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Modeling and Interpretation of Exponentiated Stretched Exponential Distribution

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Highlights

- This paper focuses on generalization of Stretched Exponential Distribution via exponentiation.
- Main statistical properties and particularize special models of subject model are derived.
- Estimation and evaluation of parameters are formed through R software.
- Asymptotic confidence intervals for unknown parameters are suggested and checked characterization.
- The potential of subject model has been proved for climate and covid-19 data by evaluation criteria.

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Abstract

An innovative model titled as Exponentiated stretched exponential distribution is introduced. The main statistical properties of subject distribution are derived and special models are particularized. The most general technique of maximum likelihood estimation is focused to obtain the parameter estimates of new innovative model. A simulation study is presented to evaluate the behavior of the proposed estimators. Asymptotic confidence intervals for unknown parameters of new model are also suggested. The characterization of model is also checked. The competency of the subject distribution is demonstrated by fitting four real data sets through evaluation criteria.

1. INTRODUCTION

Specification to Generalization of discrete probability distributions as well as continuous probability densities covers huge dimension of Statistical literature based on Modeling and interpretation of real-life data. Innovatively, unifying approach of Modeling and interpretation is leading mechanism in probability theory. It provides knowledge that helps to draw conclusion about crucial characteristics of random occurrence of phenomena. Particularly, baseline probability models are required for this purpose. As a result, numerous classical and existing probability models have been generalized to pick up flexibility through fitness of natural real life data sets. This significant innovation contributes stretchy proficiency.

Indeed, new innovative distribution holds a stronger structure than the baseline distribution and thus it yields much better performance. Johnson et al. [1] stated that a distribution with four parameters is sufficient for most of the practical purposes. Therefore, according to description of Johnson et al. alongwith at least three parameters, noticeable improvements are evaluated about the new model. Hence, induction of one or more parameter(s) in baseline model is one of the famous innovations for improvement the analysis of baseline probability models. Motivationaly, the baseline distribution generators are occupying interesting role for developing probablity models. These generators are frequently available for Modeling and interpretion. Some of these generators are listed in this regard.
Eugene et al. [2] created a generator on the basis of Beta distribution and named it beta generated distribution: $F(x)=\int_0^{G(x)} b(x)\,dx$; $G(x)$ denotes cdf of any arbitrary probability model; and

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$b(x)$ denotes pdf of beta distribution. Jones [3] gave the idea of creating a generator for mixing any arbitrary continuous distribution with Kumaraswamy distribution by:

$F(x)=1-\left[1-G^{\alpha}(x)\right]^{\beta}\ ;\alpha>0,\beta>0;$

$G(x)$ denotes cdf of any arbitrary continuous distribution, α and β are two additional shape parameters. Zografos and Balakrishnan [4] suggested generator of univariate distributions' family based on gamma random variable:

$F(x)=\int_{0}^{-log[1-G(x)]} g(x)d(x)\,;x\in R,$

$G(x)$ denotes any baseline cdf for x and $g(x)$is derivative of $G(x)$. Kareema and Abdalhussain [5] introduced a generator depending on survival function of any arbitrary continuous distribution and pdf of baseline distribution:

$F(x)=\int_{-\infty}^{\frac{1}{R_1(x)}} f_2(x)dx\ ;$

$f_2(x)$ represents pdf belonging to parent distribution and $R_1(x)$represents survival function belonging to distribution used for mixture. Alzaatreh et al. [6] introduced the T-X family cdf:

$F(x)=\int_{-\infty}^{W(F(x))} g(x)dx,$

$g(x)$ denotes probability density function of a random variable $X>0$ and $W(F(x))$denotes function of cumulative distribution function that satisfies three conditions. Bourguignon et al. [7] contributed a generator based on Weibull G-distributions:

$F(x)=\int_{0}^{G(x,\epsilon)/1-G(x,\epsilon)} w(x)d(x),$

$w(x)$ denotes pdf of Weibull distribution with positive parameters and $G(x;\epsilon)$ is a baseline cumulative distribution function depending on vector of parameter ϵ. Moreover, some researchers adopted mixture techniques for developing probability densities such as: [8-12], etc. Logically, there are adequate methods for introducing more flexible and stretchy probability models by additional of shape parameter(s). Hence, the main goal of this study is to extend from baseline distribution to new model through simple technique of Exponentiation, called Exponentiated Stretched Exponential Distribution. Gupta et al. [13] endorsed a proposal to add an auxiliary shape parameter in the baseline model to expose a new family of distributions. The cdf for Exponentiated family is

$$F(x)=[G(x)]^{\alpha},x\in R,\alpha>0. \tag{1}$$

The corresponding pdf for Exponentiated family is

$$f(x)=\alpha[G(x)]^{\alpha-1}g(x). \tag{2}$$

Progressively, a variety of several statistical distributions have been introduced as Exponentiated models with applications. The Exponentiated models seem more flexible than its classical models via demonstration of real data. Therefore, the statistical literature is engaged with interesting and demanding Exponentiated distributions along with applications. Moreover, to extend this effort, numerous distributions have been obtained, see [14- 23] and among others. Having kept in view the flexibility of these models, the collective contribution of this study is outlined as follows: (i) selection of baseline distribution due to its importance in nature via review of literature, named Stretched Exponential distribution (SED), (see Laherr`ere and Sornette [24]). (ii) transformation of baseline distribution into exponentiated model by adding a positive shape parameter. Then the new innovative model, Exponentiated Stretched Exponential Distribution (ESED) introduces. (iii) specification of its sub models and provide properties. (iv) investigation of its parameters through most popular technique of estimation, known as maximum likelihood method. (v) demonstration and fitness of this model through climate data of three zones of Pakistan collected from Met. office and confirmed cases of COVID-19 observed by John Hopkins CSSE.

Accordingly, this paper has been organized as follows: Firstly, ESED is defined with graphical presentations. Secondly, the special models of ESED are presented in form of specific models. Thirdly, the statistical properties of ESED are derived. Fourthly, the most famous method of maximum likelihood is used to estimate parameters of ESED. The asymptotic confidence intervals for unknown parameters are suggested. Fifthly, the characterization of new model is also confirmed before data analyses. Sixthly, the applicability of ESED is demonstrated via real life data sets. Its comparison with other existing models is also elaborated numerically as well as graphically. Seventhly, the simulated values of ESED's parameters for purpose of evaluation is discussed in simulation section. Finally concluding remarks about ESED are quoted. Now, modeling of ESED is followed next.

2. EXPONENTIATED STRETCHED EXPONENTIAL DISTRIBUTION

Probability density function and cumulative distribution function is defined in this section with general shapes of model.

2.1. Probability Density and Cumulative Distribution Functions

A random variable X is said to have Exponentiated Stretched Exponential distribution with distribution function (cdf), $G(x;\lambda,\theta,\alpha)$, corresponding to its probability density function (pdf), $g(x;\lambda,\theta,\alpha)$, are defined, respectively by

$$G(x;\lambda,\theta,\alpha)=\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right]^{\alpha} \tag{3}$$

$$g(x;\lambda,\theta,\alpha)=\frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left[1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right]^{\alpha-1}\ ;x>0,\lambda\in R,\theta>0,\alpha>0. \tag{4}$$

By using binomial series expansion, $(1-x)^n=\sum_{i=0}^{\infty}\binom{n}{i}(-1)^i x^i$, the cdf and pdf of ESED can be written as

$$G(x;\lambda,\theta,\alpha)=\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{-m\left(\frac{x}{\lambda}\right)^{\theta}} \tag{5}$$

$$g(x;\lambda,\theta,\alpha)=\frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}e^{-(m+1)\left(\frac{x}{|\lambda|}\right)^{\theta}}\ ;x>0,\lambda\in R,\theta>0,\alpha>0. \tag{6}$$

Thus, ESED consists of a real scale parameter λ and two shape parameters θ and α. If λ is negative then ESED is inverse ESED. Figures 1(a) and 2(c) show that the ESED is unimodal.

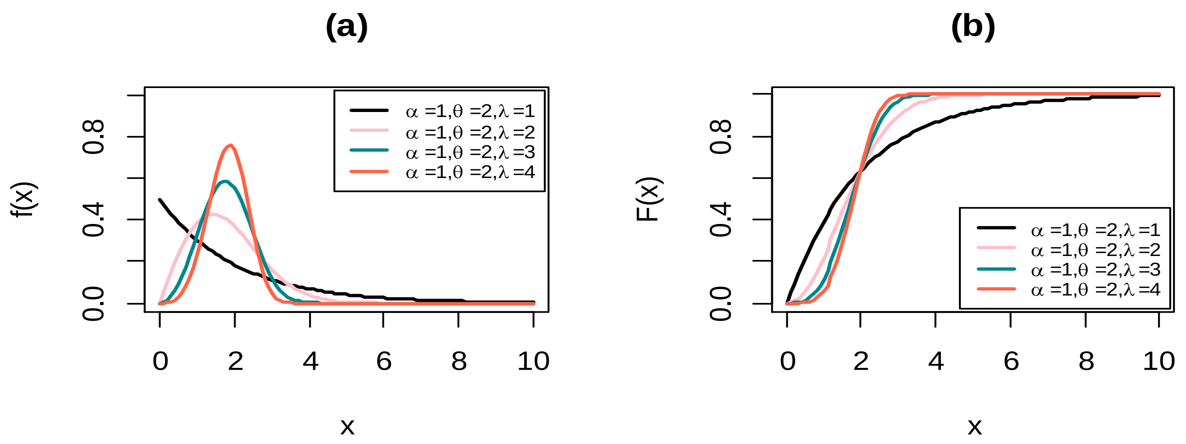

Figure 1. *Plots of $g(x;\lambda,\theta,\alpha)$ and $G(x;\lambda,\theta,\alpha)$ for fixed values of $\theta=2$, $\alpha=1$ with different values of λ*

