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APPLICATIONS OF HERMITE POLYNOMIALS AND MEIJER'S G-FUNCTION IN OUANTUM MECHANICS AND HEAT CONDUCTION

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ABSTRACT

In this paper, we employ an integral involving Hermite polynomials and Meijer's G-function to obtain solutions of the problems of simple harmonic oscillator and heat conduction.

INTRODUCTION

In this paper, we establish three integrals involving Hermite polynomials in terms of gamma functions, which are extremely useful for evaluating integrals involving generalized hypergeometric functions. We use one of the three integrals to evaluate an integral involving Hermite polynomials and Miejer's G-function and employ this integral to obtain a solution of the simple harmonic oscillator problem occuring in quantum mechanics and a solution of a heat conduction problem given by Bhonsle (1).

On specializing the parameters, Meijer's G-function may be reduced to almost all special functions and elementary functions appearing in applied mathematics and engineering (2). Therefore the solutions given in this paper are of a general character and hence may encompass several cases of interest.

The following formulae[3] are required in the proofs.

$$x^{2k+1} = \frac{(2k+1)!}{2^{2k+1}} \sum_{n=0}^{k} \frac{H_{2n+1}(x)}{(2n+1)!\ (k-n)!} \tag{1}$$

$$x^{1k} = \frac{(2k)!}{2^{2k}} \sum_{n=0}^{k} \frac{H_{2n}(x)}{2n!\ (k-n)!} \tag{2}$$

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_k(x) dx = \begin{cases} 0, \ k = m; \\ 2^m m! \sqrt{\pi}, \ k \neq m. \end{cases} \tag{3}$$

In what follows for sake of brevity a_p stands for $a_1, \ldots, a_p$, μ is a positive integer and the symbol $\Delta(\mu, \alpha)$ represents the set of parameters $\frac{\alpha}{\mu}, \frac{\alpha+1}{\mu}, \ldots, \frac{\alpha+\mu-1}{\mu}$.

INTEGRALS INVOLVING HERMITE POLYNOMIALS

The integrals to be evaluated are

$$\int_{-\infty}^{\infty} x^{2k+1} e^{-x^2} H_{2m+1}(x) dx = \frac{2^{2m-2k} \sqrt{\pi}\, \Gamma(2k+2)}{\Gamma(k-m+1)}, \tag{4}$$

where $k \geq m$, k and m are zero or positive integers.

$$\int_{-\infty}^{\infty} x^{2k} e^{-x^2} H_{2m}(x) dx = \frac{2^{2m-2k} \sqrt{\pi}\, \Gamma(2k+1)}{\Gamma(k-m+1)}, \tag{5}$$

where $k \geq m$, k and m are zero or positive integers.

$$\int_{-\infty}^{\infty} x^{n} e^{-x^2} H_{m}(x)\, dx = \frac{2^{m-n} \sqrt{\pi}\, \Gamma(n+1)}{\Gamma\left(\frac{n-m}{2} + 1\right)}, \tag{6}$$

where $n \geq m$, n and m are either 1,3,5,. . . or 0, 2, 4, ...

Proof: On multiplying both sides of (1) by $e^{-x^2} H_{2m+1}(x)$, integrating with respect to x from $-\infty$, to ∞ and using (3), the result (4) is obtained.

On multiplying both sides of (2) by $e^{-x^2} H_{2m}(x)$, integrating with respect to x from $-\infty$, to ∞, the formula (5) is obtained.

The integral (6) follows from (4) and (5).

INTEGRAL INVOLVING HERMITE POLYNOMIALS AND MEIJER'S G-FUNCTION

The integral to be evaluated is

$$\int_{-\infty}^{\infty} x_2^{k+1} e^{-x^2} H_{2m+1}(x)\, G_{p,q}^{u,v}\left[zx^{2\mu} \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] dx$$

$$= \frac{2^{2m+1}\mu^{k+m+2}}{(2\pi)^{\mu/2-1/2}} G^{u,v+2\mu}_{p+2\mu,q+\mu}\left[z\mu^{\mu} \middle| \begin{matrix} \Delta(2\mu,-2k-1),a_p \\ b_q, \Delta(\mu,m-k) \end{matrix} \right] \quad (7)$$

where $2(u+v) < p+q$, $|\arg z| < u+v - \frac{1}{2}p - \frac{1}{2}q)\pi$.

