Fuzzy $\overline{X}$ and S Control Charts Based on Confidence Intervals

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Abstract − There have been changes since the companies have realized the important role of quality improvement in their success. If they are able to produce high quality products and satisfy demands, then they can survive in competitive global markets. Quality improvement applications aim to decrease variability, which leads to less cost, production time, number of defects, scrap, rework and more customer satisfaction. Quality can be improved by reducing product variability. On the other hand, uncertainty or subjectivity is a part of many engineering and real life problems. However, these problems cannot be solved by traditional methods. This study focuses on constructing Xbar and S control charts in fuzzy environment. The approach is developed by considering the theoretical structure of the Shewhart control charts. The core of the approach depends on the combination of parametric interval estimation and fuzzy statistics. Control limits and samples are presented by fuzzy numbers which ensures to maintain fuzziness in control charts. An important property of the approach is that the fuzzy charts can be reduced to Shewhart control charts. A simulation study was conducted for the performance evaluation of fuzzy Xbar and S control charts. The proposed fuzzy control chart is sensitive to process mean shifts and variance changes, and outperforms the traditional control charts under the changes of variance. In addition, an example from the literature shows that the approach is an effective way of presenting fuzziness in the quality characteristics, which enables the approach to have high applicability to the real life problems.

Keywords − *Confidence intervals, fuzzy numbers, fuzzy set theory, fuzzy statistics, quality control charts,*

1. Introduction

Quality improvement is a systemic approach aims to improve the performance of systems. It makes controllable and measurable changes that lead to better outcomes and can be applied to any area. Understanding and improving quality are the main factors for a company or organization to achieve its goals such as increasing profit, efficiency, market share, growth ...etc. Quality of production processes can be improved by the reducing the variability of products. Quality and variability are inversely proportional. Having low variability is the main way of reducing waste, production time and costs, and improving the performance and quality of processes.

Monitoring is the first step of quality improvement methods. In 1924 Walter A. Shewhart who is often considered the pioneer of statistical process control, developed quality control charts. Shewhart control charts are the most commonly referred statistical quality control tools used to monitor processes. These charts are graphical displays of quality characteristics that have been measured or observed over time. If sample values fall within the control limits which are two horizontal lines calculated by the sample data, with a random pattern, the process is said to be in control, otherwise the process is out of control and necessary actions should be taken.

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These traditional control charts are based on random precise data of the quality characteristics. However, uncertainty is the nature of the real life problems, systems, and ignored by the traditional control charts. Sources of uncertainty arise from lack of precise information, such as not fully reliable information due to measurement error, incomplete data, instrument noise or environmental factors. These sources can occur at any point of data collection. For example, a measurement of 0.01 might necessitate a greater precision such as 0.01008. Lack of calibration of measuring instruments or calibration which is not suitable for the production environment might cause inaccurate measurements. Classification of products such as defective non-defective, conforming non-conforming might not be appropriate in some situations. Intermediate states might be required. Judgement of human inspectors might be another reason for uncertainty: a defective product might be non-defective for another person.

Since fuzzy set theory was introduced by Zadeh in 1965 it has been integrated with statistical process control to handle uncertainty in data or human subjectivity. Combination of fuzzy theory and statistical quality control has been studied by many researchers. Various different approaches, methods and applications have been proposed. These studies have been developed from many different points of view but the main objective is same: dealing with uncertainty and fuzziness in process control.