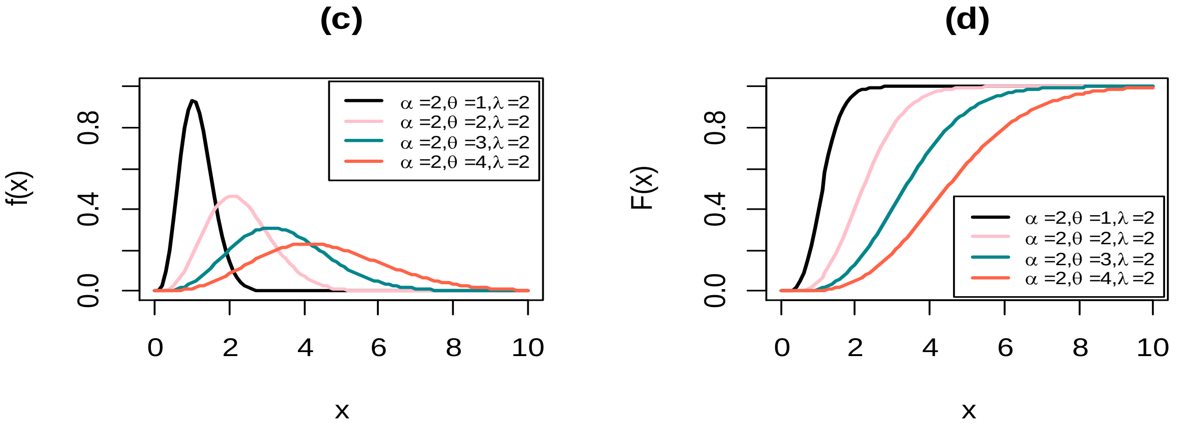

Figure 2. *Plots of $g(x;\lambda,\theta,\alpha)$ and $G(x;\lambda,\theta,\alpha)$ for fixed values of $\lambda=2$, $\alpha=2$ with different values of θ*

Figures **1(a), 1(b), 2(c)** and **2(d)** remark the possible shapes of the ESED with distinct values of λ,θ and α respectively. We have observed the following effects on the basis of pdf plots: For fixed θ and $\alpha$with $\lambda>0$, the peak of the curve flatters, presenting an exponential and positively skewed shapes in Figure **1(a)**: For fixed λ and α with $\theta>0$, the peakedness of the curve decreases gradually in Figure **2(c)**.

3. SPECIAL MODELS

With suitable selection of parameters, different classical and existing models are configured. Accordingly, following special models of ESED are given by

- For $|\lambda|=\lambda,\theta=\theta$ *and* $\alpha=1$, we get two parameter stretched exponential distribution (SED) which is also resembled two parameter Weibull distribution and the pdf is given by
$$g(x;\lambda,\theta,1)=\frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}};x>0,\lambda\in R,\theta>0. \quad (7)$$

- For $|\lambda|=1,\theta=\theta$ *and* $\alpha=1$, we get one parameter exponential distribution (ED) with rate parameter which is special case of Weibull distribution and the pdf is obtained as
$$g(x;1,\theta,1)=\theta e^{-x^{\theta}};x>0,\theta>0. \quad (8)$$

- For $|\lambda|=\lambda,\theta=1$ and $\alpha=1$, we get one parameter exponential distribution with scale parameter (also known as negative exponential distribution) and the density function is obtained as
$$g(x;\lambda,1,1)=\frac{1}{\lambda}e^{\frac{-x}{\lambda}};x>0,\lambda>0. \quad (9)$$

- For $|\lambda|=1,\theta=1$ and $\alpha=1$, we get the standard exponential distribution and the density function is given by
$$g(x;1,1,1)=e^{-x};x>0. \quad (10)$$

4. STATISTICAL PROPERTIES

Important statistical properties for ESED are settled in this section.

4.1. Non-Central Moments

Moment is defined as specific statistical quantity of shape of a function that form a set of quantities. The properties of a density function can be conveniently characterized through the technique of moments. The expected value of specified quantity power of deviation from central location (commonly from mean) of a random variable is defined as central moments. The central moments of higher order narrate the dispersion as well as shape of the distribution. On the other hand, non-central moments are defined as the expected value of specified quantity power of a random variable. Hence, the r[th] non-central moment of ESED is given by

$$\mu_r' = E(X^r) = \lambda^r \alpha \Gamma\left(\frac{r}{\theta}+1\right)\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\frac{1}{(m+1)^{\frac{r}{\theta}+1}}\,,\; r=0,1,2,... \tag{11}$$

Since (11) is a convergent series for all $r \geq 0$. Then, the moment generating function (*m.g.f*) of X is given by

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\left\{\lambda^r \alpha \Gamma\left(\frac{r}{\theta}+1\right)\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\frac{1}{(m+1)^{\frac{r}{\theta}+1}}\right\} \tag{12}$$

and the cumulant generating function, $lnM_X(t)$, and negative moments are given, respectively, by

$$K_X(t) = ln\left[\sum_{r=0}^{\infty}\frac{t^r}{r!}\left\{\lambda^r \alpha \Gamma\left(\frac{r}{\theta}+1\right)\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\frac{1}{(m+1)^{\frac{r}{\theta}+1}}\right\}\right] \tag{13}$$

$$E[X^{-r}] = \lambda^{-r}\alpha\Gamma\left(1-\frac{r}{\theta}\right)\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\frac{1}{(m+1)^{1-r/\theta}}\,.$$

4.2. Incomplete Moments

The incomplete moments of the ESED can be expressed as

$$M_r(u) = \int_0^u x^r g(x;\lambda,\theta,\alpha)dx\,.$$

From (6), by substituting $g\left(x;\lambda,\theta,\alpha\right)$ in above expression, we get

$$M_r(u) = \alpha\int_0^u x^r\frac{\theta}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}e^{-(m+1)\left(\frac{x}{\lambda}\right)^{\theta}}dx$$

$$= \alpha\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\int_0^u x^r\frac{\theta}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-(m+1)\left(\frac{x}{\lambda}\right)^{\theta}}dx\,.$$

By using lower incomplete gamma function, $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$, the incomplete moments of ESED is given by

$$M_r(u) = \alpha\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\left[\Gamma\left(\frac{r}{\theta}+1\right)-\Gamma\left(\frac{r}{\theta}+1,(m+1)\left(\frac{u}{\lambda}\right)^{1/\theta}\right)\right]. \tag{14}$$

4.3. Quantile Function

The quantile function, q, specifies the probability of the random variable that is less than or equal to that value equivalents to the specified probability. By definition, the qth quantile of any arbitrary pdf of x_q is given by $q = P\left(X \leq x_q\right) = F\left(x_q\right), x_q > 0, 0 < q < 1$

For ESED:

$q = \left[1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right]^{\alpha}$, $x > 0, \lambda \in R, \theta > 0, \alpha > 0$, after simplification, we get

$$x_q = |\lambda|\sqrt[\theta]{ln\left(1-\sqrt[\alpha]{q}\right)^{-1}}. \tag{15}$$

By substituting $q = 0.25, 0.5$ *and* 0.75, we get 1[st], 2[nd] and 3[rd]quartiles respectively as

$$x_{0.25} = |\lambda|\sqrt[\theta]{ln\left(1-\sqrt[\alpha]{0.25}\right)^{-1}},\; x_{0.5} = |\lambda|\sqrt[\theta]{ln\left(1-\sqrt[\alpha]{0.5}\right)^{-1}},\; x_{0.75} = |\lambda|\sqrt[\theta]{ln\left(1-\sqrt[\alpha]{0.75}\right)^{-1}}.$$

4.4. Mode

Let X~ESED(x: λ, θ,α).Then, the mode of 'X' is $x_o = \sqrt[\theta]{\frac{\lambda^\theta}{(i+1)}\left(1-\frac{1}{\theta}\right)}$.

Proof: The pdf of ESED is

$$g(x;\lambda,\theta,\alpha) = \frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}e^{-(m+1)\left(\frac{x}{|\lambda|}\right)^\theta}$$

$$g(x;\lambda,\theta,\alpha) = \frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}E_1 e^{-(m+1)\left(\frac{x}{|\lambda|}\right)^\theta}. \quad (16)$$

Here $E_1 = \sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}$

Taking "ln" on both sides of the Equation (16), we have $lng_E(x;\lambda,\theta,\alpha)$, mode will exist if $\frac{d^2}{dx^2}lng(x;\lambda,\theta,\alpha) < 0$, and mode exists at a point where $\frac{d}{dx}lng_E(x;\lambda,\theta,\alpha) = 0$.

,

Now for mode, differentiate $lng_E(x;\lambda,\theta,\alpha)$ with respect to x, we obtain $\frac{b-1}{x} - \frac{b(i+1)}{a}\left(\frac{x}{a}\right)^{b-1}$, equating this expression to zero and after simplification, we get

$$x_o = \sqrt[\theta]{\frac{\lambda^\theta}{(i+1)}\left(1-\frac{1}{\theta}\right)}, \theta > 0, i \geq 0. \quad (17)$$

It is computationally observed that for fixed θ and α with $\lambda > 0$, the mode of ESED increases, for fixed λ and α with $\theta > 1$, the mode of ESED increases, for fixed value of a and θ with different values of $\alpha \geq 0$, the mode of ESED remains fixed.