Proof. On expressing the G-function in the integrand as a Mellin-Barnes type integral[2], interchanging the order of integrations, evaluating the inner-integral with the help of (4), and using the multiplication formula for gamma function[2] and the Mellin-Barnes type integral representation of the G-function[2], the value of the integral (7) is obtained.

Note 1. On applying the above procedure and using (5) and (6), we can easily establish two other integrals involving Hermite polynomials and Meijer's G-function.

SIMPLE-HARMONIC OSCILLATOR PROBLEM

One of the fundamental problems in quantum mechanics involving Schrodinger's equation belongs to the one- dimensional motion of a particle bounded in a potential well. The bounded solution of Schrodinger's equation for such a problem is possible only for certain discrete energy levels of the particle within the well. A particular problem of this important category of problems is the simple harmonic oscillator problem, the solution of which involves Hermite polynomials.

Schrodinger's equation for the simple harmonic-oscillator problem in terms of dimensionless parameters takes the form:

$$\varnothing'' + (\lambda - x^2)\ \varnothing = 0, \quad -\infty < x < \infty, \qquad (8)$$

where $\varnothing$ is related to the corresponding wave function and λ is proportional to the possible energy levels.

The solution of $\varnothing$ must satisfy the boundary condition:

$$\lim_{|x| \to \infty} \varnothing(x) = 0. \qquad (9)$$

To obtain a bounded solution of (8), we see that for large values of x, λ becomes negligible compared with x^2. Therefore, asymptotically we expect the solution of (8) to behave as

$$\varnothing(x) \backsim e^{\pm x^2}, \quad |x| \to \infty, \qquad (10)$$

where the negative sign in the exponent is appropriate in order that (9) is satisfied. Therefore, the solution of (8) may be assumed of the form:

$$\varnothing(x) = y(x)\, e^{-x^2/2} \tag{11}$$

Substituting the value of $\varnothing$ (x) from (1.1) into (8), yields the following differential equation :

$$y'' - 2xy' + (\lambda - 1)\, y = 0 \tag{12}$$

It has been established[4] that the only solution of (12) satisfying (9) are those for which $\lambda - 1 = 2n$, i.e.

$$\lambda = \lambda_n = 2n + 1,\ n=0,1,2,\ldots\ . \tag{13}$$

The above values of λ are called energy levels or eigenvalues of the oscillator. Now (12) becomes

$$y'' - 2xy' + 2ny = 0, \tag{14}$$

which is Hermite's equation with solutions $y = H_n(x)$.

We see that to each eigenvalue λ_n given by (13), there corresponds the solutions of (8) given by

$$\varnothing_n(x) = H_n\, e^{-x^2/2},\ n = 0,1,2,\ \ldots\ . \tag{15}$$

The linear combination of the solutions of (8) takes the form:

$$\varnothing(x) = \sum_{n=0}^{\infty} C_n\, H_n(x)\, e^{-x^2/2}\ . \tag{16}$$

SOLUTION OF THE SIMPLE-HARMONIC OSCILLATOR PROBLEM

Let us consider

$$\varnothing(x) = x^{2k+1}\, e^{-x^2/2}\, G^{u,v}_{p,q}\left[zx^{2\lambda} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right] \tag{17}$$

From (16) and (17), we have

$$x^{2k+2} e^{-x^2/2}\, G^{u,v}_{p,q}\left[zx^{2\mu} \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right] = \sum_{n=0}^{\infty} C_n H_n(x) e^{-x^2/2} \tag{18}$$