The previous studies were mainly developed on linguistic data. Raz & Wang (1990), Wang & Raz (1990), Kanagawa Tamaki & Ohta (1993) and Gulbay, Kahraman & Ruan (2004) proposed approaches for monitoring attribute data which are in linguistic forms. Chen & Yeh (2004) developed a two stages fuzzy approach for variable sampling interval. In another study, Cheng (2005) developed fuzzy control charts for attributes and generated the fuzzy data by combining the experts' opinion and measurements. Gulbay & Kahraman (2007) introduced direct fuzzy approach (DFA) for c control chart and compared with the previous approaches using defuzzification methods. Hsieh, Tong & Wang, (2007) proposed a fuzzy control chart which is based on engineering experiences and fuzzy theory. A fuzzy approach for constructing attributes control chart in multistage process was presented by Engin, Elik & Kaya (2008). The approach is based on acceptance sampling and solved by genetic algorithms. Amirzadeh, Mashinchi & Parchami (2009) proposed an approach for attributes. Although the approach was proposed for constructing fuzzy p control charts; as data represents the degree of nonconformity of items, the technique of the fuzzy chart is similar to traditional $\bar{x}$ and s control charts. Shu & Wu (2010) emphasized the critical role of fuzzy data in measurements of quality characteristics and they proposed a fuzzy p chart with permissibly function. Beside these, Wang, Li & Yasuda (2014) introduced fuzzy c chart monitored with weighted possibilistic mean and weighted interval valued possibilistic mean of fuzzy numbers. Thaga & Sivasamy (2015) developed transition probability approach based on markov chain theory for linguistic data. Senturk & Antucheviciene (2017) proposed an interval type-2 fuzzy c-control chart with an application to a food company. Recently, Aslam, Bantan & Khan (2020) developed a neutrosophic NEWNA np chart.

Applications of fuzzy set theory have been investigated in the field of fuzzy variable control charts by many other authors. The former study on monitoring variable control charts for uncertain observations was developed by Fazel Zarandi, Turksen & Kashanet, (2006). They proposed plotting control charts by considering uncertain process parameters for both variables and attributes. Another fuzzy chart approach for variables includes a warning line besides its upper control limit (Faraz & Moghadam, 2007). The shifts between the subgroups of the fuzzy control chart for variables are designed with Pearson goodness of fit statistic. Use of α - cut midrange transformation technique for fuzzy $\bar{X}$, R and S control charts are proposed by Senturk & Erginel (2009).

Faraz & Shapiro (2010) developed an approach for fuzzy $\bar{X}$ and S charts by considering the assumptions of the Shewhart control charts. The approach is based on a fuzzy region and a graded exclusion measure which defines the degree of a sample out of the in-control region by the percentage area of under fuzzy sets. Shu &

Wu (2011) proposed dominance approach for constructing fuzzy $\overline{X}$ and R control charts. Dominance results of the resolution identity are used to calculate the control limits of the fuzzy chart. Studies have been carried out in the field of fuzzy control charts in which transformation methods are not used. A Fuzzy mean and range control charts are proposed by Mojtaba Zabihinpour, Ariffin, Tang & Azfanizam, (2014). Observations and control limits are in the form of triangular fuzzy numbers and the processes are defined by percentage of area of the sample mean which remains outside the control limits. They extended their study for a fuzzy $\overline{X}$ and *S* charts (Mojtaba Zabihinpour, Arif, Tang & Azfanizam, 2015). Another variable control chart is proposed for non-crisp data by Panthong & Pongpullponsak, (2016). Shu, Dang, Nguyen, Hsu & Phan (2017) proposed fuzzy control limits based on results of the resolution identity in fuzzy set theory. Soleymani & Amiri, (2017) proposed a fuzzy cause selecting control chart to monitor the second stage of a two-stage process where the quality characteristic of the second stage belongs to fuzzy sets in Phase I analysis. Ercan Teksen & Anagun, (2018) considered interval type-2 fuzzy sets based on different methods for $\overline{X}$-R control charts. Recently, Mendes, Machado & Rizol, (2019) compared fuzzy control charts and traditional control charts and presented that fuzzy control chart has greater efficiency than the traditional control charts. Al-Refaie, Obaidat, Fouad & Hanayneh, (2019) proposed fuzzy variables control charts under linguistic data and development process capability analysis.

