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On Parameterized Simpson, Midpoint and Trapezoid Type Inequalities for Differentiable $(\eta_1,\eta_2)-$Convex Functions via Generalized Fractional Integrals

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Abstract

In this paper, we first obtain an identity for differentiable mappings. Then we establish some new generalized inequalities for differentiable $(\eta_1,\eta_2)-$ convex functions involving some parameters and generalized fractional integrals. We show that these results reduces to several new Simpson, midpoint and trapezoid type inequalities. Some special cases are also discussed.

Keywords: *simpson's $\frac{1}{3}$ formula, Integral inequalities, fractional calculus, convex functions.*
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1. Introduction

Simpson's inequality plays an important role in many areas of mathematics. The classical Simpson's inequality is expressed as follows for four times continuously differentiable functions:

Theorem 1.1. *Suppose that $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable mapping on (a,b), and let $\left\|f^{(4)}\right\|_\infty=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4$$

In recent years, many authors ha ve focused on Simpson's type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson's and Newton's type results for convex mappings, because convexity theory is an effective and powerful method for solving a large number of problems which arise within different branches of pure and applied mathematics. For example, Dragomir et al. [7] presented new Simpson's type results and their applications to quadrature formulas in numerical integration. What is more, new Newton's type inequalities for functions whose local fractional derivatives are generalized convex are given by Iftikhar et al. in [14]. For more recent developments, one can consult [1,2,3,9,10,11,19,20,21,29].

2. Preliminaries

In this section, we first summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [24].
Let's define a function $\varphi:[0,\infty)\to[0,\infty)$ satisfying the following conditions :

$$\int_0^1\frac{\varphi(t)}{t}dt<\infty.$$

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We define the following left-sided and right-sided generalized fractional integral operators, respectively, as follows:

$$_{a^+}I_{\varphi}f(x)=\int_a^x \frac{\varphi(x-t)}{x-t}f(t)dt,\ \ x>a, \tag{2.1}$$

$$_{b^-}I_{\varphi}f(x)=\int_x^b \frac{\varphi(t-x)}{t-x}f(t)dt,\ \ x<b. \tag{2.2}$$

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (2.1) and (2.2) are mentioned below.
i) If we take $\varphi(t)=t$, the operator (2.1) and (2.2) reduce to the Riemann integral
ii) If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$ the operator (2.1) and (2.2) reduce to the Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f(x)$ and $J_{b-}^{\alpha}f(x)$, respectively. Here Γ is Gamma function.
iii) If we take $\varphi(t)=\frac{1}{k\Gamma_k(\alpha)}t^{\frac{\alpha}{k}}$, $\alpha,k>0$ the operator (2.1) and (2.2) reduce to the k-Riemann-Liouville fractional integrals $J_{a+,k}^{\alpha}f(x)$ and $J_{b-,k}^{\alpha}f(x)$, respectively. Here Γ_k is k-Gamma function.
Sarıkaya and Ertuğral also establish the following Hermite-Hadamard inequality for the generalized fractional integral operators:

Theorem 2.1 (24). *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $a<b$, then the following inequalities for fractional integral operators hold*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2\Lambda(1)}\left[_{a+}I_{\varphi}f(b)+_{b-}I_{\varphi}f(a)\right]\le\frac{f(a)+f(b)}{2} \tag{2.3}$$

where the mapping $\Lambda:[0,1]\to\mathbb{R}$ is defined by

$$\Lambda(t)=\int\limits_0^t \frac{\varphi((b-a)u)}{u}du.$$

In the literature, there are several papers on inequalities for generalized fractional integrals. Budak et al. proved Midpoint type inequalities and extensions of Hermite-Hadamard inequalities in [4] and [5] , respectively. In [11], Ertuğral and Sarikaya presented some Simpson type inequalities for these fractional integral operators. For some of other papers on inequalities for generalized fractional integrals, please refer to [13,15,16,25,26,27,29].

Definition 2.2. *[31] A set $I\subseteq\mathbb{R}$ is invex with respect to a real bifunction $\eta:I\times I\to\mathbb{R}$, if*

$$x,y\in I,\ \lambda\in[0,1]\Longrightarrow y+\lambda\eta(x,y)\in I. \tag{2.4}$$

If I is an invex set with respect to η, then a function $f:I\to\mathbb{R}$ is called preinvex , if $x,y\in I$ and $\lambda\in[0,1]$.

$$f(y+\lambda\eta(x,y))\le\lambda f(x)+(1-\lambda)f(y). \tag{2.5}$$

Definition 2.3. *[31]A function $f:I\to\mathbb{R}$ is called convex with respect to $\eta-$convex, if*

$$f(tx+(1-t)y)\le f(y)+t\eta(f(x),f(y)) \tag{2.6}$$

for all $x,y\in I$ and $t\in[0,1]$.

Definition 2.4 (32). *Let $I\subseteq\mathbb{R}$ be an invex set with respect to $\eta_1:I\times I\to\mathbb{R}$.Consider $f:I\to\mathbb{R}$ and $\eta_2:f(I)\times f(I)\to\mathbb{R}$.The function f is said to be $(\eta_1,\eta_2)-$ convex, if*

$$f(x+\lambda\eta_1(y,x))\le f(x)+\lambda\eta_2(f(y),f(x)) \tag{2.7}$$

for all $x,y\in I$ and $\lambda\in[0,1]$.

Remark 2.5. *An $(\eta_1,\eta_2)-$convex function reduces to;*

(i) If we choose $\eta_1(x,y)=x-y$ for all $x,y\in I$ in Definition 2.4, then we obtain η-convex function.
(ii) If we choose $\eta_2(x,y)=x-y$ for all $x,y\in f(I)$ in Definition 2.4, then we obtain preinvex function.
(iii) If we choose $\eta_1(x,y)=\eta_2(x,y)=x-y$ in Definition 2.4, then we obtain classical convex function.
In this paper, simpson, midpoint and trapezoid type inequalities using (η_1,η_2)-convex function via generalized fractional integrals. Moreover, we also consider their relevances for other related known results.

