

On the Asymptotic Stability of the Nonlinear Difference Equation System

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Abstract

In this paper, we obtain some new results on the equi-boundedness of solutions and asymptotic stability for a class of nonlinear difference systems with variable delay of the form

$$x(n+1) = ax(n) + B(n)F(x(n-m(n))), \quad n = 0, 1, 2, \dots$$

where F is the real valued vector function, $m: \mathbb{Z} \rightarrow \mathbb{Z}^+$, which is bounded function and maximum value of m is k and $B(n)$ is a $k \times k$ variable coefficient matrix. We carry out the proof of our results by using the Banach fixed point theorem and we use these results to determine the asymptotic stability conditions of an example.

1. Introduction

Nonlinear difference equations are the proper mathematical representation for discrete processes, which have remarkable importance in areas such as Nicholson's blowflies model, bobwhite quail population model and predator-prey models. Recently, many researchers have investigated asymptotic behaviour of the solutions of the nonlinear difference equations. Especially, Huong et al. [5, 6, 9, 10, 11, 12, 13, 14] have done important studies in this area in the last 20 years. For instance Giang-Huong [5, 6] obtained some new results for the asymptotic behaviour of solutions of nonlinear difference equations with time-invariant delay of the form

$$x(n+1) = \lambda x(n) + F(x(n-m)) \quad n = 0, 1, \dots,$$

where $F: [0, \infty) \rightarrow [0, \infty)$ is a continuous function, $m \geq 0$ is a fixed integer and $\lambda \in (0, 1)$. In addition, Huong [10] studied nonlinear difference equation with bounded multiple delay of the form

$$x(n+1) = \lambda(n)x(n) + \sum_{i=1}^r \alpha_i(n)F(x(n-m(i))), \quad (1.1)$$

where $n \in \mathbb{N}$ with $n \geq a$ for some $a \in \mathbb{N}$, where $r, m(i) \geq 1$, $1 \leq i \leq r$ are fixed positive integers, the functions α_i are defined on \mathbb{N} and the functions F are defined on \mathbb{R} . Also, Huong [11] obtained some oscillatory results for equation (1.1). Huong and Nam [12] investigated the oscillation, convergence and boundedness of solution of some nonlinear difference equations with multiple delay of the form

$$x(n+1) = G(x(n), x(n-m_1), \dots, x(n-m_r)) \quad n = 0, 1, \dots,$$

where $m_i \in \mathbb{N}_0$, $\forall i = 1, \dots, r$ and the function $G: \mathbb{R}^{m+1} \rightarrow \mathbb{R}$. After, Huong and Mau [9] studied on the stability of the zero of autonomous nonlinear difference equation with variable delay of the form

$$x(n+1) = \lambda(n)x(n) + \alpha(n)F(x(n-m(n))) \quad n = 0, 1, \dots$$

where the functions λ , α are defined on the set of integers, the function m maps the set of integers to the set of positive integers, the function F is defined on the set of real numbers. Moreover, Huong [14] gave the some new results stability and strict boundedness conditions of non-autonomous nonlinear difference equation with time-varying delay of the form

$$x(n+1) = \lambda(n)x(n) + \alpha(n)F(n, x(n-m(n))) \quad n = 0, 1, \dots$$

In addition to Huong's investigations, in the last century, there has been many literature on stability of delay difference equations. In particular, as examples of studies, Györi, Ladas and Vlahos [8], Chen and Liu [2], Graef and Qian [7], Edwards and Neville [4], Rao and Sudha [18], Asiri and Elsayed [19], Değer and Bolat [20], Muroya, Ishiwata and Guglielmi [16], Akgul, Inc and Karatas [22], Akgul [23], Zhang [21], Liao [15] can be given. In this study, we obtain some new results on the equi-boundedness of solutions and asymptotic stability for a class of nonlinear difference systems with variable delay of the form

$$x(n+1) = ax(n) + B(n)F(x(n-m(n))), \quad n = 0, 1, 2, \dots \quad (1.2)$$

where F is the real valued vector function, $m: \mathbb{Z} \rightarrow \mathbb{Z}^+$, which is bounded function and maximum value of m is k and $B(n)$ is a $k \times k$ variable coefficient matrix. Let \mathbb{Z}_0 be a set of integers belonging to the interval $[n_0 - k, n_0]$ for each integer $n_0 \geq 0$ (if m is unbounded, then \mathbb{Z}_0 as a set of integers belonging to the interval $(-\infty, n_0]$). Also, let $\phi: \mathbb{Z}_0 \rightarrow \mathbb{R}^k$ be an initial discrete bounded vector function. Now, we can give some definitions and proposition for the proof of the Lemma and Theorems.