4.5. Entropies

The amount of uncertainty is referred to the entropy of a random variable X. Let X be the random variable of the ESED, then the Rényi entropy are given by using the following relation

$$I(\grave{e}) = \frac{1}{1-\grave{e}}log\int_0^\infty g^{\grave{e}}(x;\lambda,\theta,\alpha)dx; \quad \grave{e} > 0 \; and \; \grave{e} \neq 1$$

$$I(\grave{e}) = \frac{1}{1-\grave{e}}log\left\{\frac{(\theta\alpha)^{\grave{e}}}{|\lambda|^{\grave{e}-1}\theta(\grave{e}+m)^{\grave{e}(\theta-1)+1/\theta}}\sum_{m=0}^{\infty}(-1)^m\binom{\grave{e}(\alpha-1)}{m}\Gamma\left(\frac{\grave{e}}{\theta}(\theta-1)+\frac{1}{\theta}\right)\right\}. \quad (18)$$

The Shannon entropy and generalized entropy for $\delta \geq 1$, $\delta \neq \beta$, $\beta - 1 < \delta < \beta$of ESED are given, respectively, by

$$E[-\log(g(x;\lambda,\theta,\alpha))] = -log\left(\frac{\theta\alpha}{|\lambda|}\right) + (\theta-1)\alpha\gamma\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\frac{1}{(m+1)^{1+1/\theta}} + \alpha\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}$$

here $\gamma = 0.57722$ is Euler gamma constant.

$$V_{\delta,\beta}(X) = \frac{1}{\beta-\delta}log\left[\alpha\left(\frac{\theta\alpha}{|\lambda|}\right)^{\delta+\beta-2}\frac{\Gamma((1-1/\theta)(\delta+\beta-2)+1)}{(\delta+\beta-1)^{(1-1/\theta)(\delta+\beta-2)+1}}\left[\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}\right]^{\delta+\beta-1}\frac{1}{(m+1)^{(1-1/\theta)(\delta+\beta-2)+1}}\right]$$

$$;\beta \geq 1 \; ;\delta \neq \beta \; ;\beta - 1 < \delta < \beta.$$

4.6. Reliability Analyses

This section is based on survival, hazard, reverse hazard, cumulative hazard with explicit expressions and graphical presentation. In addition, mean residual life with reverse mean residual life of ESED are also specified.

The survival function and correspondingly hazard function of the ESED are given, respectively by

$$S(x;\lambda,\theta,\alpha) = 1 - \sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{-m\left(\frac{x}{\lambda}\right)^\theta} \quad (19)$$

$$h(x;\lambda,\theta,\alpha) = \frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\gamma-1}\frac{\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}e^{-(m+1)\left(\frac{x}{\lambda}\right)^\theta}}{1-\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{i}e^{-m\left(\frac{x}{\lambda}\right)^\theta}}. \quad (20)$$

The graphical behavior of survival and hazard functions of the ESED are observed and presented in Figures 3 and 4 respectively with remarks, (i). If $\theta = 2, \alpha = 1$,the failure rate is constant and increases for $\lambda > 0$, (ii), If $\lambda = 2, \alpha = 2$, the failure rate is constant, decreases and increases for $\theta > 0$.

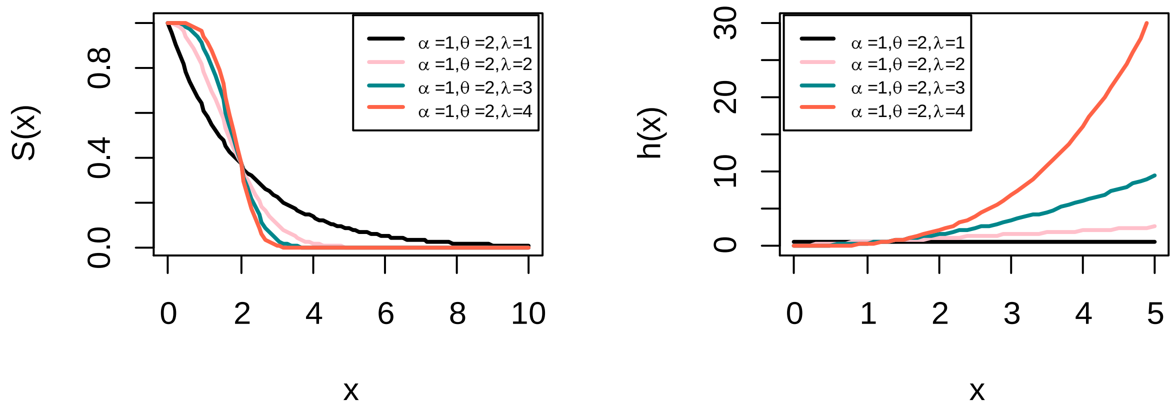

Figure 3. *Plots Survival and hazard functions of ESED when* $\theta = 2,\ \alpha = 1$ *with different values of* λ

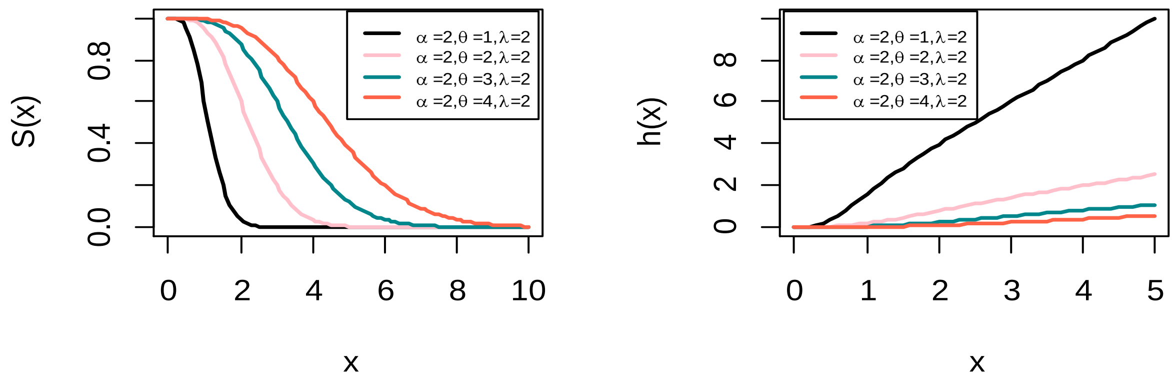

Figure 4. *Plots Survival and hazard functions of ESED when* $(\lambda, \alpha) = (2{,}2)$ *with different values of* θ

The reverse hazard function, $\tau(x; \lambda, \theta, \alpha)$, cumulative hazard function, $H_x(t)$, and mean residual life, $M(x)$, of the ESED are given, respectively by

$$\tau(x;\lambda,\theta,\alpha) = \frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1} \frac{\sum_{m=0}^{\infty}(-1)^m\binom{\alpha-1}{m}e^{-(m+1)\left(\frac{x}{\lambda}\right)^{\theta}}}{\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{-m\left(\frac{x}{\lambda}\right)^{\theta}}}$$

$$H_x(t) = -\ln\left(1 - \sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{-m(x/a)^b}\right) \tag{21}$$

$$M(x) = \frac{\lambda\alpha}{1-\sum_{m=0}^{\infty}(-1)^m\binom{\alpha}{m}e^{-m(x/\lambda)^{\theta}}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^{\frac{1}{\theta}+1}}\binom{\alpha-1}{m}\Gamma\left(\frac{1}{\theta}+1,(m+1)\left(\frac{x}{\lambda}\right)^{\theta}\right) - x$$

(22)

where $\Gamma\left(\frac{1}{\theta}+1,(i+1)\left(\frac{x}{\lambda}\right)^{\theta}\right)$ is the upper incomplete gamma function. Then, the mean waiting time also called reverse mean residual life of X is

$$M_1(x) = x - \frac{\lambda\alpha}{1-\sum_{m=0}^{\alpha}(-1)^m\binom{\alpha}{m}e^{-m(x/\lambda)^{\theta}}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)^{\frac{1}{\theta}+1}}\binom{\alpha-1}{m}\gamma\left(\frac{1}{b}+1,(m+1)\left(\frac{x}{\lambda}\right)^{\theta}\right) \quad (23)$$

where $\gamma\left(\frac{1}{\theta}+1,(m+1)\left(\frac{x}{\lambda}\right)^{\theta}\right)$ is the lower incomplete gamma function.

4.7. Order Statistics

The ESED proposed as a lifetime model, so minimum and maximum order statistics are very useful in the lifetimes of the series and parallel systems. Therefore, this subsection makes this section better and more informative about the distribution.