3. An Identity for Generalized Fractional Integrals

In this section, we offer a parameterized identity involving ordinary first derivative via generalized fractional integrals.

Lemma 3.1. *Let $f \in L_1[a, a+\eta_1(b,a)]$ be a differentiable function on $(a, a+\eta_1(b,a))$. If f' is continuous and integrable on $[a, a+\eta_1(b,a)]$, then for $\lambda, \mu \geq 0$, $\eta_1(b,a) > 0$, one has the identity;*

$$\begin{aligned} & (1-\mu) f(a) + (\mu+\lambda) f\left(a + \frac{\eta_1(b,a)}{2}\right) + (1-\lambda) f(a+\eta_1(b,a)) \qquad (3.1) \\ & -\frac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right) + {}_{a+\eta_1(b,a)^-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right] \\ = & \frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\int_0^1 \left(\Delta_{\eta_1}(t) - \Delta_{\eta_1}(1)\lambda\right) f'\left(a+\frac{1+t}{2}\eta_1(b,a)\right) dt \right. \\ & \left. + \int_0^1 \left(\Delta_{\eta_1}(1)\mu - \Delta_{\eta_1}(t)\right) f'\left(a+\frac{1-t}{2}\eta_1(b,a)\right) dt\right] \end{aligned}$$

where the mapping $\Delta_{\eta_1} : [0,1] \to \mathbb{R}$ is defined by;

$$\Delta_{\eta_1}(t) = \int_0^t \frac{\varphi\left(\frac{\eta_1(b,a)}{2}u\right)}{u} du.$$

Proof. Applying fundamental rules of integration, we have

$$\begin{aligned} & \int_0^1 \left(\Delta_{\eta_1}(t) - \Delta_{\eta_1}(1)\lambda\right) f'\left(a+\tfrac{1+t}{2}\eta_1(b,a)\right) dt \\ & = \tfrac{2}{\eta_1(b,a)}\left[\Delta_{\eta_1}(1)\left((1-\lambda) f(a+\eta_1(b,a)) + \lambda f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right)\right. \qquad (3.2) \\ & \left. - {}_{a+\eta_1(b,a)-}I_\varphi f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right] \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \left(\Delta_{\eta_1}(1)\mu - \Delta_{\eta_1}(t)\right) f'\left(a+\frac{1-t}{2}\eta_1(b,a)\right) dt \qquad (3.3) \\ = & \frac{2}{\eta_1(b,a)} \\ & \times\left[\Delta_{\eta_1}(1)\left((1-\mu) f(a) + \mu f\left(a+\frac{\eta_1(b,a)}{2}\right)\right) - {}_{a+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]. \qquad (3.4) \end{aligned}$$

By adding (3.2) and (3.3), we obtain the required equality (3.1). □

Corollary 3.2. *If we assume $\varphi(t) = t$ in Lemma 3.1, then we obtain the following equality;*

$$\begin{aligned} & \tfrac{1}{2}\left[(1-\mu) f(a) + (\mu+\lambda) f\left(a+\tfrac{\eta_1(b,a)}{2}\right) + (1-\lambda) f(a+\eta_1(b,a))\right] - \tfrac{1}{\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} f(t) dt \\ & = \tfrac{\eta_1(b,a)}{4}\left[\int_0^1 (t-\lambda) f'\left(a+\tfrac{1+t}{2}\eta_1(b,a)\right) dt \right. \\ & \left. + \int_0^1 (\mu - t) f'\left(a+\tfrac{1-t}{2}\eta_1(b,a)\right) dt\right]. \end{aligned}$$

Corollary 3.3. *In Lemma 3.1, if we set $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$, then we obtain the following new identity for Riemann-Liouville fractional integral;*

$$\begin{aligned} & (1-\mu) f(a) + (\mu+\lambda) f\left(a+\tfrac{\eta_1(b,a)}{2}\right) + (1-\lambda) f(a+\eta_1(b,a)) \\ & -\tfrac{2^\alpha \Gamma(\alpha+1)}{(\eta_1(b,a))^\alpha}\left[J_{a+}^\alpha f\left(a+\tfrac{\eta_1(b,a)}{2}\right) + J_{a+\eta_1(b,a)-}^\alpha f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right] \\ & = \tfrac{\eta_1(b,a)}{2} \\ & \times\left[\int_0^1 (t^\alpha - \lambda) f'\left(a+\tfrac{1+t}{2}\eta_1(b,a)\right) dt + \int_0^1 (\mu - t^\alpha) f'\left(a+\tfrac{1-t}{2}\eta_1(b,a)\right) dt\right]. \end{aligned}$$

Corollary 3.4. *In Lemma 3.1, if we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we obtain the following new identity for k-Riemann-Liouville fractional integral;*

$$(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))$$

$$-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-,k}f\left(\frac{a+b}{2}\right)\right]$$

$$=\frac{\eta_1(b,a)}{2}\left[\int_0^1\left(t^{\frac{\alpha}{k}}-\lambda\right)f'\left(a+\frac{1+t}{2}\eta_1(b,a)\right)dt\right.$$

$$\left.+\int_0^1\left(\mu-t^{\frac{\alpha}{k}}\right)f'\left(a+\frac{1-t}{2}\eta_1(b,a)\right)dt\right].$$

4. Some parameterized inequalities for generalized fractional integral operators

In this section, we establish some new generalized inequalities for differentiable convex functions via generalized fractional integrals.