Definition 1.1. If $x(n) = \phi(n)$ on \mathbb{Z}_0 and satisfies (1.2) for $n \geq n_0$, then $x(n) = x(n, n_0, \phi)$ is a solution of (1.2) [9].

Definition 1.2. If for any $\varepsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta(n_0, \varepsilon) > 0$ such that $\|\phi(n)\| \leq \delta$ on \mathbb{Z}_0 implies

$$\|x(n, n_0, \phi)\| \leq \varepsilon \text{ for } n \geq n_0,$$

then the zero solution of (1.2) is Liapunov stable [9].

Definition 1.3. If the zero solution of (1.2) is Liapunov stable and if for any integer $n_0 \geq 0$ there exists $\mu(n_0) > 0$ such that $\|\phi(n)\| \leq \mu(n_0)$ on \mathbb{Z}_0 implies

$$\|x(n, n_0, \phi)\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then the zero solution of (1.2) is asymptotically stable [9].

Definition 1.4. If there exists a $B(n_0, \phi) > 0$ such that $\|x(n, n_0, \phi)\| \leq B(n_0, \phi)$ for $n \geq n_0$, then a solution $x(n) = x(n, n_0, \phi)$ of (1.2) is said to be bounded [9].

Definition 1.5. If for any n_0 and any $B_1 > 0$ there exists $B_2 = B_2(n_0, B_1) > 0$ such that $\phi(n) \leq B_1$ on \mathbb{Z}_0 implies

$$\|x(n, n_0, \phi)\| \leq B_2 \text{ for } n \geq n_0,$$

then solutions of (1.2) is equi-bounded [9].

Definition 1.6. A Banach space is a complete, normed, vector space.

Definition 1.7. Let $P: X \rightarrow X$ be a mapping from a set X to itself. We call a point $x \in X$ a fixed point of P if $P(x) = x$ [3].

Proposition 1.8. Let X be a Banach space and $P: X \rightarrow X$ be a map such that

$$\|Px - Py\| \leq \alpha \|x - y\| \quad (1.3)$$

for some $0 \leq \alpha < 1$ and all $x, y \in X$. Then P has a unique fixed point in X . Moreover for any $x_0 \in X$ the sequence of iterates x_0, Px_0, PPx_0, \dots converges to the fixed point of P [3].

When $\|Px - Py\| \leq \alpha \|x - y\|$ for some $0 \leq \alpha < 1$ and all $x, y \in X$, P is called contraction. Also, the fact that P is contraction implies the existence of a solution of equation (1.3). A contraction shrinks distances by a uniform factor α less than 1 for all pairs of points. Proposition

1.8 is called the contraction mapping theorem or Banach's fixed-point theorem. Also, for any sequence (x_k) , we denote $\sum_{k=a}^b x_k = 0$, $\prod_{k=a}^b x_k = 1$

for any $a > b$.

2. Main Results

Proposition 2.1. Suppose that $a \in (-1, 1) - \{0\}$ and $B(n)$ is a nonsingular matrix function for all $n \in \mathbb{Z}$. Then $x(n)$ is a solution of system (1.2) if and only if

$$x(n) = x(n_0)a^{n-n_0} + \sum_{t=n_0}^{n-1} B(t)F(x(t-m(t)))a^{n-t-1}. \quad (2.1)$$

Proof. Consider the system (1.2). We multiply both sides of equation (1.2) by $\prod_{s=n_0}^n a^{-1}$

$$\Delta \left(x(n) \prod_{s=n_0}^{n-1} a^{-1} \right) = B(n) F(x(n-m(n))) \prod_{s=n_0}^n a^{-1} \tag{2.2}$$

is obtained. If equation (2.2) is summed from n_0 to $n-1$,

$$\begin{aligned} \sum_{t=n_0}^{n-1} \Delta \left(x(n) \prod_{s=n_0}^{n-1} a^{-1} \right) &= \sum_{t=n_0}^{n-1} B(n) F(x(n-m(n))) \prod_{s=n_0}^t a^{-1}, \\ x(n) \prod_{s=n_0}^{n-1} a^{-1} - x(n_0) &= \sum_{t=n_0}^{n-1} B(n) F(x(n-m(n))) \prod_{s=n_0}^t a^{-1}, \\ x(n) \prod_{s=n_0}^{n-1} a^{-1} &= x(n_0) + \sum_{t=n_0}^{n-1} B(n) F(x(n-m(n))) \prod_{s=n_0}^{n-1} a^{-1}. \end{aligned}$$