Let $X_{(1)}, X_{(2)},\dots,X_{(n)}$ be the order statistics from a random sample of $X_1,X_2,\dots X_n$ from a population. Then, pdf of j^{th} order statistic, say $X_{(j)}=x$ of ESED is given by

$$f_{j:n}(x) = \frac{n!}{(j-1)!\,(n-j)!}\left(\frac{b\alpha}{|a|}\left(\frac{x}{a}\right)^{b-1}e^{-(x/a)^b}\left[1-e^{-(x/a)^b}\right]^{\alpha-1}\right)\left(\left[1-e^{-(x/a)^b}\right]^{\alpha}\right)^{j-1}\left(1-\left[1-e^{-(x/a)^b}\right]^{\alpha}\right)^{n-j}$$
$$,0\le x_{(1)}\le x_{(2)}\le\cdots\le x_{(n)}\le+\infty\,.$$

For ESED, the distribution of minimum order statistic, say $X_{(1)}=x$ is given by

$$f_{1;n}(x) = \frac{nb\alpha}{|a|}\left(\frac{x}{a}\right)^{b-1}e^{-(x/a)^b}\left[1-e^{-(x/a)^b}\right]^{\alpha-1}\left(1-\left[1-e^{-(x/a)^b}\right]^{\alpha}\right)^{n-1}\quad ,0\le x_{(1)}\le+\infty\,.$$

Similarly, the distribution of maximum order statistic, say $X_{(n)}=x$ is given by

$$f_{n;n}(x) = \frac{nb\alpha}{|a|}\left(\frac{x}{a}\right)^{b-1}e^{-(x/a)^b}\left[1-e^{-(x/a)^b}\right]^{\alpha-1}\left(\left[1-e^{-(x/a)^b}\right]^{\alpha}\right)^{n-1}\quad ,0\le x_{(n)}\le+\infty\,.$$

5. ESTIMATION

The method of maximum likelihood estimation is used for the purpose of parameters estimates for ESED. Let $X_1,X_2,X_3,\dots,X_n$ be a random sample of size n from the ESED given by (4). Then, $logL(X;\Omega)=l(X;\Omega)=l(\Omega)$

with derivatives

$\frac{\partial l(X;\Omega)}{\partial\lambda}, \frac{\partial l(X;\Omega)}{\partial\theta}$ and $\frac{\partial l(X;\Omega)}{\partial\alpha}$

are shown below

$$l(\Omega) = n\log\theta + n\log\alpha - n\log|\lambda| + (\theta-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-(x_i/\lambda)^{\theta}}\right) + \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}$$

$$\frac{\partial l(X;\Omega)}{\partial\lambda} = \frac{-1}{\lambda} - \frac{n(\theta-1)}{\lambda} + \frac{\theta(\alpha-1)}{\lambda^2}\sum_{i=1}^{n}\frac{x_i\left(\frac{x_i}{\lambda}\right)^{\theta-1}e^{-(x_i/\lambda)^{\theta}}}{1-e^{-(X_i/\lambda)^{\theta}}} + \frac{\theta}{\lambda^2}\sum_{i=1}^{n}x_i\left(\frac{x_i}{\lambda}\right)^{\theta-1}$$

$$\frac{\partial l(X;\Omega)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right) + (\alpha-1)\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\log\left(\frac{x_i}{\lambda}\right)\frac{e^{-(x_i/\lambda)^{\theta}}}{1-e^{-(x_i/\lambda)^{\theta}}} - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\theta}\log\left(\frac{x_i}{\lambda}\right)$$

$$\frac{\partial l(X;\Omega)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left\{1-e^{-(x_i/\lambda)^{\theta}}\right\}$$

here, $\Omega=(\lambda,\theta,\alpha)$. By equating zero to $\frac{\partial l(X;\Omega)}{\partial\lambda}, \frac{\partial l(X;\Omega)}{\partial\theta}$ and $\frac{\partial l(X;\Omega)}{\partial\alpha}$, the resultant equations are nonlinear system of equations. This nonlinear system of likelihood equations can be estimated numerically by iterative procedure of Newton-Raphson method. Conversely, R statistical software is the best one framework that is used to obtain estimates numerically.

6. CONFIDENCE INTERVALS

The inverse Fisher information matrix is used to suggest the asymptotic confidence intervals for unknown parameters $\hat{\lambda},\hat{\theta}$and $\hat{\alpha}$ through maximum likelihood estimation. It is denoted by $I(\hat{\lambda},\hat{\theta},\hat{\alpha})$ and defined as

$$I(\hat{\lambda},\hat{\theta},\hat{\alpha}) = \begin{bmatrix} -l''_{\lambda}(X;\Omega) & -l''_{\lambda\theta}(X;\Omega) & -l''_{\lambda\alpha}(X;\Omega) \\ -l''_{\theta\lambda}(X;\Omega) & -l''_{\theta}(X;\Omega) & -l''_{\theta\alpha}(X;\Omega) \\ -l''_{\alpha\lambda}(X;\Omega) & -l''_{\alpha\theta}(X;\Omega) & -l''_{\alpha}(X;\Omega) \end{bmatrix}^{-1} = \begin{bmatrix} I_{\hat{\lambda}} & I_{\hat{\lambda}\hat{\theta}} & I_{\hat{\lambda}} \\ I_{\hat{\theta}\hat{\lambda}} & I_{\hat{\theta}} & I_{\hat{\lambda}} \\ I_{\hat{\alpha}\hat{\lambda}} & I_{\hat{\alpha}\hat{\theta}} & I_{\hat{\alpha}} \end{bmatrix}.$$

The expressions for the following derivatives are given

$$l''_{\lambda}(X;\Omega) = \frac{\partial l^2(X;\Omega)}{\partial\lambda^2}, \qquad l''_{\theta}(X;\Omega) = \frac{\partial l^2(X;\Omega)}{\partial\theta^2}, \qquad l''_{\alpha}(X;\Omega) = \frac{\partial l^2(X;\Omega)}{\partial\alpha^2}.$$

Here

$$l''_{\lambda}(X;\Omega) = \frac{\partial l^2(X;\Omega)}{\partial\lambda^2} = \frac{1}{\lambda^2} + \frac{n(\theta-1)}{\lambda^2} - \frac{\theta(\theta-1)}{\lambda^4}\sum_{i=1}^{n} x_i^2\left(\frac{x_i}{\lambda}\right)^{\theta-2} - \frac{2\theta}{\lambda^3}\sum_{i=1}^{n} x_i\left(\frac{x_i}{\lambda}\right)^{\theta-1} - \frac{2\theta(\alpha-1)}{\lambda^3}\sum_{i=1}^{n}\frac{e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}} x_i\left(\frac{x_i}{\lambda}\right)^{\theta-1}}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}}} +$$
$$+\frac{\theta(\alpha-1)}{\lambda^2}\left[\sum_{i=1}^{n}\left\{-(\theta-1)\frac{e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}} x_i^2\left(\frac{x_i}{\lambda}\right)^{\theta-2}}{\theta^2\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{b}}\right)} + \frac{e^{-2\left(\frac{x_i}{\lambda}\right)^{\theta}}\theta x_i^2\left(\frac{x_i}{\lambda}\right)^{2\theta-2}}{\theta^2\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}}\right)^2} + \frac{\theta e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}} x_i^2\left(\frac{x_i}{\lambda}\right)^{2\theta-2}}{\lambda^2\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}}\right)}\right\}\right]$$

$$l''_{\theta}(X;\Omega) = \frac{\partial l^2(X;\Omega)}{\partial\theta^2} = -\frac{n}{\theta^2} - \sum_{i=1}^{n} log\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta} log\left(\frac{x_i}{\lambda}\right) +$$
$$+(\alpha-1)\sum_{i=1}^{n}\left[\frac{e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}} log\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{\theta} log\left(\frac{x_i}{\lambda}\right)}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}}} - \frac{e^{-2\left(\frac{x_i}{\lambda}\right)^{\theta}} log\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{2\theta} log\left(\frac{x_i}{\lambda}\right)}{\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}}\right)^2} - \frac{e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}} log\left(\frac{x_i}{\lambda}\right)\left(\frac{x_i}{\lambda}\right)^{2\theta} log\left(\frac{x_i}{\lambda}\right)}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{\theta}}}\right]$$

$$l''_{\alpha}(X;\Omega) = \frac{\partial l^2(X;\Omega)}{\partial\alpha^2} = -\frac{n}{\alpha^2}.$$

By solving the above matrix, the solution will give the asymptotic variance covariances of maximum likelihood estimators for $\hat{\lambda},\hat{\theta}$ and $\hat{\alpha}$. Hence two sided $(1-\alpha)100\%$ asymptotic confidence intervals for λ,θ and α can be determined as:

$$\hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{\hat{\lambda}}},\ \hat{\theta} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{\hat{\theta}}},\ \hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{I_{\hat{\alpha}}}\ .$$

where Z_α is 100α th percentile of standard normal distribution $N(0,1)$.

7. SIMULATION STUDY

Simulation is applied to evaluate the maximum likelihood estimation (MLE) for different parameters of ESED. The simulated values of parameters of the ESED including MLE with mean square error (MSE) are reported in Table 1. The succeeding steps are adopted for simulation.

Specify the actual values of parameters λ,θ and α of ESED; Select the sample size, n; Generate an algorithm to create a random sample of size n from ESED(x: λ,θ,α) in the following manners:

- Generate$U_i \sim U(0,1); i = 1,2,3,\dots,n; U$ means Uniform distribution.

- Design $X_i = |\lambda| \sqrt[\theta]{ln\left(1-\sqrt[\alpha]{U}\right)^{-1}}$; $\lambda \in R, \theta > 0, \alpha > 0$
- Simulate the values of MLE of λ, θ and α;
- Repeat steps ii and iii, N times;
- Calculate MSE of parameters of λ, θ and α.