Theorem 4.1. *We assume that the conditions of Lemma 3.1 hold. Let $I\subseteq\mathbb{R}$ be an invex set with respect to η_1 and η_2 is an integrable bifunction on $f(I)\times f(I)$, for any $a,b\in I$ with $\eta_1(b,a)>0$. If the mapping $|f'|$ is (η_1,η_2)-convex on $[a,a+\eta_1(b,a)]$, then the following inequality holds for generalized fractional integrals;*

$$\begin{aligned}&\Big|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\tfrac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|\\&\leq\tfrac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left[\eta_2(|f'(b)|,|f'(a)|)\left(\Pi_1^{\varphi}(\mu)+\Pi_2^{\varphi}(\lambda)\right)\right.\\&\left.+|f'(a)|\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right|+\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|\right)\right]\end{aligned}\tag{4.1}$$

where

$$\Pi_1^{\varphi}(\kappa)=\int_0^1(1-t)\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\kappa\right|dt$$

and

$$\Pi_2^{\varphi}(\kappa)=\int_0^1(1+t)\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\kappa\right|dt.$$

Proof. By taking the modulus in Lemma 3.1 and using the properties of the modulus, we obtain that

$$\begin{aligned}&\Big|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\tfrac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|\\&=\tfrac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right|\left|f'\left(a+\tfrac{1+t}{2}\eta_1(b,a)\right)\right|dt\right.\\&\left.+\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|\left|f'\left(a+\tfrac{1-t}{2}\eta_1(b,a)\right)\right|dt\right].\end{aligned}\tag{4.2}$$

Since the mapping $|f'|$ is $(\eta_1,\eta_2)-$ convex on $[a,a+\eta_1(b,a)]$, therefore, we have

$$\begin{aligned}&\Big|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\tfrac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi,\eta_1}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|\\&\leq\tfrac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left[|f'(a)|\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right|dt+\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|dt\right)\right.\\&\left.+\eta_2(|f'(b)|,|f'(a)|)\left(\int_0^1(1+t)\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right|dt+\int_0^1(1-t)\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|dt\right)\right]\\&=\tfrac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left[\eta_2(|f'(b)|,|f'(a)|)\left(\Pi_1^{\varphi}(\mu)+\Pi_2^{\varphi}(\lambda)\right)+|f'(a)|\left(\Pi_3^{\varphi}(\lambda)+\Pi_3^{\varphi}(\mu)\right)\right]\end{aligned}$$

which ends the proof. □

Corollary 4.2. *Under assumption of Theorem 4.1, with $\varphi(t)=t$, then we obtain the following inequality;*

$$\begin{aligned}&\left|\tfrac{1}{2}\left[(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\right]\right.\\&\left.-\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\leq\tfrac{\eta_1(b,a)}{8}\\&\times\left[|f'(a)|\left(\mu^2+\lambda^2-\lambda-\mu+1\right)+\eta_2(|f'(b)|,|f'(a)|)\left(\Pi_1^{\alpha}(\mu)+\Pi_2^{\alpha}(\lambda)\right)\right]\end{aligned}$$

$$\Pi_1(\kappa)=\kappa^2-\frac{\kappa^3}{3}-\frac{\kappa}{2}+\frac{1}{6},$$

and

$$\Pi_2(\kappa)=\frac{\kappa^3}{3}+\kappa^2-\frac{3\kappa}{2}+\frac{5}{6}.$$

Corollary 4.3. *Under assumption of Theorem 4.1, with $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$, then we obtain the following inequality for Riemann-Liouville fractional integrals;*

$$\Big|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))$$

$$-\tfrac{2^{\alpha}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|$$

$$\leq\tfrac{\eta_1(b,a)}{4}\Big[|f'(a)|\left(\tfrac{2\alpha}{\alpha+1}\left(\lambda^{\frac{\alpha+1}{\alpha}}+\mu^{\frac{\alpha+1}{\alpha}}\right)-\lambda-\mu+\tfrac{2}{(\alpha+1)}\right)$$

$$+\eta_2(|f'(b)|,|f'(a)|)\left(\Pi^{\alpha}_1(\mu)+\Pi^{\alpha}_2(\lambda)\right)\Big]$$

where

$$\Pi^{\alpha}_1(\kappa)=\frac{2\alpha}{\alpha+1}\kappa^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{\alpha+2}\kappa^{\frac{\alpha+2}{\alpha}}-\frac{\kappa}{2}+\frac{1}{(\alpha+2)(\alpha+1)},$$

and

$$\Pi^{\alpha}_2(\kappa)=\frac{\alpha}{\alpha+2}\kappa^{\frac{\alpha+2}{\alpha}}-\frac{3\kappa}{2}+\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\kappa^{\frac{\alpha+1}{\alpha}}$$

Corollary 4.4. *In Theorem 4.1, if we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(a)}$, then we obtain the following inequality for k-Riemann-Liouville fractional integrals;*

$$\Big|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))$$

$$-\tfrac{2^{\frac{\alpha}{k}}\Gamma(\alpha+k)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+,k}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-,k}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|$$

$$\leq\tfrac{\eta_1(b,a)}{4}\Big[|f'(a)|\left(\tfrac{2\alpha}{\alpha+k}\left(\lambda^{\frac{\alpha+k}{\alpha}}+\mu^{\frac{\alpha+k}{\alpha}}\right)-\lambda-\mu+\tfrac{2k}{(\alpha+k)}\right)$$

$$+\eta_2(|f'(b)|,|f'(a)|)\Pi^{\frac{\alpha}{k}}_1(\mu)+\Pi^{\frac{\alpha}{k}}_2(\lambda)\Big]$$

where

$$\Pi^{\frac{\alpha}{k}}_1(\kappa)=\frac{2\alpha}{\alpha+k}\kappa^{\frac{\alpha+k}{\alpha}}-\frac{\alpha}{\alpha+2k}\kappa^{\frac{\alpha+2k}{\alpha}}-\frac{\kappa}{2}+\frac{k^2}{(\alpha+2k)(\alpha+k)},$$

and

$$\Pi^{\frac{\alpha}{k}}_2(\kappa)=\frac{\alpha}{\alpha+2k}\kappa^{\frac{\alpha+2k}{\alpha}}-\frac{3\kappa}{2}+\frac{2\alpha k+3k^2}{(\alpha+2k)(\alpha+k)}+\frac{2\alpha}{\alpha+k}\kappa^{\frac{\alpha+k}{\alpha}}.$$

Theorem 4.5. *We assume that the conditions of Lemma 3.1 hold. Let $I\subseteq\mathbb{R}$ be an invex set with respect to η_1 and η_2 is an integrable bifunction on $f(I)\times f(I)$, for any $a,b\in I$ with $\eta_1(b,a)>0$.If the mapping $|f|^{p_1}$, $p_1\geq 1$ is (η_1,η_2)-convex on $[a,a+\eta_1(b,a)]$ then the following inequality holds for generalized fractional integrals;*

$$\Big|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))$$

$$-\tfrac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|$$

$$\leq\tfrac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\Big[\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right|dt\right)^{1-\frac{1}{p_1}}$$

$$\times\left(\tfrac{\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right||f'(a)|^{p_1}+\Pi^{\varphi}_2(\lambda)\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{2}\right)^{\frac{1}{p_1}}$$

$$+\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|dt\right)^{1-\frac{1}{p_1}}$$

$$\times\left(\tfrac{\Pi^{\varphi}_1(\mu)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|dt\right)|f'(a)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\Big].$$

where $\Pi^{\varphi}_1(\kappa)$ and $\Pi^{\varphi}_2(\kappa)$ are defined as in Theorem 4.1.