At last we have

$$x(n) = x(n_0) \prod_{s=n_0}^{n-1} a + \sum_{t=n_0}^{n-1} B(n) F(x(n-m(n))) \prod_{s=t+1}^{n-1} a. \tag{2.3}$$

The proof is completed. □

Theorem 2.2. *Suppose that the following conditions are satisfied:*

(i) $F(0) = 0$ and F is locally Lipschitz in x . That is, there is a $M > 0$ such that if $\|x\|, \|y\| \leq M$, then

$$\|F(x) - F(y)\| \leq L \|x - y\| \tag{2.4}$$

for positive constant L .

(ii) There exists $a \in (-1, 1) - \{0\}$, $b \in (0, 1)$ and $\|B(t)\| = \|B(t)\|_1$ such that

$$L \sum_{t=n_0}^{n-1} \|B(t)\| \leq b, \tag{2.5}$$

for $n \geq n_0$. Then the solutions of (1.2) are equi-bounded.

Proof. Let B_1 be a positive constant. We choose $B_2 > 0$ such that $B_1 \leq (1-b)B_2$. Also, let $\phi(n)$ be an initial discrete bounded function which satisfies $\|\phi(n)\| \leq B_1$ on \mathbb{Z}_0 of (1.2) for $n \geq n_0$. We define

$$H = \left\{ \mu : \mathbb{Z} \rightarrow \mathbb{R}^k \mid \mu(n) = \phi(n) \text{ on } \mathbb{Z}_0 \text{ and } \|\mu\| \leq B_2 \right\}, \tag{2.6}$$

where $\|\mu\| = \max_{n \in \mathbb{Z}} \|\mu(n)\|$. Firstly, we must show that $(H, \|\cdot\|)$ is a complete metric space. $(H, \|\cdot\|)$ is a metric space. It is clear that the metric space conditions are provided. Now we shall prove that every Cauchy sequence of points in H has a limit that is also in H . Assume that (μ^β) is a Cauchy sequence in H . Then, we have

$$\forall \varepsilon > 0, \exists \beta_0 : \forall \alpha, \beta \geq \beta_0 : \|\mu^\beta - \mu^\alpha\| < \varepsilon,$$

or

$$\forall \varepsilon > 0, \exists \beta_0 : \forall \alpha, \beta \geq \beta_0 : \max_{n \in \mathbb{Z}} \|(\mu^\beta - \mu^\alpha)(n)\| < \varepsilon,$$

or

$$\forall \varepsilon > 0, \exists \beta_0 : \forall \alpha, \beta \geq \beta_0 : \|(\mu^\beta - \mu^\alpha)(n)\| < \varepsilon, \quad \forall n \in \mathbb{Z}.$$

$(\mu^\beta(n))$ is a Cauchy sequence in \mathbb{R}^k for a fixed $n \in \mathbb{Z}$. we know that \mathbb{R}^k is a complete metric space and thus, we can write

$$\mu(n) = \lim_{\beta \rightarrow \infty} \mu^\beta(n), \quad \exists \mu(n) \in \mathbb{R}^k. \tag{2.7}$$

Since $\mu^\beta \in H$, we get $\mu^\beta(n) = \phi(n)$ on \mathbb{Z}_0 . By (2.6), we know that

$$\mu(n) = \phi(n).$$

Furthermore, we have $\|\mu\| \leq B_2$ with $\|\mu^\beta\| \leq B_2$. That is $\mu \in H$. We define a mapping $P : H \rightarrow H$ such that

$$(P\mu)(n) = \phi(n) \text{ on } \mathbb{Z}_0,$$

and

$$(P\mu)(n) = \phi(n_0)a^{n-n_0} + \sum_{t=n_0}^{n-1} B(t)F(\mu(t-m(t)))a^{n-t-1}. \quad (2.8)$$