Table 1. *MLE and MSE at different values of parameters*

Actual Values of Parameters	Sample Size	MLE			MSE		
$\lambda = 0.5$ $\theta = 0.5$ $\alpha = 2$	n	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$
	100	0.5415	0.4610	1.9865	0.0182	0.0037	0.0069
	200	0.5741	0.4464	1.9584	0.0258	0.0051	0.0242
	300	0.5733	0.4547	1.9597	0.0182	0.0039	0.0206
	500	0.5400	0.4533	1.9913	0.0049	0.0035	0.0025
	1000	0.5680	0.4343	1.9763	0.0130	0.0055	0.0117
$\lambda = 0.05$ $\theta = 0.5$ $\alpha = 2$	100	0.0518	0.4744	1.9817	0.0003	0.0039	0.0227
	200	0.0513	0.4875	1.9799	0.0001	0.0014	0.0118
	300	0.0508	0.4830	1.9801	0.0002	0.0024	0.0138
	500	0.0492	0.4828	1.9935	0.00007	0.0031	0.0035
	1000	0.0502	0.4930	1.9955	0.00009	0.0008	0.0020

Table 1 represents simulated values parameters of the ESED for specified values of parameters with MSE. The simulation study is based on $N = 10{,}000$. The sizes of sample are $n = 100, 200, 300, 500$ &1000 with actual values of parameters $(\lambda, \theta, \alpha) = (0.5, 0.5, 2)$ *and* $(0.05, 0.5, 2)$ The simulated values of parameters improved with increasing sample size as decreasing values of MSE generally.

8. CHARACTERIZATION

The characterization of a distribution confirms essential role in mathematical statistics before data analysis. The characterization of a distribution is a main characteristic to check if suggested model is the correct. The general theory of characterization is popularized by truncated moments of Galambos and Kotz [25].

8.1. Characterization Based on Truncated Moments

Assume X be an absolutely continuous random variable has$G(x)$; cumulative distribution function (cdf) and $g(x)$; probability density function (pdf) for $x > 0$ such that $g'(x)$ and $E(X|X \le x)$ exist for all $x > 0$. Then X has the Exponentiated Stretched Exponential Distribution

$$g(x;\lambda,\theta,\alpha) = \frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-\left(\frac{x}{\lambda}\right)^{\theta}} \left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)^{\alpha-1}, \; x > 0, \lambda \in R, \theta > 0, \alpha > 0$$

if and only if $E(X|X \le x) = D(x)\eta(x)$ here $\eta(x) = \frac{g(x)}{G(x)}$ and $D(x)$ is differentiable function with respect to x for all real $x \in (l, m)$. According to Lemma 1; (see Ahsanullah et al. [26]).

Proof: Assume that $\int_0^x u \frac{\theta\alpha}{|\lambda|}\left(\frac{u}{\lambda}\right)^{\theta-1} e^{-\left(\frac{u}{\lambda}\right)^{\theta}} \left(1-e^{-\left(\frac{u}{\lambda}\right)^{\theta}}\right)^{\alpha-1} du = D(x)g(x)$

differentiating both sides of equation give the result

$$x\frac{\theta\alpha}{|\lambda|}\left(\frac{x}{\lambda}\right)^{\theta-1}e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)^{\alpha-1}=D'(x)g(x)+D(x)g'(x),\text{ here}|\lambda|=\lambda,\ x=D'(x)+D(x)\frac{g'(x)}{g(x)}$$

$$\frac{g'(x)}{g(x)}=\frac{x-D'(x)}{D(x)}. \tag{24}$$

Integrating (24), we get $g(x)=ce^{\int_0^x A(u)du}$,

Here $A(u)=\frac{\theta-1}{u}-\frac{\theta}{\lambda}\left(\frac{u}{\lambda}\right)^{\theta-1}+\frac{\theta(\alpha-1)e^{-\left(\frac{u}{\lambda}\right)^{\theta}}}{\lambda\left(1-e^{-\left(\frac{u}{\lambda}\right)^{\theta}}\right)}\left(\frac{u}{\lambda}\right)^{\theta-1}$

$g(x)=ce^{-\left(\frac{x}{\lambda}\right)^{\theta}}x^{\theta-1}\left(1-e^{-\left(\frac{x}{\lambda}\right)^{\theta}}\right)^{\alpha-1}$, where $c=\frac{\theta^{\alpha}}{|\lambda|\lambda^{\theta-1}}$. This completes the proof.

Now the rest of study is based on the applicability with evaluation criteria of introduced and considered models. It is interpreted by three climate parameters and COVID-19 data for drawing conclusion.

9. APPLICATIONS

Visibly, climate change is the major global issue that has occurred strongly during the last two decades on the environment at vulnerable states. Pakistan is one of the states that has recorded vulnerability signal to natural disaster. Seemingly, climate of Pakistan is becoming volatile due to different signals of change. Likewise, Corona virus COVID-19 pandemic is the recorded challenge on the most vulnerable for our time. Every country requires immediate proceed to react and recover. Therefore, analyses were conducted and discussed about these issues as current challenges of time in this study. So, data interpretation bases on climate in Pakistan and total confirmed cases of COVID-19 through subject model. An analogy among subject model along with other models is also pointed up through evaluation criteria for drawing conclusion.

9.1. Evaluation Criteria

The potentiality of subject model ESED was demonstrated by four real data sets. An analogy among models was also put up to evaluate the comparative quality of these statistical models through evaluation criteria including Maximized Log Likelihood $(-2lnL)$, Akaike Information Criterion(AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion $(CAIC)$, Hannan Quinn Information Criteria $(HQIC)$ and Kolmogorov Smirnov (KS) test. These measures are given in the following sequence: Maximized Log likelihood $(-2lnL)$, $AIC=2p-2lnL$, $BIC=pln(n)-2lnL$, $CAIC=\frac{2pn}{n-p-1}-2lnL$, $HQIC=2pln(ln(n))-2lnL$. Where, L,p,n denote the maximized likelihood function, the number of parameters in the model and the total number of observations respectively. The analyses of these measures were obtained trough R software. The list of probability models is decorated in Table 2.

Table 2. *List of classical, modern and new proposed models*

Models		Density Functions
Stretched Exponential Distribution	SED	$f_{SE}(x;a,b)=\frac{b}{a}\left(\frac{x}{a}\right)^{b-1}e^{-\left(\frac{x}{a}\right)^{b}};x>0,a\,in R,b>0$
Exponential distribution	ED	$f_E(x;\lambda)=\lambda e^{-\lambda x};\ x\geq 0,\lambda>0$
Lindley distribution	LD	$f_L(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};x>0,\theta>0$
Weibull distribution	WD	$g_W(x;k,\theta)=\frac{k}{\theta}\left(\frac{x}{k}\right)^{\theta-1}e^{-\left(\frac{x}{k}\right)^{\theta}};x\geq 0,k>0,\theta>0$
A Three Parameter Lindley Distribution	ATPLD	$g_{ATPL}(x;\theta,\alpha,\beta)=\frac{\theta^2}{\alpha\theta+\beta}(\alpha+\beta x)e^{-\theta x}$ $x>\theta,\theta>0\ ,\alpha>0,\beta>0,\alpha\theta+\beta>0$
Transmuted Exponentiated Pareto-I Distribution	TEPID	$g_{TEPI}(x;a,k,\lambda)=ak^{a}e^{-ax}\left[1-\lambda\left(1-2k^{a}e^{-ax}\right)\right]$ $;x>lnk,a>0,k>0,\lvert\lambda\rvert\leq 1$
Lindley Weibull Distribution	LWD	$g_{LW}(x;a,b,\theta)=\frac{a\theta^2}{b(\theta+1)}\left(\frac{x}{b}\right)^{a-1}\left[1+\left(\frac{x}{b}\right)^{a}\right]e^{-\theta\left(\frac{x}{b}\right)^{a}}$ $;x\geq\theta,a>0\ ,b>0,\theta>0$
Exponentiated Weibull Distribution	EWD	$g_{EW}(x;a,b,\theta)=a\alpha\theta^{\alpha}x^{\alpha-1}e^{-(\theta x)^{\alpha}}\left[1-e^{-(\theta x)^{\alpha}}\right]^{a-1}$ $;x>0,a>0,b>0,\theta>0$
Lindley Stretched Exponential distribution	LSED	$g_L(x;a,b,\theta)=\frac{b\theta^2}{\lvert a\rvert(1+\theta)}\left(\frac{x}{a}\right)^{b-1}\left[1+\left(\frac{x}{a}\right)^{b}\right]e^{-\theta\left(\frac{x}{a}\right)^{b}}$ $;x\geq\theta,a\,in\,\mathbb{R},b>0,\theta>0$
Exponentiated Stretched Exponential distribution	ESED	$g_E(x;a,b,\alpha)=\frac{b\alpha}{\lvert a\rvert}\left(\frac{x}{a}\right)^{b-1}e^{-(x/a)^{b}}\left[1-e^{-(x/a)^{b}}\right]^{\alpha-1}$ $;x>0,a\,in\ \mathbb{R},b>0,\alpha>0$

9.2. Climate Data

The conditions of climate change in Pakistan were observed due to global warming. Therefore three data sets of climate parameters including Temperature (ºC) for observed zone of Cherat, Humidity (%) for Gilgit and Wind Speed (knots) for observed zone of Gilgit in Pakistan were selected. Accordingly, the data regarding these three climate parameters were analysed as follows:

Temperature

Temperature numerically indicates hot and cold physical particular of surface. It is the signal or measure of heat that available in every matters. There are frequent measuring scales of temperature including kelvin

(K), Celsius (°C) "OR" Centigrade and Fahrenheit (F). As well, Rankine (R) is also scale of temperature that is less used. Hence, the data of temperature of Cherat, Pakistan from 2007-2014 was analyzed on basis of monthly mean minimum temperature (°C). Data was colleced from MET Office (for detailed study, one can See Majid and Akhter [27]). The Descriptive Statistics with Box and Dot Plots are reported in Table 3.