Proof. Reutilizing inequality (4.2) and from power mean inequality, we have

$$\begin{aligned}
&\left|(1-\mu) f(a)+(\mu+\lambda) f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda) f(a+\eta_1(b,a))\right. \\
&\left.-\frac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right| \\
\leq\; & \frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right| dt\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right|\left|f'\left(a+\frac{1+t}{2}\eta_1(b,a)\right)\right|^{p_1} dt\right)^{\frac{1}{p_1}} \\
&+\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right| dt\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right|\left|f'\left(a+\frac{1-t}{2}\eta_1(b,a)\right)\right|^{p_1} dt\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Using the $(\eta_1,\eta_2)-$ convexity of $|f'|^{p_1}$, we have

$$\begin{aligned}
&\left|(1-\mu) f(a)+(\mu+\lambda) f\left(\frac{a+b}{2}\right)+(1-\lambda) f(a+\eta_1(b,a))\right. \\
&\left.-\frac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(\frac{a+b}{2}\right)+{}_{a+\eta_1(b,a)-}I_\varphi f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\; & \frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right| dt\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\left|f'(a)\right|^{p_1}\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right| dt+\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)^{p_1}\int_0^1\left(\frac{1+t}{2}\right)\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right| dt\right)^{\frac{1}{p_1}} \\
&+\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right| dt\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(\left|f'(a)\right|^{p_1}\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right| dt+\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)^{p_1}\int_0^1\left(\frac{1-t}{2}\right)\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right| dt\right)^{\frac{1}{p_1}}\right] \\
=\; & \frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right| dt\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\frac{\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda\right||f'(a)|^{p_1}+\Pi_2^{\varphi}(\lambda)\,\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{2}\right)^{\frac{1}{p_1}} \\
&+\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right| dt\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(\frac{\Pi_1^{\varphi}(\mu)\,\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\int_0^1\left|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)\right| dt\right)|f'(a)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

which finishes the proof. □

Corollary 4.6. *If we assume that $\varphi(t)=t$ in Theorem 4.5, then we obtain the following inequality;*

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-\mu) f(a)+(\mu+\lambda) f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda) f(a+\eta_1(b,a))\right]\right. \\
&\left.-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)} f(t)dt\right| \\
&\leq \frac{\eta_1(b,a)}{8} \\
&\times\left[\left(\Pi_1(\lambda)+\Pi_2(\lambda)\right)^{1-\frac{1}{p_1}}\left(\left(\lambda^2-\lambda+\tfrac{1}{2}\right)|f'(a)|^{p_1}+\Pi_2(\lambda)\,\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\left.\left(\Pi_1(\mu)+\Pi_2(\mu)\right)^{1-\frac{1}{p_1}}\left(\Pi_1(\mu)\,\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\mu^2-\mu+\tfrac{1}{2}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where $\Pi_1(\kappa)$ and $\Pi_2(\kappa)$ are defined as in Corollary 4.2.

Corollary 4.7. *If we take $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ in Theorem 4.5, then we have the following inequality for Riemann-Liouville fractional integrals;*

$$\begin{aligned}&\Bigg|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\frac{2^{\alpha}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\Bigg|\\\leq\;&\frac{\eta_1(b,a)}{4}\Bigg[\left(\Pi_1^{\alpha}(\lambda)+\Pi_2^{\alpha}(\lambda)\right)^{1-\frac{1}{p_1}}\\&\times\left(\left(\frac{2\alpha}{\alpha+1}\lambda^{\frac{\alpha+1}{\alpha}}-\lambda+\frac{1}{\alpha+1}\right)|f'(a)|^{p_1}+\Pi_2^{\alpha}(\lambda)\eta_2\left(|f'(b)|,|f'(a)|\right)^{p_1}\right)^{\frac{1}{p_1}}\\&+\left(\Pi_1^{\alpha}(\mu)+\Pi_2^{\alpha}(\mu)\right)^{1-\frac{1}{p_1}}\\&\times\left(\Pi_1^{\alpha}(\mu)\eta_2\left(|f'(b)|,|f'(a)|\right)^{p_1}+\left(\frac{2\alpha}{\alpha+1}\mu^{\frac{\alpha+1}{\alpha}}-\mu+\frac{1}{\alpha+1}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\Bigg]\end{aligned}\tag{4.3}$$

where $\Pi_1^{\alpha}(\kappa)$ and $\Pi_2^{\alpha}(\kappa)$ are defined as in Corollary 4.3.