Notice that P maps from H to itself. Indeed, the properties of norm

$$\|(P\mu)(n)\| = \left\| \phi(n_0)a^{n-n_0} + \sum_{t=n_0}^{n-1} B(t)F(\mu(t-m(t)))a^{n-t-1} \right\|,$$

we have

$$\|(P\mu)(n)\| \leq \|\phi(n_0)\| |a^{n-n_0}| + \sum_{t=n_0}^{n-1} \|B(t)\| \|F(\mu(t-m(t)))a^{n-t-1}\|. \quad (2.9)$$

By $\|\mu\| \leq B_2$, $\|\mu(t-m(t))\| \leq B_2$. Hence, we can write

$$\|F(\mu(t-m(t)))\| \leq L\|\mu(t-m(t))\| \leq LB_2. \quad (2.10)$$

Then, from (2.9) and (2.10), we obtain

$$\begin{aligned} \|(P\mu)(n)\| &\leq B_1 + B_2L \left\| \sum_{t=n_0}^{n-1} B(t)a^{n-t-1} \right\| \\ &\leq B_1 + B_2L \left\| \sum_{t=n_0}^{n-1} B(t) \right\| \\ &\leq B_1 + B_2b \leq B_2. \end{aligned}$$

So, we can say that P maps from H to H . Now, let $\mu, \nu \in H$, then we get

$$\begin{aligned} \|(P\mu)(n) - (P\nu)(n)\| &= \left\| \sum_{t=n_0}^{n-1} B(t)F(\mu) a^{n-t-1} - \sum_{t=n_0}^{n-1} B(t)F(\nu) a^{n-t-1} \right\| \\ &\leq L \sum_{t=n_0}^{n-1} \|B(t)\| \|\mu - \nu\| |a^{n-t-1}| \\ &\leq b\|\mu - \nu\|. \end{aligned}$$

Thus, we have shown that the mapping P is a contraction under the supremum norm. According to the contraction mapping principle, P has a unique fixed point $\mu^* \in H$. So, we have $(P\mu^*)(n) = \mu^*(n)$. By $n_0 \in \mathbb{Z}_0$ and $\mu^* \in H$, we can write $\phi(n_0) = \mu^*(n_0)$. Hence

$$\mu^*(n) = \phi(n_0)a^{n-n_0} + \sum_{t=n_0}^{n-1} B(t)F(\mu^*(t-m(t)))a^{n-t-1},$$

that is, $\mu^* \in H$ is a solution of (1.2) and solutions of (1.2) is equi-bounded. The proof is completed. \square

Corollary 2.3. Suppose that the conditions of Theorem 2.1 are satisfied. Then the zero solution of (1.2) is Liapunov stable.

Proof. Let $\varepsilon > 0$ be such that $b\varepsilon \in (0, 1)$. If we choose $0 < \delta \leq \varepsilon(1-b)$, then $\delta + b\varepsilon \leq \varepsilon$. Also, let $\phi(n)$ be an initial discrete bounded function which satisfies $\|\phi(n)\| \leq \delta$ on \mathbb{Z}_0 of (1.2) for $n \geq n_0$. We define

$$H = \left\{ \mu : \mathbb{Z} \rightarrow \mathbb{R}^k \mid \mu(n) = \phi(n) \text{ on } \mathbb{Z}_0 \text{ and } \|\mu\| \leq \varepsilon \right\}, \quad (2.11)$$

where $\|\mu\| = \max_{n \in \mathbb{Z}} \|\mu(n)\|$. It can be shown easily that $(H, \|\cdot\|)$ is a complete metric space similar to proof of Theorem 2.1. Let us define the mapping $P : H \rightarrow H$ as follows

$$(P\mu)(n) = \phi(n_0)a^{n-n_0} + \sum_{t=n_0}^{n-1} B(t)F(\mu(t-m(t)))a^{n-t-1}.$$

Also, by proof of Theorem 2.1, we know that P is a contraction and $\|P\mu\| \leq \varepsilon$ for any $\mu \in H$. Hence, the zero solution of (1.2) is Liapunov stable. Therefore, the proof is completed. \square

Theorem 2.4. Suppose that the following conditions are satisfied:

(i) $F(0) = 0$ and F is locally Lipschitz in x . That is, there is a $M > 0$ such that if $\|x\|, \|y\| \leq M$, then

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad (2.12)$$

for positive constant L .

(ii) There exist $a \in (-1, 1) - \{0\}$, $b \in (0, 1)$ and $\|B(t)\| = \|B(t)\|_1$ such that

$$L \sum_{t=n_0}^{n-1} \|B(t)\| \leq b, \quad (2.13)$$

for $n \geq n_0$.