Many other researchers have contributed to fuzzy process control works from different point of view including skewed data in fuzzy control charts (Atta, Shoraim, Yahaya, Zain & Ali, 2016; Yimnak & Intaramo, 2020), nonparametric fuzzy charts (Momeni & Shokri, 2019; Wang & Hryniewicz, 2015), flexible control charts (Pekin Alakoc & Apaydin, 2018), economic design of individual control chart (Wang & Chen, 1995; Chen, Chang & Chiu, 2008), fuzzy inference control system (Saricicek & Cimen, 2011), charts for auto correlated fuzzy observations (Sadeghpour Gildeh & Shafiee, 2015), performance of FEV theory control charts with α – cut level fuzzy midrange method for three skewed distributions (Intaramo, 2012), nonrandom patterns of fuzzy control charts and fuzzy run rules (Hsu & Chen, 2001; Tannock, 2003; Gulbay & Kahraman, 2006; Chih & Kuo, 2007; Fazel Zarandi, Alaeddini & Turksen 2008; Demirli & Vijayakumar, 2010; Pekin Alakoc & Apaydin, 2013), detecting mean and variance shifts of a process (Chang & Aw, 1996; Moameni, Saghaei, & Ghorbani Salnghooch, 2012; Salnghooch, 2015; Kaya, Erdogan & Yildiz, 2017), fuzzy multivariate control charts (Taleb Limam & Hirota, 2006; Moheb Alizadeh, Arshadi Khamseh & Fatemi Ghomi, 2010; Pastuizaca Fernandez, Carrion Garcia, A. & Ruiz Barzola, 2015), multi objective design of control charts (Morabi, Owlia, Bashiri & Doroudyan, 2015), fuzzy CUSUM and EWMA control charts (Senturk, Erginel, Kaya, & Kahraman, 2014; Akhundjanov & Pascual, 2015).

Although there are many studies in literature, only a few of them focused on using fuzzy statistics. Fuzzy statistical theory is based on both probability theory and fuzzy set theory. The major contribution of this study is to use fuzzy statistical theory to express uncertain data in quality control charts. Therefore, fuzzy control chart values are thoroughly determined in accordance with the uncertain measurements and the probability theory. The main difficulty of designing a fuzzy control chart is to develop appropriate fuzzy control limits and to present uncertain data. The selection of the limits may lead to different judgments on the process state. In order to overcome this issue, in some existing works it is suggested to use transformation methods, but it is known that these methods cause loss of information. On the other hand, using crisp values directly causes a strict classification and may lead to a wrong decision. In this study, we propose an approach for constructing fuzzy $\overline{X}$ and *S* control charts, based on confidence interval estimation and fuzzy sets. We conduct a computational simulation study and compare the proposed fuzzy control charts with Shewhart control charts for various mean and variance shifts. The average run length statistics present that the fuzzy control chart is more sensitive to variance shifts than the Shewhart control chart.

2. Materials and Methods

2.1. Statistical Background

Let X_1, X_2,…, X_n be a random sample with probability density function N(μ, σ^2) and mean and standard deviation of the sample are crisp numbers. If all of the 100(1-α)% confidence intervals on μ for different significance levels are placed, one on top of the other, then a triangular shaped fuzzy number is produced as fuzzy estimator of μ (Buckley, 2006). For this estimator, the confidence limits are continuous nonlinear functions of the significance level and a 100(1-α)% confidence interval equals to α - cut set of a fuzzy number denoted by $\tilde{\mu} = \left[\mu_1(\alpha), \mu_2(\alpha)\right]$ where $\tilde{\mu}$ is a crisp set of elements whose membership degrees are greater than or equal to the specified value of α. An example triangular fuzzy estimator when 0.001$\leq \alpha \leq$1 is displayed in Figure 1 where 0.001 is an arbitrary value.
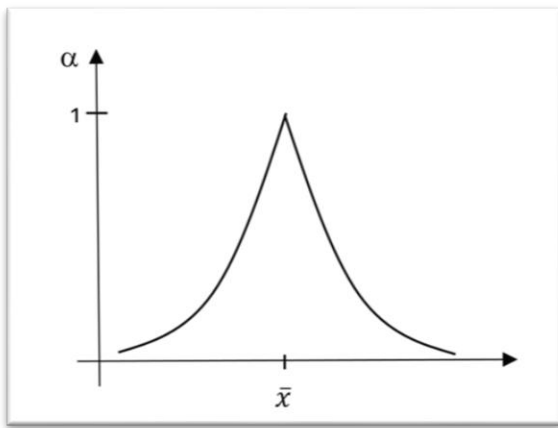