Table 3. *Descriptive Statistics of temperature (°C) data for zone of Cherat, Pakistan*

Zone	Min.	Max.	Q_1	Median	Q_3	Mean	Variance	SD
Cherat	0.400	22.500	7.125	15.000	19.125	13.326	42.8434	6.5455

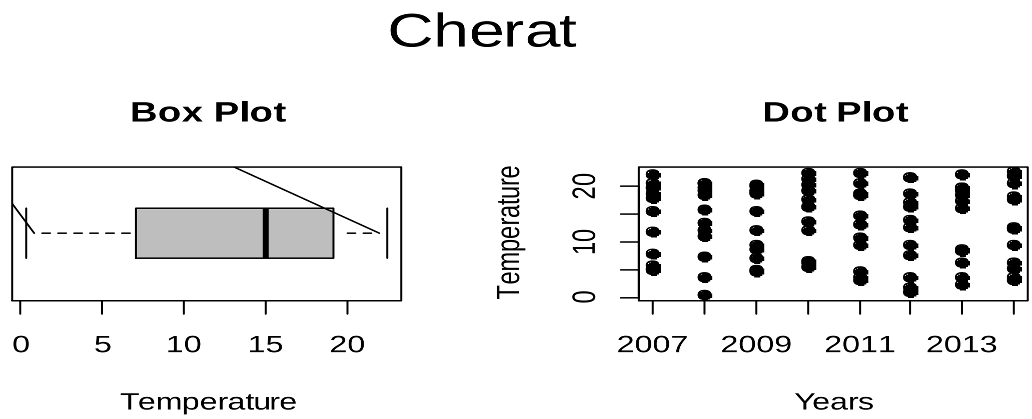

Figure 5. *Box and Dot Plots of temperature (°C) data for zone of Cherat, Pakistan*

Box plot displays skewed right whereas Dot plot represents long diagonals that ascertain the sequence in Figure 5. Descriptive statistics also confirm the right skewness due to greater median than mean. The ML estimates for all considered models are tabulated in Table 4. ESED fits best with minimum measures of evaluation criteria among other models analogically in Table 4.

Table 4. *ML Estimates with Evaluation Criteria for Temperature (°C) data*

Model	ML Estimates			Evaluation Criteria					
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	-2lnL	AIC	CAIC	BIC	HQIC	KS Statistic
LD	0.1408	---	---	656.6507	658.6507	658.6507	658.7063	660.9548	0.0098
ED	13.3280	---	---	685.2263	691.2263	693.7906	691.2688	692.2628	0.0099
LED	348.3428	27.0713	---	689.1330	693.1330	693.2621	698.2617	695.2061	0.0099
WD	14.9454	2.0960	---	633.2894	639.2894	644.4181	639.4184	641.3625	0.0095
EWD	0.1792	8.2922	0.0472	605.2146	611.2146	611.4755	618.9076	614.3242	0.0104
LWD	1.9447	12.6808	1.0985	632.3115	638.3115	638.5723	646.0045	641.4211	0.0897
ATPLD	1.9447	-0.1190	1.4092	653.3171	659.3171	667.0101	659.5779	662.4267	0.0095
TEPID	0.1220	114.1249	0.1003	605.0799	611.0799	618.7730	611.3408	614.1896	0.0117
LSED	12.7122	1.9444	1.1028	632.3115	638.3115	638.5723	646.0045	641.4211	0.0096
ESED	**21.1782**	**8.8773**	**0.1665**	**604.0212**	**610.0212**	**610.2821**	**617.7143**	**613.1309**	**0.0094**

The graphical behavior of ESED is observed in Figure 6.

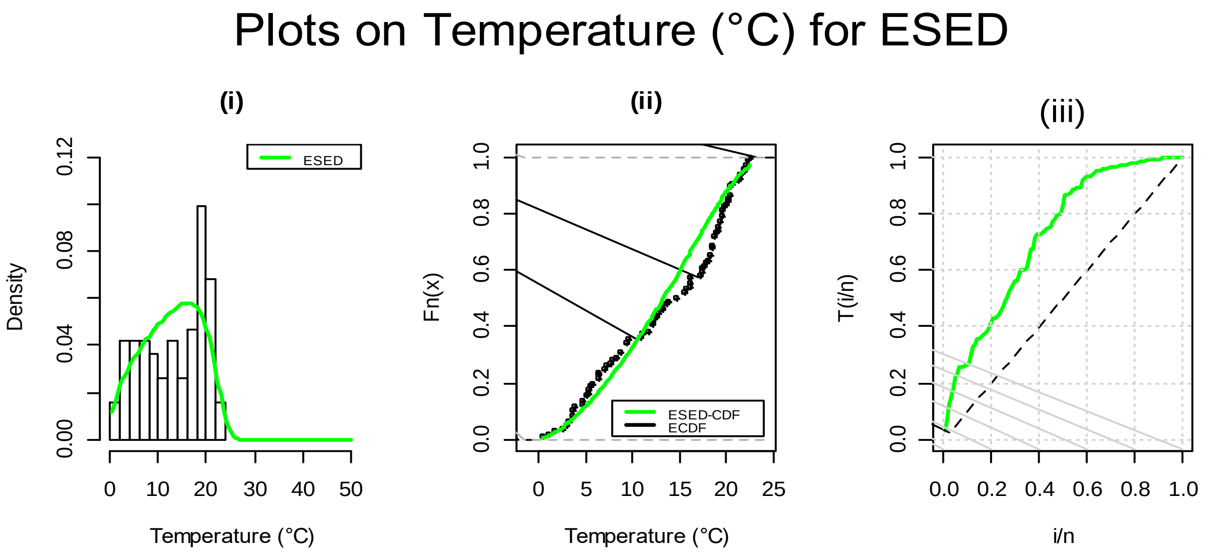

Figure 6. *Estimated Density, Empirical Cumulative Distribution Function and TTT Plots of ESED for zone of Cherat, Pakistan on basis of Temperature (ºC) data*

The estimated density plot, empirical cumulative distribution function (ECDF) plot and TTT plot for ESED on Temperature (ºC) data was shown in Figure 6 respectively. The appearance of TTT plot is concave shaped in Figure 6(iii), thus the hazard rate is increasing for Temperature (ºC) data of Cherat, Pakistan. Figure 6(i) also indicates that ESED is good fit regarding Temperature (ºC) data for zone of Cherat, Pakistan.

Humidity

Humidity assigns to the amount of moisture in the atmosphere at a specific temperature. Humidity is reported in form of percentage. Therefore, humidity (%) data for zone of Muzaffarabad, Pakistan from 2004-2014 was analyzed and interpreted. Data was collected from MET Office. The Descriptive Statistics with Box-Plot and Dot Plot for Humidity (%) data are summarized in Table 5.

Table 5. *Descriptive Statistics of Humidity (%) Data for zone of Muzaffarabad, Pakistan*

Zone	Min.	Q_1	Median	Mean	Q_3	Max.	Variance	SD
Muzaffarabad	60.00	76.75	82.00	80.48	85.00	89.00	37.5952	6.1315

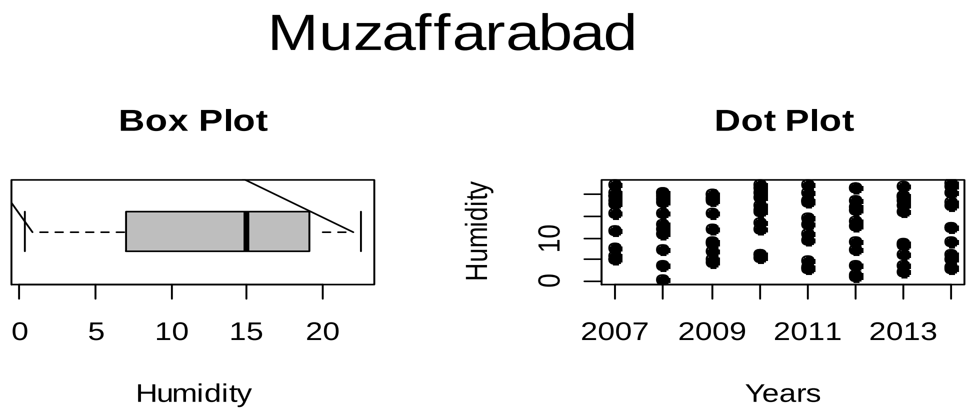

Figure 7. *Box Plot with Dot Plot of Humidity (%) Data for zone of Muzaffarabad, Pakistan*

Descriptive statistics confirm the right skewness according to greater median than mean. Box plot also displays skewed right in Figure 7. Dot plot denotes increasing diagonal in Figure 7. The ML Estimates of considered distributions are specified in Table 6 for humidity (%) data of Muzaffarabad, Pakistan. ESED interprets the best model to fit data for humidity (%) according to its minimum measures of evaluation Criteria in Table 6.