Corollary 4.8. *If we take $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ in Theorem 4.5, then we have the following inequality for k-Riemann-Liouville fractional integrals;*

$$\begin{aligned}&\Bigg|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\Bigg|\\\leq\;&\frac{\eta_1(b,a)}{4}\Bigg[\left(\Pi_1^{\frac{\alpha}{k}}(\lambda)+\Pi_2^{\frac{\alpha}{k}}(\lambda)\right)^{1-\frac{1}{p_1}}\\&\times\left(\left(\frac{2\alpha}{\alpha+k}\lambda^{\frac{\alpha+k}{\alpha}}-\lambda+\frac{k}{\alpha+1}\right)|f'(a)|^{p_1}+\Pi_2^{\frac{\alpha}{k}}(\lambda)\eta_2\left(|f'(b)|,|f'(a)|\right)^{p_1}\right)^{\frac{1}{p_1}}\\&+\left(\Pi_1^{\frac{\alpha}{k}}(\mu)+\Pi_2^{\frac{\alpha}{k}}(\mu)\right)^{1-\frac{1}{p_1}}\\&\times\left(\Pi_1^{\frac{\alpha}{k}}(\mu)\eta_2\left(|f'(b)|,|f'(a)|\right)^{p_1}+\left|\left(\frac{2\alpha}{\alpha+k}\mu^{\frac{\alpha+k}{\alpha}}-\mu+\frac{k}{\alpha+1}\right)f'(a)\right|^{p_1}\right)^{\frac{1}{p_1}}\Bigg]\end{aligned}\tag{4.4}$$

where $\Pi_1^{\frac{\alpha}{k}}(\kappa)$ and $\Pi_2^{\frac{\alpha}{k}}(\kappa)$ are described in Corollary 4.3.

Theorem 4.9. *We assume that the conditions of Lemma 3.1 hold. Let $I\subseteq\mathbb{R}$ be an invex set with respect to η_1 and η_2 is an integrable bifunction on $f(I)\times f(I)$, for any $a,b\in I$ with $\eta_1(b,a)>0$.If the mapping $|f|^{r_1}$, $r_1>1$ is (η_1,η_2)-convex on $[a,a+\eta_1(b,a)]$ then the following inequality holds for generalized fractional integrals;*

$$\begin{aligned}&\Bigg|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\frac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\Bigg|\\\leq\;&\frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\Bigg[\left(\int_0^1|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\\&+\left(\int_0^1|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\Bigg]\end{aligned}\tag{4.5}$$

where $\frac{1}{p_1}+\frac{1}{r_1}=1$.

Proof. Reutilizing inequality (4.2) and from well-known Hölder's inequality, we have

$$\begin{aligned}&\Bigg|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\\&-\frac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\Bigg|\\\leq\;&\frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\Bigg[\left(\int_0^1|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_0^1\left|f'\left(a+\frac{1+t}{2}\eta_1(b,a)\right)\right|^{r_1}dt\right)^{\frac{1}{r_1}}\\&+\left(\int_0^1|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_0^1\left|f'\left(a+\frac{1-t}{2}\eta_1(b,a)\right)\right|^{r_1}dt\right)^{\frac{1}{r_1}}\Bigg].\end{aligned}$$

Using the fact that $|f'|^{r_1}$ is $(\eta_1,\eta_2)-$ convex, we have

$$\begin{aligned}
&\Bigg|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a)) \\
&-\frac{1}{\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\Bigg| \\
\leq\;& \frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\left(\int_0^1|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda|^{p_1}dt\right)^{\frac{1}{p_1}}\right. \\
&\times\left(|f'(a)|^{r_1}\int_0^1 dt+\eta_2(|f'(b)|,|f'(a)|)^{r_1}\int_0^1\left(\frac{1+t}{2}\right)dt\right)^{\frac{1}{r_1}} \\
&+\left(\int_0^1|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)|^{p_1}dt\right)^{\frac{1}{p_1}} \\
&\left.\times\left(|f'(a)|^{r_1}\int_0^1 dt+\eta_2(|f'(b)|,|f'(a)|)^{r_1}\int_0^1\left(\frac{1-t}{2}\right)dt\right)^{\frac{1}{r_1}}\right] \\
=\;& \frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left[\left(\int_0^1|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\lambda|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\left.+\left(\int_0^1|\Delta_{\eta_1}(1)\mu-\Delta_{\eta_1}(t)|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right]
\end{aligned}$$

which completes the proof. □

Corollary 4.10. *In Theorem 4.9, if we set $\varphi(t)=t$, then we obtain the following inequality;*

$$\begin{aligned}
&\left|\frac{1}{2}\left[(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\right]\right. \\
&\left.-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right| \\
&\leq\frac{\eta_1(b,a)}{4}\left[(\Pi_3(\lambda))^{\frac{1}{p_1}}\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\left.+(\Pi_3(\mu))^{\frac{1}{p_1}}\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right]
\end{aligned}$$

where

$$\Pi_3(\kappa)=\frac{\kappa^{p_1+1}+(1-\kappa)^{p_1+1}}{p_1+1}.$$

Corollary 4.11. *In Theorem 4.9, if we take $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, then we obtain the following inequality for Riemann-Liouville fractional integrals;*

$$\begin{aligned}
&\left|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a))\right. \\
&\left.-\frac{2^\alpha\Gamma(\alpha+1)}{(\eta_1(b,a))^\alpha}\left[{}_{a^+}I_\alpha f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_\alpha f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\right| \\
&\leq\frac{\eta_1(b,a)}{2}\left[\left(\int_0^1|t^\alpha-\lambda|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\left.+\left(\int_0^1|\mu-t^\alpha|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

Corollary 4.12. *In Theorem 4.9, if we set $\varphi(t)=\frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$, then we obtain the following inequality for k-Riemann-Liouville fractional integrals;*

$$\begin{aligned}
&\Bigg|(1-\mu)f(a)+(\mu+\lambda)f\left(a+\frac{\eta_1(b,a)}{2}\right)+(1-\lambda)f(a+\eta_1(b,a)) \\
&-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^\alpha_{a+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^\alpha_{a+\eta_1(b,a)-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\Bigg| \\
\leq\;&\frac{\eta_1(b,a)}{2}\left[\left(\int_0^1\left|t^{\frac{\alpha}{k}}-\lambda\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right. \\
&\left.+\left(\int_0^1\left|\mu-t^{\frac{\alpha}{k}}\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].
\end{aligned}$$

5. Special Cases

In this section, we give some special cases of our main results.