(iii) $|n - m(n)| \rightarrow \infty$ as $n \rightarrow \infty$.

Then the zero solution of (1.2) is asymptotically stable.

Proof. Let $\phi(n)$ be an initial discrete bounded function which satisfies $\|\phi(n)\| \leq \eta(n_0)$ on \mathbb{Z}_0 of (1.2) for $n \geq n_0$. We define

$$H^* = \left\{ \mu : \mathbb{Z} \rightarrow \mathbb{R}^k \mid \mu(n) = \phi(n) \text{ on } \mathbb{Z}_0, \|\mu\| \leq \varepsilon \text{ and } \|\mu(n)\| \rightarrow 0 \text{ as } n \rightarrow \infty \right\},$$

where $\|\mu\| = \max_{n \in \mathbb{Z}} \|\mu(n)\|$. It can be easily shown that $(H^*, \|\cdot\|)$ is a complete metric space similar to proof of Theorem 2.1. Let us define the mapping $P : H^* \rightarrow H^*$ as follows

$$(P\mu)(n) = \phi(n_0) a^{n-n_0} + \sum_{t=n_0}^{n-1} B(t) F(\mu(t-m(t))) a^{n-t-1}. \tag{2.14}$$

From the proof of Theorem 2.1, P is a contraction and it maps from H to itself. For the asymptotic stability, we shall show that

$$(P\mu)(n) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.15}$$

Now, since $|a| < 1$,

$$\prod_{s=n_0}^{n-1} a = a^{n-n_0} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.16}$$

Thus, $\phi(n_0) a^{n-n_0} \rightarrow 0$. In that case, it will be enough to show that

$$\sum_{t=n_0}^{n-1} B(t) F(\mu(t-m(t))) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Let $\mu \in H^*$, then $\|\mu(t-m(t))\| \leq \varepsilon$. Since $\mu(t-m(t)) \rightarrow 0$ as $n-m(n) \rightarrow \infty$, there exists an $\varepsilon_1 > 0$ and $n_1 > 0$ such that $\|\mu(t-m(t))\| < \varepsilon_1$ for $n > n_1$. Also, from (2.16), there exist an $n_2 > n_1$ such that $|a^{n-n_0}| \leq \frac{\varepsilon_1}{b\varepsilon}$ for $n > n_2$. Hence, for all $n > n_2$

$$\begin{aligned} & \left\| \sum_{t=n_0}^{n-1} B(t) F(\mu(t-m(t))) a^{n-t-1} \right\| \leq \sum_{t=n_0}^{n-1} \left\| B(t) F(\mu(t-m(t))) a^{n-t-1} \right\|, \\ & \leq \sum_{t=n_0}^{n_1-1} \left\| B(t) F(\mu(t-m(t))) a^{n-t-1} \right\| + \sum_{t=n_1}^{n-1} \left\| B(t) F(\mu(t-m(t))) a^{n-t-1} \right\| \\ & \leq L \sum_{t=n_0}^{n_1-1} \left\| B(t) \mu(t-m(t)) a^{n-t-1} \right\| + L \sum_{t=n_1}^{n-1} \left\| B(t) \mu(t-m(t)) a^{n-t-1} \right\| \\ & \leq \varepsilon L \sum_{t=n_0}^{n_1-1} \left\| B(t) a^{n-t-1} \right\| + \varepsilon_1 L \sum_{t=n_1}^{n-1} \left\| B(t) a^{n-t-1} \right\| \\ & \leq \varepsilon L \sum_{t=n_0}^{n_1-1} \left\| B(t) a^{n-t-1} \right\| + \varepsilon_1 L \sum_{t=n_1}^{n-1} \left\| B(t) a^{n-t-1} \right\| \\ & \leq \varepsilon L \sum_{t=n_0}^{n_1-1} \left\| B(t) \prod_{s=t+1}^{n-1} a \prod_{s=n_1}^{n-1} a \right\| + \varepsilon_1 b \\ & \leq \varepsilon L \left| \prod_{s=n_1}^{n-1} a \right| \sum_{t=n_0}^{n_1-1} \|B(t)\| \left| \prod_{s=t+1}^{n_1-1} a \right| + \varepsilon_1 b \\ & \leq \varepsilon_1 + \varepsilon_1 b. \end{aligned}$$