Figure 1. An example: Fuzzy estimator of μ

The concept of confidence interval estimation on a single parameter can be extended to several parameters. Let X_1, X_2,…, X_n be a random sample from a normal distribution with mean μ and variance σ^2, then simultaneous confidence region for the mean and the variance is estimated by a parameter space, which includes all possible combinations of the parameters. A joint confidence region for the mean and variance of a normal distribution is formed by the independence property of $\bar{x}$ and s^2. Since, $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ and $\frac{(n-1)s^2}{\sigma^2}$ are each pivotal quantities, we may find numbers q_1, q_2' and q_2'' such that 2.1 and 2.2.

$$P\left(-q_1 \leq \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \leq q_1\right) = 1 - \gamma_1 \tag{2.1}$$

$$P\left(q_2' \leq \frac{(n-1)s^2}{\sigma^2} \leq q_2''\right) = 1 - \gamma_2 \tag{2.2}$$

Also since, $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ and $\frac{(n-1)s^2}{\sigma^2}$ are independent (Mood, Graybill & Boes 2001) we have joint probability which given in 2.3:

$$P\left(-q_1 \leq \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \leq q_1; q_2' \leq \frac{(n-1)s^2}{\sigma^2} \leq q_2''\right) = (1 - \gamma_1)(1 - \gamma_2) \tag{2.3}$$

where $100(1-\gamma_1)\%$ and $100(1-\gamma_2)\%$ are confidence levels. The inequalities in 2.3 present a region in the parameter space, which can be drawn by plotting its boundaries where the x axis and y axis are μ and σ^2, respectively. Figure 2 presents simultaneous confidence region of mean and variance (Mood, Graybill & Boes 2001).
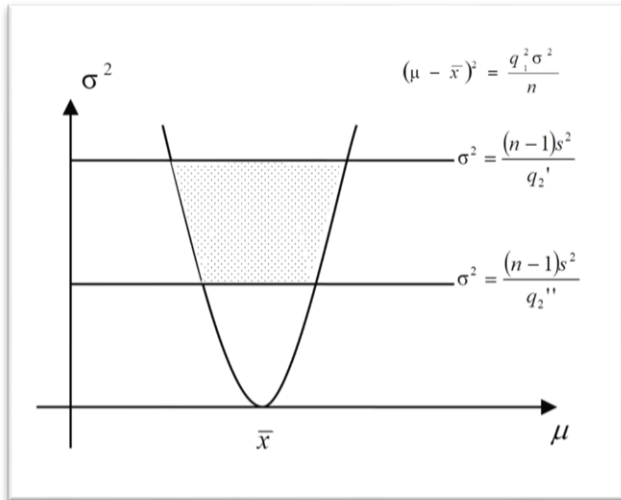

Figure 2. Simultaneous confidence region of μ and σ^2

2.2. Design of Fuzzy $\overline{X}$ Control Chart

Let *m* samples with size n are selected (randomly) from a normal distribution with mean μ and variance σ^2, then; i^{th} sample mean is estimated by fuzzy estimation method. Based on fuzzy estimation method, $100(1-\alpha)\%$ confidence intervals on the mean of each sample can be calculated for all significance levels. Then, the fuzzy estimate of each sample's mean is a triangular fuzzy number whose α - cut is defined by 2.4.

$$\left[\bar{x}_i^l[\alpha], \bar{x}_i^u[\alpha]\right] = \left[\bar{x}_i - t_{\alpha/2,n-1}\frac{s_i}{\sqrt{n}}, \bar{x}_i + t_{\alpha/2,n-1}\frac{s_i}{\sqrt{n}}\right] \tag{2.4}$$

where $\bar{x}_i^l[\alpha]$ and $\bar{x}_i^u[\alpha]$ are the lower and the upper limits of $100(1-\alpha)\%$ interval.