Remark 5.1. *From Theorem 4.1, we have following new inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have the following inequality;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+2f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{8\Delta_{\eta_1}(1)}\left[\Pi_1^\varphi\left(\frac{2}{3}\right)+\Pi_2^\varphi\left(\frac{2}{3}\right)\right]\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)\\&+\left|f'(a)\right|\left[\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\frac{2}{3}\right|+\int_0^1\left|\Delta_{\eta_1}(1)\frac{2}{3}-\Delta_{\eta_1}(t)\right|\right)\right].\end{aligned}$$

Particularly, if we choose $\eta_1(x,y)=\eta_2(x,y)=x-y$, then we obtain [11, Theorem 4].
2. For $\lambda=\mu=0$, we have the following inequality;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.\\&\left.-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)\right|dt\right)\left[\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)+\left|f'(a)\right|\right].\end{aligned}\tag{5.1}$$

Particularly, the inequalty (5.1) for $\eta_1(x,y)=\eta_2(x,y)=x-y$ is proved by Ertugral et al. in [12].
3. For $\lambda=\mu=1$, we have the following inequality;

$$\begin{aligned}&\left|2f\left(a+\frac{\eta_1(b,a)}{2}\right)\right.\\&\left.-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|dt\right)\left[\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)+\left|f'(a)\right|\right].\end{aligned}\tag{5.2}$$

Particularly, the inequalty (5.2) for $\eta_1(x,y)=\eta_2(x,y)=x-y$ is proved by Ertugral et al. in [12].

Remark 5.2. *From Corollary 4.2, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have the following Simpson's inequality for Riemann integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\le\frac{\eta_1(b,a)}{8}\left[\left|f'(a)\right|\left(\frac{5}{9}\right)+\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)\left(\Pi_1^\alpha\left(\frac{2}{3}\right)+\Pi_2^\alpha\left(\frac{2}{3}\right)\right)\right].\end{aligned}\tag{5.3}$$

Particularly, the inequalty (5.3) for $\eta_1(x,y)=\eta_2(x,y)=x-y$ is proved by Sarikaya et al. in [22,23].
2. For $\lambda=\mu=0$, we have the following trapezoid inequality for Riemann integrals;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\le\frac{\eta_1(b,a)}{8}\left[\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)\int_0^1\left|\Delta_{\eta_1}(t)\right|dt+\left|f'(a)\right|\right].\end{aligned}\tag{5.4}$$

Particularly, the inequalty (5.4) for $\eta_1(x,y)=\eta_2(x,y)=x-y$ is proved by Dragomir and Agarwal [8].
3. For $\lambda=\mu=1$, we have the following midpoint inequality for Riemann integrals;

$$\begin{aligned}&\left|f\left(a+\frac{\eta_1(b,a)}{2}\right)-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\le\frac{\eta_1(b,a)}{8}\left[\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|dt\right)\eta_2\left(\left|f'(b)\right|,\left|f'(a)\right|\right)+\left|f'(a)\right|\right].\end{aligned}\tag{5.5}$$

Particularly, the inequalty (5.5) for $\eta_1(x,y)=\eta_2(x,y)=x-y$ is proved by Kirmaci in [17].

Remark 5.3. *From Corollary 4.3, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have the following Simpson's inequality for Riemann-Liouville fractional integrals;

$$\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{\eta_1(b,a)}{4}\left(\Pi^{\alpha}_1\left(\tfrac{2}{3}\right)+\Pi^{\alpha}_2\left(\tfrac{2}{3}\right)\right)[\eta_2(|f'(b)|,|f'(a)|)$$
$$\left.+|f'(a)|\left(\frac{4\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{4}{3}+\frac{2}{(\alpha+1)}\right)\right].$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for Riemann-Liouville fractional integrals;

$$\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{\eta_1(b,a)}{2}\left(\int_0^1\left|\Delta_{\eta_1}(t)\right|dt\right)\left[\eta_2(|f'(b)|,|f'(a)|)+\frac{2}{(\alpha+1)}|f'(a)|\right].$$

3. For $\lambda=\mu=1$, we have the following midpoint type inequality for generalized fractional integrals;

$$\left|f\left(a+\frac{\eta_1(b,a)}{2}\right)\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{\eta_1(b,a)}{2}\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|dt\right)\left[\eta_2(|f'(b)|,|f'(a)|)+\frac{2\alpha}{(\alpha+1)}|f'(a)|\right].$$

Remark 5.4. *From Corollary 4.4, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$,we have following Simpson's type inequality for k-Riemann-Liouville fractional integrals;

$$\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}}\Gamma(\alpha+k)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{\eta_1(b,a)}{4}\left(\Pi^{\frac{\alpha}{k}}_1\left(\tfrac{2}{3}\right)+\Pi^{\frac{\alpha}{k}}_2\left(\tfrac{2}{3}\right)\right)[\eta_2(|f'(b)|,|f'(a)|)$$
$$\left.+|f'(a)|\left(\frac{4\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{\alpha+k}{\alpha}}-\frac{4}{3}+\frac{2k}{(\alpha+k)}\right)\right].$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for k-Riemann-Liouville fractional integrals;

$$\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{k\eta_1(b,a)}{4(\alpha+k)}[\eta_2(|f'(b)|,|f'(a)|)+8|f'(a)|].$$

3. For $\lambda=\mu=1$ we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals;

$$\left|2f\left(a+\frac{\eta_1(b,a)}{2}\right)\right.$$
$$\left.-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{\eta_1(b,a)}{4}\left[\frac{\alpha}{(\alpha+k)}\eta_2(|f'(b)|,|f'(a)|)+\left(\frac{2\alpha}{\alpha+k}\right)|f'(a)|\right].$$

Remark 5.5. *From Theorem 4.5, we have following new inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for generalized fractional integrals;

$$\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.$$
$$\left.-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|$$
$$\leq\frac{\eta_1(b,a)}{2\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\tfrac{2}{3}\right|dt\right)^{1-\frac{1}{p_1}}$$
$$\times\left[\left(\frac{\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\frac{2}{3}\right|dt|f'(a)|^{p_1}+\Pi^{\varphi}_2\left(\frac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{2}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(\frac{\Pi^{\varphi}_1\left(\frac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\frac{2}{3}\right|dt|f'(a)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right].$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for generalized fractional integrals;

$$\begin{aligned}
&\Big|\tfrac{f(a)+f(a+\eta_1(b,a))}{2}\\
&-\tfrac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|\\
&\leq \tfrac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)\right|dt\right)^{1-\frac{1}{p_1}}\\
&\times\left[\left(\tfrac{\left(\int_0^1\left|\Delta_{\eta_1}(t)\right|dt\right)|f'(a)|^{p_1}+\Pi_2^{\varphi}(0)\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{2}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\tfrac{\Pi_1^{\varphi}(0)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\int_0^1\left|\Delta_{\eta_1}(t)\right|dt\right)|f'(a)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