Multiplying both sides of (18) by $e^{-x^2}\, H_{2m+1}(x)$, integrating with respect to x from $-\infty$ to ∞, and using (3) and (7), we get

$$C_{2m+1} = \frac{(2\pi)^{1/2-\mu/2}\mu^{k+m+1}}{\sqrt{\pi}(2m+1)!} G^{u,v+2\mu}_{p+2\mu,q+\mu}\left[z\mu^{\mu} \middle| \begin{matrix} \Delta(2\mu,-2k-1), a_p \\ b_q, \Delta(\mu, m-k) \end{matrix} \right] \quad (19)$$

From (18) and (19), for the value of $\varnothing(x)$ given in (17), the following solution of the problem is obtained

From (18) and (19), for the value of $\varnothing(x)$ given in (17), the following solution of the problem is obtained

$$\varnothing(x) = \frac{(2\pi)^{1/2-\mu/2}\mu^{k+1/2}}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\mu^{n/2} H_n(x)\, e^{-x^2/2}}{n!}$$

$$\times\, G^{u,v+2\mu}_{p+2\mu,q+\mu}\left[z\mu^{\mu} \middle| \begin{matrix} \Delta(2\mu, -2k-1),\ a_p \\ b_q, \Delta\left(\mu, \frac{n}{2} - \frac{1}{2} - k\right) \end{matrix} \right], \quad (20)$$

where $n = 1,3,5,\ldots, 2(u+v) < p+q$, $|\arg z| < (u+v-\frac{1}{2}p - \frac{1}{2}q)\pi$.

Note 2. In view of Note 1, two other solutions of the problem can be obtained easily.

SOLUTION OF THE HEAT CONDUCTION PROBLEM

Bhonsle[1] obtained the following solution:

$$u(x,t) = \sum_{r=0}^{\infty} A_r\, e^{-(1+2r)kt - x^2/2} H_r(x), \quad (21)$$

for the following partial differential equtaion

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - kux^2, \quad (22)$$

which is related to the following problem of heat conduction[5].

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - h(u - u_0), \quad (23)$$

provided $u_0 = 0$ and $h = kx^2$.

Now, we consider the problem of determining the function

$$u(x,0) = x^{2\omega+1} e^{-x^2/2} G^{u,v}_{p,q}\left[zx^{2\mu} \middle| \begin{matrix} a_p \\ b_q \end{matrix} \right] = \sum_{r=0}^{\infty} A_r e^{-x^2/2} H_r(x) \quad (24)$$

Multiplying both sides of (24) by $e^{-x^2/2} H_{2m+1}(x)$, integrating with respect to x from $-\infty$ to ∞, and using (3) and (7) we get

$$A_{2m+1} = \frac{(2\pi)^{1/2-\mu/2} \mu^{\omega+m+1}}{\sqrt{\pi}(2m+1)!} G^{u,v+2\mu}_{p+2\mu,q+\mu}\left[z\mu^{\mu} \middle| \begin{matrix} \Delta(2\mu,-2w-1), a_p \\ b_g, \Delta(\mu,m-w) \end{matrix}\right] \quad (25)$$

From (21) and (25), for the value of u(x,0) given in (24), the following solution of the problem is obtained

$$u(x,t) = \frac{(2\pi)^{1/2-\mu/2} \mu^{w+1/2}}{\sqrt{\pi}} \sum_{r=1}^{\infty} \frac{\mu^{r/2} e^{-(1+2r)kt-x^2/2} H_r(x)}{r!}$$

$$\times G^{u,v+2\mu}_{p+2\mu,q+\mu}\left[x\mu^{\mu} \middle| \begin{matrix} \Delta(2\mu, -2w-1), a_p \\ b_q, \Delta(\mu, \frac{r}{2} - \frac{1}{2} - w) \end{matrix}\right], \quad (26)$$

where $r = 1,3,5,\ldots, 2(u+v) < p+q$, $|\arg z| < (u+v-\frac{1}{2}p-\frac{1}{2}q)\pi$.

Note 3. In view of Note 1, two other solutions of the problem can be obtained easily.

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