Triangular fuzzy control limits are based on the confidence interval estimations on $\mu + 3\frac{\sigma}{\sqrt{n}}$ and $\mu - 3\frac{\sigma}{\sqrt{n}}$ which are obtained by modifying the simultaneous confidence region of mean and variance approach. In order to have a simultaneous confidence region for μ and variance σ, the parabola given in Figure 2 is replaced by the straight lines $\mu = \bar{\bar{x}} + q_1\frac{\sigma}{\sqrt{nm}}$ and $\mu = \bar{\bar{x}} - q_1\frac{\sigma}{\sqrt{nm}}$ which intersect with the horizontal lines $\sigma = \sqrt{\frac{(n-1)\sum s_i^2}{q_2'}}$ and $\sigma = \sqrt{\frac{(n-1)\sum s_i^2}{q_2''}}$. Figure 3 shows the simultaneous confidence region which is confined by these lines.
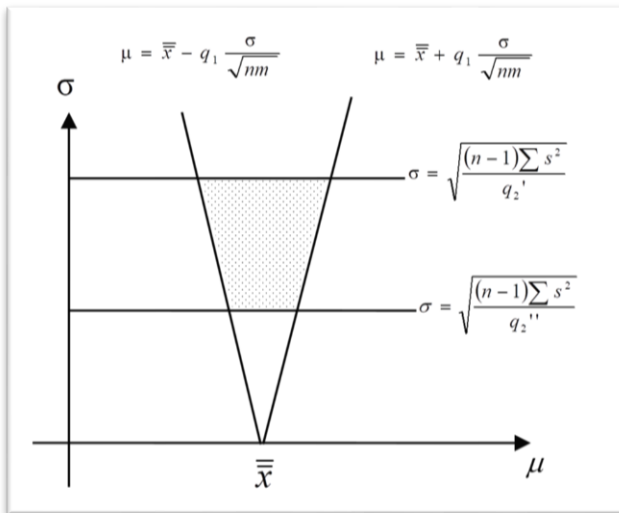
Figure 3. Simultaneous confidence region of μ and σ

Derivation of confidence intervals for the limits is an optimization problem in which the simultaneous confidence region is the feasible region. The lower and upper bound values of the confidence interval on the upper fuzzy control limit are obtained by minimizing and maximizing $\mu + 3\frac{\sigma}{\sqrt{n}}$, respectively. Confidence interval estimation on the lower control limit is derived by a similar calculation; minimizing and maximizing $\mu - 3\frac{\sigma}{\sqrt{n}}$. The corner points of the confidence region are the optimum points, which give the required confidence limits. A $100(1-\gamma_1)(1-\gamma_2)\%$ confidence interval estimators of the upper and lower limits are defined in 2.5 and 2.6:

$$\bar{\bar{x}} + \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi^2_{\gamma_2/2,m(n-1)}}}\left(3 - \frac{z_{\gamma_1/2}}{\sqrt{m}}\right) < \mu + 3\frac{\sigma}{\sqrt{n}} < \bar{\bar{x}} + \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi^2_{1-\gamma_2/2,m(n-1)}}}\left(\frac{z_{\gamma_1/2}}{\sqrt{m}} + 3\right) \qquad (2.5)$$

$$\bar{\bar{x}} - \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi^2_{1-\gamma_2/2,m(n-1)}}}\left(\frac{z_{\gamma_1/2}}{\sqrt{m}} + 3\right) < \mu - 3\frac{\sigma}{\sqrt{n}} < \bar{\bar{x}} - \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi^2_{\gamma_2/2,m(n-1)}}}\left(3 - \frac{z_{\gamma_1/2}}{\sqrt{m}}\right) \qquad (2.6)$$

where $\bar{\bar{x}}$ is the mean of m sample means, $z_{\gamma_1/2}$ is the upper $100(\gamma_1/2)\%$ point of the standard normal distribution and $\chi^2_{1-\gamma_2/2,m(n-1)}$ and $\chi^2_{\gamma_2/2,m(n-1)}$ are the lower and upper $100(\gamma_2/2)\%$ points of the chi-square distribution with *m(n-1)* degrees of freedom, respectively.