Table 6. *ML Estimates with Evaluation Criteria for Humidity (%) data*

Model	ML Estimates			Evaluation Criteria					
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	-2lnL	AIC	CAIC	BIC	HQIC	KS Statistic
LD	0.0246	---	---	1324.4347	1326.4347	1326.4900	1328.739	1327.354	0.0044
ED	80.4849	---	---	1422.4500	1424.4500	1424.4810	1427.3330	1425.6220	0.0046
LED	136.6530	2.1961	---	1408.9070	1412.9070	1413.0000	1418.6720	1415.2500	0.0044
WD	83.0938	17.9639	---	824.8475	828.8475	828.9405	834.6131	831.1904	0.0061
EWD	77.6189	2.6316	0.0225	880.0710	886.0710	886.2585	894.7195	889.5854	0.0059
LWD	16.7225	82.4355	1.2893	822.5538	828.5538	828.7413	837.2022	832.0681	0.0913
ATPLD	0.0024	34118	89840	1332.3560	1338.3560	1338.5430	1347.0040	1341.8700	0.0044
TEPID	0.0073	0.2037	0.6554	1440.4430	1446.4430	1446.6300	1455.0910	1449.9570	0.0044
LSED	82.4337	16.7140	1.2892	822.5538	828.5538	828.7413	837.2022	832.0682	0.0063
ESED	**87.0982**	**63.3503**	**0.1881**	**807.9783**	**813.9783**	**814.1658**	**822.6267**	**817.4926**	**0.0062**

The graphical view is also accorded in Figure 8 for Humidity (%) data.

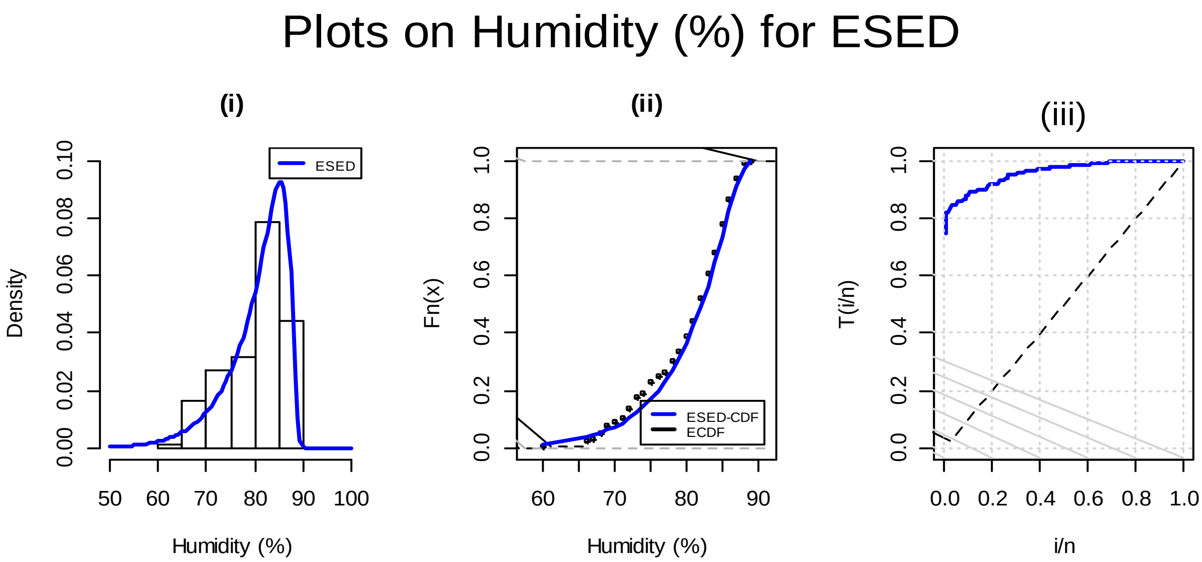

Figure 8. *Estimated Density, Empirical Cumulative Distribution Function and TTT Plots of ESED for zone of Muzaffarabad, Pakistan*

Figure 8(i) frames the estimated density plot, Figure 8(ii) shows empirical cumulative distribution function (ECDF) plot whereas Figure 8(iii) indicates TTT plot for ESED on Humidity (%) data. The hazard rate is increasing due to concave shaped of TTT plot for Humidity (%) data for zone of Muzaffarabad, Pakistan.

Wind Speed

Wind speed is a basic atmospheric measure caused by air moving from high to low pressure, frequently affected by changes in temperature of a zone or surface of earth. Wind speed is usually measured with an anemometer. The most common unit of wind speed is meters per second (m/s) whereas, the SI unit for velocity, kilometres per hours (km/h). Some units are used for historical reasons such as miles per hour (mph), knots (kn) or feet per seconds (ft/s). So, the data of Wind speed (knots) of Gilgit, Pakistan was analyzed from 2010-2014. Data Source is MET Office. The Descriptive Statistics with Box and Dot Plots for Wind speed (knots) data are summarized in Table 7.

Table 7. *Descriptive Statistics of Wind Speed (knots) data for zone of Gilgit, Pakistan*

Zone	Min.	Max.	Q_1	Median	Q_3	Mean	Variance	SD
Gilgit	0.600	2.600	1.200	1.600	2.000	1.577	0.2357	0.4855

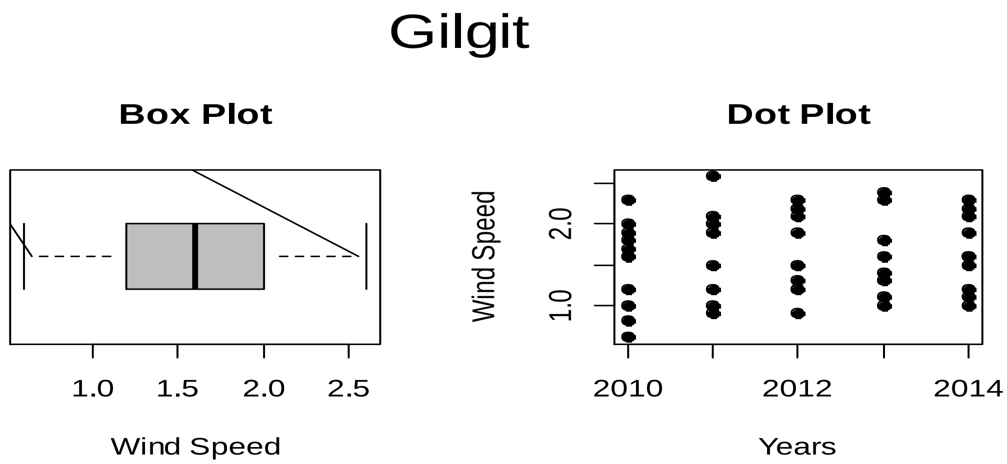

Figure 9. *Box and Dot Plots of Wind Speed (knots) data for zone of Gilgit, Pakistan*

Box plot confers that Wind speed (knots) data is skewed right for Gilgit in Figure 9. Descriptive statistics of Wind speed (knots) data of Gilgit also confirm the right skewness because median is higher than mean in Table 7. Dot plot signifies long diagonals as well as frequently continuous sequences are linked with full length in Figure 9. The ML estimates for all considered models are indexed in Table 8. ESED fits and interprets the best model on account of minimum measures of evaluation criteria among other models analogically in Table 8.

Table 8. *ML Estimates with Evaluation Criteria for Wind Speed (knots) data*

Model	ML Estimates			Evaluation Criteria					
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	-2lnL	AIC	CAIC	BIC	HQIC	KS Statistic
LD	0.9581	---	---	160.7400	162.7400	166.9071	173.6526	169.4978	0.0088
ED	1.5766	---	---	174.6375	176.6375	176.7065	178.7319	177.4568	0.0122
LED	295.6023	188.4766	---	174.6360	178.6360	178.8466	182.8247	180.2745	0.0122
WD	1.8425	2.6514	---	92.8452	96.8452	97.0557	101.0339	98.4836	0.0134
EWD	1.2942	3.1601	0.6106	81.9296	87.9296	87.6382	93.4927	89.6673	0.1058
LWD	3.6305	2.7669	6.0249	81.4489	87.4489	87.6775	93.5319	89.7065	0.1224
ATPLD	0.6352	500.000	0.5009	174.6374	180.6374	181.0660	186.9204	183.0951	0.0122
TEPID	1.2941	3.1601	0.6106	81.9620	87.9620	87.6382	93.4927	89.6673	0.0485
LSED	2.7526	3.6293	5.9212	81.4290	87.4290	87.6776	93.5321	89.7067	0.0150
ESED	**1.6382**	**3.1618**	**1.2930**	**81.3628**	**87.3628**	**89.6673**	**93.4927**	**80.8978**	**0.04845**

Table 8 specifies the ML Estimates of introduced as well as considered distributions for wind speed (knots) data of Gilgit, Pakistan. Table 8 interprets that ESED provides best competitor to other models used for fitting wind speed (knots) data of Gilgit, Pakistan. Hence, ESED results the best fit as reported by its minimum measures of evaluation criteria. The graphical behavior of ESED is also appeared in Figure 10.