This implies that (2.15) is occurs. According to the contraction mapping principle, P has a unique fixed point which solves (1.2). The proof is completed. \square

Example 2.5. Consider difference equation system

$$x(n+1) = \frac{1}{2}x(n) + B(n)F\left(x\left(n - \left\lceil \frac{n}{2} \right\rceil\right)\right), \quad n \geq 0, \tag{2.17}$$

where $\lceil \cdot \rceil$ denotes greater integer function, $a = \frac{1}{2}$, $F(x) = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $2^{-2n+4m(n)} > 1$ for $m(n) = \lceil \frac{n}{2} \rceil$ and

$$B(n) = \begin{pmatrix} \frac{1+2^{-2n+2m(n)}-2^{-2m(n)}}{2^{-m(n)}(1+2^{-2n+2m(n)})} & \frac{2^{-m(n)}}{1+2^{-2n+2m(n)}} \\ \frac{3+4 \cdot 2^{-2n+2m(n)}}{2^{2m(n)+2}(1+2^{-2n+2m(n)})} & \frac{1}{2^{2m(n)+2}(1+2^{-2n+2m(n)})} \end{pmatrix}, \text{ for } \forall n \in \mathbb{N}.$$

then the zero solution of (2.17) is asymptotically stable. Indeed,

(i) there is a $L = 1$ such that $\|x\|_1 = |x_1| + |x_2| \leq 1$, then

$$\begin{aligned} \|F(x)\|_1 &= |x_1^2| + |x_2^2| = |x_1|^2 + |x_2|^2 \\ &\leq (|x_1| + |x_2|)^2 \\ &\leq |x_1| + |x_2| = \|x\|_1 \end{aligned}$$

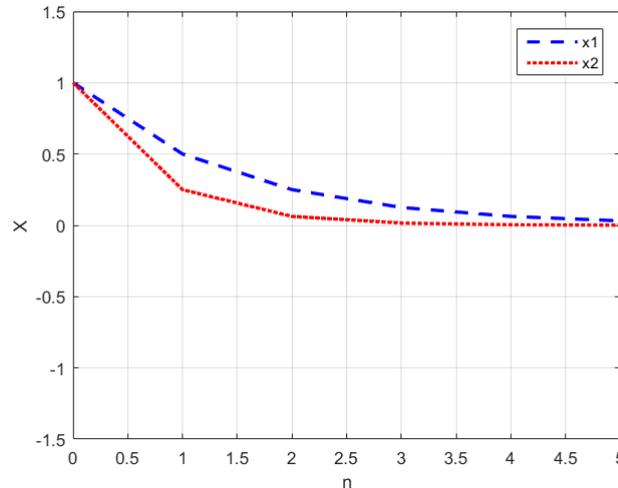


Figure 2.1: The behavior of the solution $x(n) = \begin{pmatrix} 2^{-n} \\ 4^{-n} \end{pmatrix}$ of system (2.17)

for $M = 1$.

$$(ii) \|B(n)\|_1 = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}| \right) = \frac{1}{2^{m(n)}} \leq 1$$

and $\lim_{n \rightarrow \infty} \frac{1}{2^{m(n)}} = 0$. Furthermore, there exist $b \in (0, 1)$ such that

$$\sum_{t=n_0}^{n-1} \|B(t)\| \leq b, \quad (2.18)$$

for $n \geq n_0$.

$$(iii) \left| n - \left\lfloor \frac{n}{2} \right\rfloor \right| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Since the conditions of the theorem are provided, the system (2.17) is asymptotically stable. The behavior of the solutions of the system (2.17) is as Figure 1.

3. Conclusion

In this paper, we investigate the asymptotic stability of zero solution of system (1.2) which as an generalization of [9]. We used the Banach's fixed point theorem for the proof of the results. In this study, by using the definition of equi-bounded, initially, we showed that equation (1.2) is Liapunov stable after that we proved that this equation is asymptotically stable. Finally, we applied the these results to an example. As a result of this application we verified that equation (2.17) is suitable to the example

$$x(n+1) = \frac{1}{2}x(n) + B(n)F\left(x\left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)\right), \quad n \geq 0,$$

where F is the real valued vector function, $m: \mathbb{Z} \rightarrow \mathbb{Z}^+$, which is bounded function and maximum value of m is k and $B(n)$ is a $k \times k$ variable coefficient matrix.

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