Similarly, we estimate the process mean by the same approach. 2.7 gives a $100(1-\alpha)\%$ confidence interval on the mean, which is used to define fuzzy centre number of the fuzzy $\overline{X}$ control chart.

$$\bar{\bar{x}} - z_{\alpha/2}\sqrt{\frac{\sum s_i^2}{n\,m^2}} < \mu < \bar{\bar{x}} + z_{\alpha/2}\sqrt{\frac{\sum s_i^2}{n\,m^2}} \qquad (2.7)$$

Consequently, we use the confidence intervals given in 2.5, 2.6 and 2.7 to plot fuzzy $\overline{X}$ control chart. We define the three sigma α - cut of control limits and center number by 2.8, 2.9 and 2.10:

$$\left[\widehat{\text{UCL}}_{\bar{\bar{x}}}^{l}[\alpha], \widehat{\text{UCL}}_{\bar{\bar{x}}}^{u}[\alpha]\right] = \left[\bar{\bar{x}} + \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi_{\gamma_2/2,m(n-1)}^2}}\left(3 - \frac{z_{\gamma_1/2}}{\sqrt{m}}\right), \bar{\bar{x}} + \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi_{1-\gamma_2/2,m(n-1)}^2}}\left(\frac{z_{\gamma_1/2}}{\sqrt{m}} + 3\right)\right] \tag{2.8}$$

$$\left[\widehat{\text{CL}}_{\bar{x}}^{l}[\alpha], \widehat{\text{CL}}_{\bar{x}}^{u}[\alpha]\right] = \left[\bar{\bar{x}} - z_{\alpha/2}\sqrt{\frac{\sum s_i^2}{n\,m^2}}, \bar{\bar{x}} + z_{\alpha/2}\sqrt{\frac{\sum s_i^2}{n\,m^2}}\right] \tag{2.9}$$

$$\left[\widehat{\text{LCL}}_{\bar{x}}^{l}[\alpha], \widehat{\text{LCL}}_{\bar{x}}^{u}[\alpha]\right] = \left[\bar{\bar{x}} - \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi_{1-\gamma_2/2,m(n-1)}^2}}\left(\frac{z_{\gamma_1/2}}{\sqrt{m}} + 3\right), \bar{\bar{x}} - \sqrt{\frac{(n-1)\sum s_i^2}{n\,\chi_{\gamma_2/2,m(n-1)}^2}}\left(3 - \frac{z_{\gamma_1/2}}{\sqrt{m}}\right)\right] \tag{2.10}$$

where $\alpha = 1 - (1 - \gamma_1)(1 - \gamma_2)$.

α - cut triangular fuzzy numbers are used to graph the fuzzy chart. If $\widehat{\text{UCL}}_{\bar{x}}^{u}[\alpha] \leq \bar{x}_i^{l}[\alpha]$ or $\bar{x}_i^{u}[\alpha] \leq \widehat{\text{LCL}}_{\bar{x}}^{l}[\alpha]$, then α - cut of the ith fuzzy number is beyond the control limits and the process is out of control. Similarly, if $\bar{x}_i^{u}[\alpha] \leq \widehat{\text{UCL}}_{\bar{x}}^{l}[\alpha]$ and $\widehat{\text{LCL}}_{\bar{x}}^{u}[\alpha] \leq \bar{x}_i^{l}[\alpha]$, then the α - cut of the ith fuzzy number is completely between the fuzzy limits. Otherwise, the fuzzy number intersects with fuzzy limits, that is $\widehat{\text{UCL}}_{\bar{x}}^{l}[\alpha] < \bar{x}_i^{u}[\alpha] \leq \widehat{\text{UCL}}_{\bar{x}}^{u}[\alpha]$ or $\widehat{\text{UCL}}_{\bar{x}}^{l}[\alpha] < \bar{x}_i^{l}[\alpha] \leq \widehat{\text{UCL}}_{\bar{x}}^{u}[\alpha]$ or $\widehat{\text{LCL}}_{\bar{x}}^{l}[\alpha] \leq x_i^{u}[\alpha] < \widehat{\text{LCL}}_{\bar{x}}^{u}[\alpha]$ or $\widehat{\text{LCL}}_{\bar{x}}^{l}[\alpha] \leq x_i^{l}[\alpha] < \widehat{\text{LCL}}_{\bar{x}}^{u}[\alpha]$, and the state of the process is not as certain. In this study, the proposed approach defines the process by a decision function, which is developed by considering all possible situations of fuzzy control limits and fuzzy numbers. The process is defined as in control as long as the fuzzy numbers fall within the fuzzy limits under some specified conditions. 2.11 gives the decision criterion.