3. For $\lambda=\mu=1$, we have the following midpoint type inequality for generalized fractional integrals;

$$\begin{aligned}
&\Big|f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\\
&-\tfrac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_{\varphi}f\left(a+\tfrac{\eta_1(b,a)}{2}\right)\right]\Big|\\
&\leq \tfrac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|dt\right)^{1-\frac{1}{p_1}}\\
&\times\left[\left(\tfrac{\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|dt\right)|f'(a)|^{p_1}+\Pi_2^{\varphi}(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{2}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\tfrac{\Pi_1^{\varphi}(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|dt\right)|f'(a)|^{p_1}}{2}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Remark 5.6. *From Corollary 4.6, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for Riemann integrals;

$$\begin{aligned}
&\Big|\tfrac{1}{6}\left[f(a)+4f\left(a+\tfrac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\\
&-\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\Big|\\
&\leq\tfrac{\eta_1(b,a)}{8}\left(\Pi_1\left(\tfrac{2}{3}\right)+\Pi_2\left(\tfrac{2}{3}\right)\right)^{1-\frac{1}{p_1}}\\
&\times\left[\left(\left(-\tfrac{1}{6}\right)|f'(a)|^{p_1}+\Pi_2\left(\tfrac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\Pi_1\left(\tfrac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\tfrac{1}{6}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

2. For $\lambda=\mu=0$, we have following trapezoid type inequality for Riemann integrals;

$$\begin{aligned}
&\left|\tfrac{f(a)+f(a+\eta_1(b,a))}{2}-\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\
&\leq\tfrac{\eta_1(b,a)}{8}\left(\Pi_1(0)+\Pi_2(0)\right)^{1-\frac{1}{p_1}}\\
&\times\left[\left(\left(\tfrac{1}{2}\right)|f'(a)|^{p_1}+\Pi_2(0)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\
&+\left(\Pi_1(0)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(-\tfrac{1}{2}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}
\end{aligned}$$

Remark 5.7. *3. For $\lambda=\mu=1$, we have following midpoint type inequality for Riemann integrals;*

$$\begin{aligned}
&\left|f\left(a+\tfrac{\eta_1(b,a)}{2}\right)-\tfrac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\
&\leq\tfrac{\eta_1(b,a)}{8}\left(\Pi_1(1)+\Pi_2(1)\right)^{1-\frac{1}{p_1}}\\
&\times\left[\left(\left(-\tfrac{1}{2}\right)|f'(a)|^{p_1}+\Pi_2(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\Pi_1(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\tfrac{1}{2}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Remark 5.8. *From Corollary 4.7, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta_1(b,a)}{4}\left(\frac{4\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{2}{\alpha+1}-\frac{4}{3}\right)^{1-\frac{1}{p_1}}\\&\times\left[\left(\left(\frac{2\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{2}{3}\right)|f'(a)|^{p_1}+\Pi_2^{\alpha}\left(\frac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\left.+\left(\Pi_1^{\alpha}\left(\frac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\frac{2\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{1}{\alpha+1}-\frac{2}{3}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta_1(b,a)}{4}\left(\frac{2\alpha}{\alpha+1}\right)^{1-\frac{1}{p_1}}\\&\times\left[\left(\frac{1}{\alpha+1}|f'(a)|^{p_1}+\frac{(2\alpha+3)\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{p_1}}\right.\\&\left.+\left(\frac{\eta_2(|f'(b)|,|f'(a)|)^{p_1}}{(\alpha+1)(\alpha+2)}+\frac{1}{\alpha+1}|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

3. For $\lambda=\mu=1$, we have the following midpoint type inequality for generalized fractional integrals;

$$\begin{aligned}&\left|f\left(a+\frac{\eta_1(b,a)}{2}\right)\right.\\&\left.-\frac{2^{\alpha}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[J^{\alpha}_{a^+}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta_1(b,a)}{4}\left(\frac{2\alpha}{\alpha+1}\right)^{1-\frac{1}{p_1}}\\&\times\left[\left(\left(\frac{\alpha}{\alpha+1}\right)|f'(a)|^{p_1}+\Pi_2^{\alpha}(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\left.+\left(\left(\Pi_1^{\alpha}(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\frac{\alpha}{\alpha+1}\right)|f'(a)|^{p_1}\right)\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

Remark 5.9. *From Corollary 4.8, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for k-Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta_1(b,a)}{4}\left(\frac{4\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{\alpha+k}{\alpha}}+\frac{2k}{\alpha+k}-\frac{4}{3}\right)^{1-\frac{1}{p_1}}\\&\times\left[\left(\left(\frac{2\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{\alpha+k}{\alpha}}-\frac{2}{3}+\frac{k}{\alpha+1}\right)|f'(a)|^{p_1}+\Pi_2^{\frac{\alpha}{k}}\left(\frac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\left.+\left(\Pi_1^{\frac{\alpha}{k}}\left(\frac{2}{3}\right)\eta_2(|f'(b)|,|f'(a)|)^{p_1}+\left(\frac{2\alpha}{\alpha+k}\left(\frac{2}{3}\right)^{\frac{\alpha+k}{\alpha}}-\frac{2}{3}+\frac{k}{\alpha+1}\right)|f'(a)|^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for k-Riemann-Liouville fractional integrals;

$$\begin{aligned}&|f(a)+f(a+\eta_1(b,a))\\&\left.-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+1)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)^-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta_1(b,a)}{4}\left(\frac{2\alpha}{\alpha+k}\right)^{1-\frac{1}{p_1}}\left[\left(\frac{k}{\alpha+k}|f'(a)|^{p_1}+\Pi_2^{\frac{\alpha}{k}}(0)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\left.+\left(\frac{k}{\alpha+1}|f'(a)|^{p_1}+\Pi_1^{\frac{\alpha}{k}}(0)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

