii) and the hypothesis (i). Further, it is easy to see that

$$\Sigma'_n(2) + \Sigma'_n(3) = 0 \left\{ \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} \mid T_n (x) \mid \right\} = N (x), \text{ say.}$$

Again, Abel's transformation gives that

$$N (x) = 0 \left\{ \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \mid T_k (k) \mid \right) . \triangle \left(\frac{\mu_n}{\lambda_n} \right) \right\}$$

$$= 0 \left\{ \sum_{n=1}^{\infty} n (\log n)^{(1-\alpha)} . \triangle \left(\frac{\mu_n}{\lambda_n} \right) , \right\} \text{ by Lemma 3.}$$

$$= 0 \left\{ \sum_{n=1}^{\infty} \frac{n\mu_n (\log n)^{1-\alpha}}{\lambda^2 n} \right\} + 0 \left\{ \sum_{n=1}^{\infty} \frac{n (\log n)^{1-\alpha} . \triangle\mu_n}{\lambda_n} \right\}$$

(4.2) $= 0$ (1), by hypotheses (i) and (ii).

The estimation of Σ''_n is somewhat more trickly. We get, with the aid of Abel's transformation, that

$$\Sigma''_n = \Sigma''_n \frac{1}{\lambda_n \lambda_{n+1}} \mid \sum_{k=n-\lambda_n+2}^{n+1} (\lambda_n - n - 1 + k) \frac{\mu_k}{k} k A_k (x) \mid$$

$$= 0 \Bigg[\Sigma''_n \frac{1}{\lambda^2_n} \left\{ \sum_{k=n-\lambda_n+2}^{n} k|T_k(x)| . | \triangle((\lambda_n - n - 1 + k) \frac{\mu_k}{k})) | + \right.$$

$$+ (n-\lambda_n+1) \; |T_{n-\lambda_{n+1}} (x) \mid \frac{\mu_{n-\lambda_n+2}}{n-\lambda_n+2} +$$

$$\left. + (n+1) \; |T_{n+1} (x) \mid \frac{\lambda_n \mu_{n+1}}{n+1} \right\} \Bigg]$$

$= \Sigma''_n (1) + \Sigma''_n (2) + \Sigma''_n (3)$, say.

Since

$$| \triangle ((\lambda_n - n - 1 + k) \frac{\mu_k}{k} | = | (\lambda_n - n - 1 + k) \frac{\mu_k}{k} - (\lambda_n - n - k) \frac{\mu_{k+1}}{k+k} |$$

$$\leq \lambda_k \left(\frac{\mu_k}{k} - \frac{\mu_{k-1}}{k+1}\right) + \frac{\mu_k}{k},$$

we have

$$\Sigma''_n (1) \leq \sum_{k=2}^{\infty} | T_k (x) | \left\{ k\lambda_k \left(\frac{\mu_k}{k} - \frac{\mu_{k+1}}{k+1}\right) + \mu_k \right\}$$

$$\sum_{n\geq k}'' \frac{1}{\lambda^2_n}.$$

Further, since Σ''_n has only the indices n having the property $\lambda_{n+1} > \lambda_n$, it follows that

$$\sum_{n\geq k}'' \frac{1}{\lambda^2_n} \leq \sum_{v=\lambda_k}^{\infty} \frac{1}{v^2} = 0 \left(\frac{1}{\lambda_k}\right).$$

Hence

$$\Sigma_n'' (1) = 0 \left\{ \sum_{k=1}^{\infty} k \mid T_k (x) \mid . \triangle \left(\frac{\mu_k}{k}\right) \right\}$$

$$+ 0 \left\{ \sum_{k=2}^{\infty} \mid T_k (x) \mid . \frac{\mu_k}{\lambda_k} \right\}$$

$$= 0 (1), \text{ by (4.1) and (4.2)}$$

Also,

$$\Sigma_n'' (2) + \Sigma_n'' (3) = 0 \left\{ \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} \mid T_n (x) \mid \right\}$$

$$= 0 (1), \text{ by (4.2)}$$

This completes the proof.

REFERENCES

[1] L. Leindler: *Uber die veraligemeinerte de 'a Vallee Poussinsche Summierbarkeit allgemeiner Orthonalreihen*, Acta Math. Hung. 16 (1965), 375–387.

[2] L. Leindler: *On the absolute summability factors of Fourier series*, Acta Sci. Math. Szeged 28 (1967), 323–336.

[3] V. P. Verma: *On the absolute Cesaro summability factors of Fourier series*, Abstract, Proc. Indian Sci. Congress (1972).

[4] T. Pati: *Absolute Cesaro summability factors of infinite series*, M.Z. 78 (1962), 293–297.

[5] T.S. Liu: *On the absolute Cèsaro summability factors of Fourier series*, Proc. Japan Acad. 41 (1965), 757–762.

[6] P.L. Sharma; R.K.Jain: *On* $|V, \lambda|$ *summability of a factored Fourier series*, Mathematical Versnik. 7 (22) 1970, 37–42.

[7] M.L. Misra: *On the determination of the jump of the function by its Fourier coefficients*, Quart. J. Math, 18 (1947), 129–134.

[8] E.C. Daniel: *On absolute summability factors of infinite series*, Proc. Japan Acad. 40 (1964), 65–69.

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