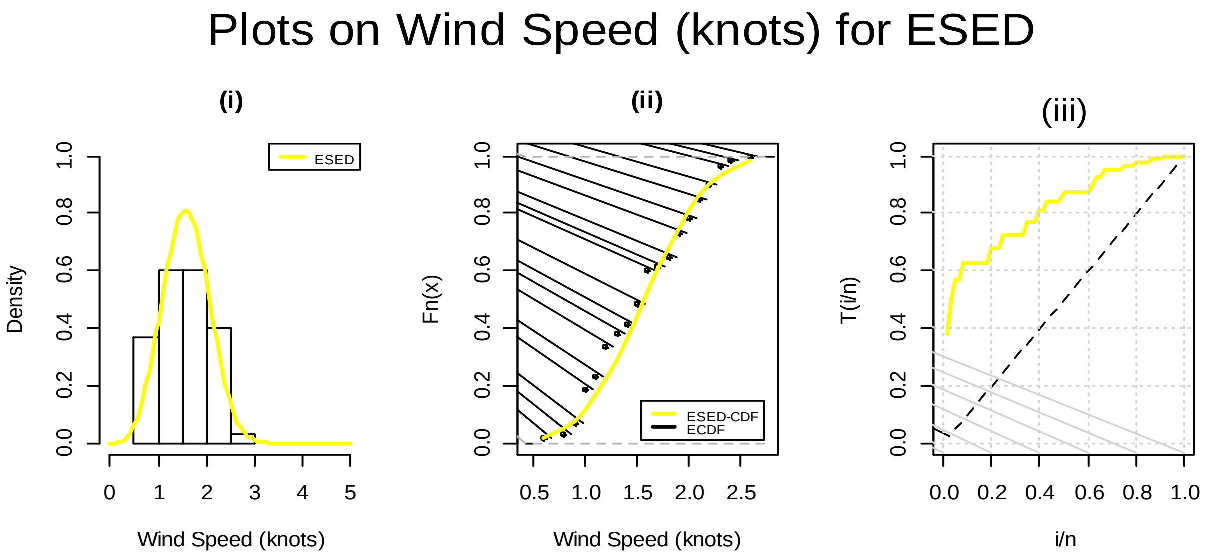

Figure 10. *Estimated Density, Empirical Cumulative Distribution Function and TTT Plots of ESED for zone of Gilgit, Pakistan on basis of Wind Speed (knots) data*

Figure 10(i) exhibits estimated density plot, 10(ii) reveals empirical cumulative distribution function (ECDF) plot and 10(iii) indicates TTT plot for ESED on Wind speed (knots) data respectively. Since TTT plot is concave shaped, thus the shape of hazard rate is increasing for ESED on Wind speed (knots) data for the zone of Gilgit, Pakistan.

9.3. COVID-19

The corona virus COVID-19 pandemic is the greatest challenge on the most vulnerable for our time. Every country needs immediate act to respond and recover. An analysis is conducted about total confirmed cases of COVID-19 in this study. Data represent the countries, territories and areas with reported laboratory-confirmed COVID-19 cases from 30 December 2019 to 16 June 2020 by WHO- region complied by John Hopkins CSSE. Descriptive statistics are reported in Table 9.

Table 9. *Descriptive Statistics for total Confirmed Cases of COVID-19 data*

Minimum	Q_1	Median	Mean	Q_3	Maximum	Variance
1.0	202.5	1715.0	37602.5	13779.5	2079592.0	27370715189

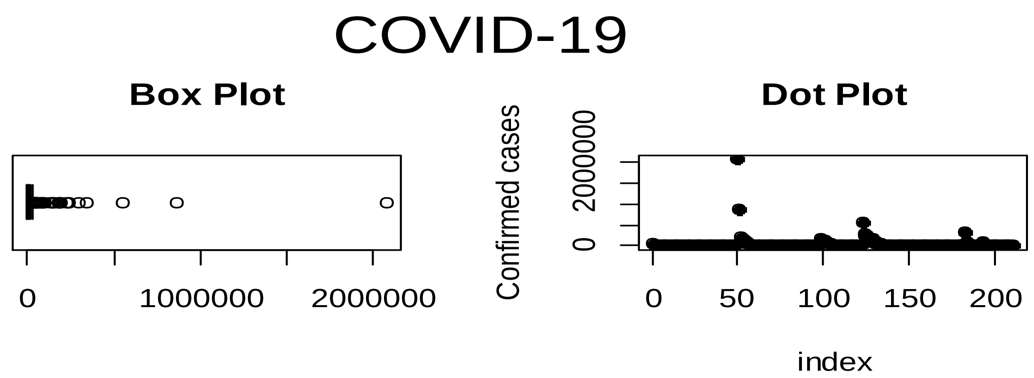

Figure 10. *Box Plot for total Confirmed Cases of COVID-19 data*

Box plot disposes that total confirmed cases of COVID-19 data is skewed left in Figure 10. Descriptive statistics also confirm the left skewness due to median (1715.0) is lower than mean (37602.5). Table 10 identifies the ML Estimates of all models to interprets the total confirmed cases of COVID-19 data. Table 10 highlights that ESED yields the best model than other models on the comparison for fitting total confirmed cases of COVID-19 data concerning minimum measures of evaluation criteria.

Table 10. *ML Estimates with Evaluation Criteria for total Confirmed Cases of COVID-19 data*

Model	ML Estimates			Evaluation Criteria					
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	-2*lnL*	AIC	CAIC	BIC	HQIC	KS Statistic
LD	0.0090	---	---	144446.4	144448.4	144448.5	144450.7	144449.3	0.0047
ED	27122.1400	---	---	4892.8880	4894.8870	4894.9060	4898.2390	4896.2420	0.0047
LED	99900.1515	3.2778	---	4917.1140	4921.1150	4921.1730	4927.8190	4923.8250	0.0047
WD	9999	0.3005	---	4331.6300	4335.6300	4335.6880	4342.3340	4338.3400	0.0044
EWD	0.4818	32.7629	1.4334	188294.6	188296.6	188296.7	188298.9	188297.5	0.0023
LWD	0.3232	998.2238	0.8693	4196.078	4202.0770	4202.1930	4212.1330	4206.1420	0.3130
ATPLD	4144	3890	0.0065	104249.2	104251.2	104251.3	104253.5	104252.1	0.0017
TEPID	999.8679	0.2450	1.4262	4186.4440	4198.3860	4198.5800	4202.7940	194.8060	0.0044
LSED	38.9596	0.2983	0.3963	4193.856	4199.8560	4199.9720	4209.912	4203.912	0.0043
ESED	**999.9770**	**0.2659**	**2.0792**	**4179.2200**	**4185.2190**	**4185.3350**	**4195.2750**	**4189.2840**	**0.0044**

Figure 11(i) recommends estimated density plot, 11(ii) specifies empirical cumulative distribution function (ECDF) plot and 11(iii) points out TTT plot for ESED on total confirmed cases COVID-19 data. Since TTT plot is convex shaped, thus hazard rate is decreasing for ESE D on total confirmed cases COVID-19 data.

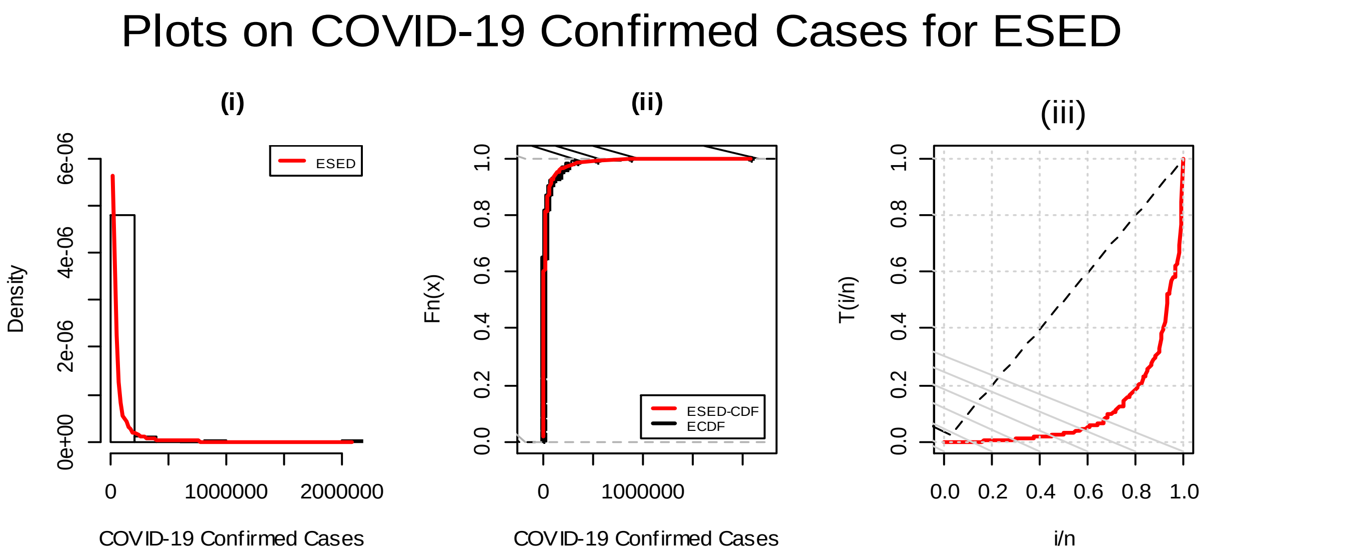

Figure 11. *Estimated Density, Empirical Cumulative Distribution Function and TTT Plots of ESED for total Confirmed Cases of COVID-19 data*

10. CONCLUSION

In this paper, Stretched Exponential distribution is extended through addition of a positive shape parameter by applying exponentiation technique. Special models of ESED are discussed and slight derivation of its statistical properties in modeling section are made. Method of maximum likelihood estimation is used to estimate its parameters. Further, the usefulness of this innovative distribution was demonstrated by four real data sets. It was observed numerically and graphically from findings and results that ESED interprets best model to the monthly mean minimum Temperature (ºC) data for zone of Cherat, Humidity (%) data for zone of Gilgit, Wind Speed (knots) data for zone of Gilgit, Pakistan as well as total confirmed cases of COVID-19 with minimum measures of evaluation criteria i.e. $-2lnL, AIC, BIC, CAIC, HQIC$ on the comparison. Finally, it was concluded that ESED has proved the best fit among all the competing distributions for all real data sets. Further, ESED can be studied and applied on other parameters of climate for other countries as well as states and diseases.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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