3. For $\lambda=\mu=1$ we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals

$$\begin{aligned}&\left|2f\left(\frac{a+b}{2}\right)-\frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+1)}{(b-a)^{\frac{\alpha}{k}}}\left[{}_{a^+}I_k^{\alpha}f\left(\frac{a+b}{2}\right)+{}_{b^-}I_k^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{4}\left(\frac{2\alpha}{\alpha+1}\right)^{1-\frac{1}{p_1}}\left[\left(\left(\frac{\alpha}{\alpha+k}\right)|f'(a)|^{p_1}+\Pi_2^{\frac{\alpha}{k}}(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right.\\&\left.+\left(\left(\frac{\alpha}{\alpha+k}\right)|f'(a)|^{p_1}+\Pi_1^{\frac{\alpha}{k}}(1)\eta_2(|f'(b)|,|f'(a)|)^{p_1}\right)^{\frac{1}{p_1}}\right].\end{aligned}$$

Remark 5.10. *From Theorem 4.9, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for generalized fractional integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{b-a}{2\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)-\frac{2}{3}\Delta_{\eta_1}(1)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\\&\times\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for generalized fractional integrals;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.\\&\left.-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\\&\times\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

3. For $\lambda=\mu=1$, we have the following midpoint type inequality for generalized fractional integrals;

$$\begin{aligned}&\left|f\left(a+\frac{\eta_1(b,a)}{2}\right)-\frac{1}{2\Delta_{\eta_1}(1)}\left[{}_{a^+}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)-}I_\varphi f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4\Delta_{\eta_1}(1)}\left(\int_0^1\left|\Delta_{\eta_1}(t)-\Delta_{\eta_1}(1)\right|^{p_1}dt\right)^{\frac{1}{p_1}}\\&\times\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

Remark 5.11. *From Corollary 4.10, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for Riemann integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(b)\right]\right.\\&\left.-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\frac{2^{p_1+1}+1}{(p_1+1)3^{p_1+1}}\right)^{\frac{1}{p_1}}\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\left.+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

2. For $\lambda=\mu=0$, we have following trapezoid type inequality for Riemann integrals;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\frac{1}{p_1+1}\right)^{\frac{1}{p_1}}\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\left.+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

3. For $\lambda=\mu=1$, we have following midpoint type inequality for Riemann integrals

$$\begin{aligned}&\left|f\left(a+\frac{\eta_1(b,a)}{2}\right)-\frac{1}{\eta_1(b,a)}\int_a^{a+\eta_1(b,a)}f(t)dt\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\frac{1}{p_1+1}\right)^{\frac{1}{p_1}}\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\left.+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

Remark 5.12. *From Corollary 4.11, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.-\frac{2^\alpha\Gamma(\alpha+1)}{(\eta_1(b,a))^\alpha}\left[{}_{a^+}I_\alpha f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_\alpha f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\int_0^1\left|t^\alpha-\frac{2}{3}\right|^{p_1}dt\right)^{\frac{1}{p_1}}\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\left.+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.\\&\left.-\frac{2^{\alpha}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[{}_{a^+}I_{\alpha}f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_{\alpha}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\frac{1}{\alpha p_1+1}\right)^{\frac{1}{p_1}}\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\left.+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

3. For $\lambda=\mu=1$, we have the following midpoint type inequality for generalized fractional integrals;

$$\begin{aligned}&\left|f\left(a+\frac{\eta_1(b,a)}{2}\right)\right.\\&\left.-\frac{2^{\alpha}\Gamma(\alpha+1)}{(\eta_1(b,a))^{\alpha}}\left[{}_{a^+}I_{\alpha}f\left(a+\frac{\eta_1(b,a)}{2}\right)+{}_{a+\eta_1(b,a)^-}I_{\alpha}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\int_0^1(1-t^{\alpha})^{p_1}dt\right)^{\frac{1}{p_1}}\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}\right.\\&\left.+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

Remark 5.13. *From Corollary 4.12, we have following inequalities:*

1. For $\lambda=\mu=\frac{2}{3}$, we have following Simpson's type inequality for k-Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta_1(b,a)}{2}\right)+f(a+\eta_1(b,a))\right]\right.\\&\left.-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\int_0^1\left|t^{\frac{\alpha}{k}}-\frac{2}{3}\right|^{p_1}dt\right)^{\frac{1}{p_1}}\\&\times\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

2. For $\lambda=\mu=0$, we have the following trapezoidal type inequality for k-Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta_1(b,a))}{2}\right.\\&\left.-\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(\eta_1(b,a))^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)+J^{\alpha}_{a+\eta_1(b,a)-,k}f\left(a+\frac{\eta_1(b,a)}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\frac{k}{\alpha p_1+k}\right)^{\frac{1}{p_1}}\\&\times\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

3. For $\lambda=\mu=1$, we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals;

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{2^{\frac{\alpha}{2}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a+,k}f\left(\frac{a+b}{2}\right)+J^{\alpha}_{b-,k}f\left(\frac{a+b}{2}\right)\right]\right|\\&\le\frac{\eta_1(b,a)}{4}\left(\int_0^1\left(1-t^{\frac{\alpha}{k}}\right)^{p_1}dt\right)^{\frac{1}{p_1}}\\&\times\left[\left(\frac{3\eta_2(|f'(b)|,|f'(a)|)^{r_1}+4|f'(a)|^{r_1}}{4}\right)^{\frac{1}{r_1}}+\left(\frac{4|f'(a)|^{r_1}+\eta_2(|f'(b)|,|f'(a)|)^{r_1}}{4}\right)^{\frac{1}{r_1}}\right].\end{aligned}$$

6. Concluding Remarks

In this study, we present some generalized inequalities for differentiable $(\eta_1,\eta_2)-$ convex functions via generalized fractional integrals. It is also shown that the results proved here are the strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study and obtain similar inequalities for different kinds of convexity in their future work.

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