$$\text{Process} = \begin{cases} \text{in control,} & \text{if } f_{i_{\bar{x},1}}[\alpha]f_{i_{\bar{x},2}}[\alpha] \geq 0 \\ \text{out of control,} & \text{otherwise} \end{cases} \tag{2.11}$$

where $f_{i_{\bar{x},1}}[\alpha] = \widehat{\text{UCL}}_{\bar{x}}^{u}[\alpha] - \bar{x}_i^{u}[\alpha]$ and $f_{i_{\bar{x},2}}[\alpha] = \bar{x}_i^{l}[\alpha] - \widehat{\text{LCL}}_{\bar{x}}^{l}[\alpha]$. If any α - cut of a fuzzy number is completely or partially beyond the fuzzy control limits, then $f_{i_{\bar{x},1}}[\alpha]$ or $f_{i_{\bar{x},2}}[\alpha]$ is negative and the process is out of control, otherwise the process is in control. In addition to the decision function, we propose a membership function to increase the functionality of the control chart. The function shows degree of being out of control of a fuzzy number. In other words, the degree of deviation from in control state is given by this function. The membership function of the ith fuzzy number is given in 2.12.

$$\varphi_{i_{\bar{x}}}[\alpha] = \frac{\sum_{k=1}^{4}|f_{i_{\bar{x},k}}[\alpha]| + 2(\bar{x}_i^{u}[\alpha] - \bar{x}_i^{l}[\alpha]) - (\widehat{\text{UCL}}_{\bar{x}}^{u}[\alpha] + \widehat{\text{UCL}}_{\bar{x}}^{l}[\alpha] - \widehat{\text{LCL}}_{\bar{x}}^{u}[\alpha] - \widehat{\text{LCL}}_{\bar{x}}^{l}[\alpha])}{2\min\{\sum_{k=0}^{1}|f_{i_{\bar{x},2k+1}}[\alpha]|, \sum_{k=1}^{2}|f_{i_{\bar{x},2k}}[\alpha]|\}} \tag{2.12}$$

where $f_{i_{\bar{x},3}}[\alpha] = \widehat{\text{UCL}}_{\bar{x}}^{l}[\alpha] - \bar{x}_i^{u}[\alpha]$ and $f_{i_{\bar{x},4}}[\alpha] = \bar{x}_i^{l}[\alpha] - \widehat{\text{LCL}}_{\bar{x}}^{u}[\alpha]$. If $\varphi_{i_{\bar{x}}}[\alpha] < 1$ for all i, then the process is in control. If $\varphi_{i_{\bar{x}}}[\alpha] = 0$ then, this denotes that fuzzy number is in desired location: the number lies completely between the fuzzy limits. When $\varphi_{i_{\bar{x}}}[\alpha] \geq 1$, then this is considered as an indication of out of control. The membership degree is not ranged in interval [0, 1], which means $\sup(\varphi_{i_{\bar{x}}}[\alpha]) > 1$. This is because of the possible locations of fuzzy numbers on the chart. Although, the membership degree is not limited, a value which is greater than 1 is rarely observed. This situation is observed when the sample is

completely out of the limits with a high variability. In addition, warning state criteria which increases sensitivity of the fuzzy control chart, is defined as follows: $0 < \varphi_{i_{\bar{x}}}[\alpha] < 1$.

The value of α affects all the statistics and tightness of the control charts. In other words, α operates like sensitizing rules; selecting a small α means having a more sensitive control chart. If we increase α, both of the significance levels (γ_1 and γ_2) increase, and this yields to a narrower center number and a smaller joint confidence region. In other words, as α approaches to 1, the joint confidence region and center number approach to a point. Consequently, when α is 1, the fuzzy numbers, fuzzy control limits and the fuzzy center number become crisp numbers and the fuzzy control chart approach is simplified to Shewhart control chart.

2.3. Design of Fuzzy *S* Control Chart

Similar to the previous part, fuzzy *S* control chart is based on fuzzy estimation method: The triangular fuzzy control limits and center number are obtained by adding 100(1-α)% confidence intervals for the standard deviation on top of the others. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, then the random variable $\frac{(n-1)s_i^2}{\sigma^2}$ has a chi-square distribution with n-1 degrees of freedom. Accordingly, α - cut of fuzzy estimate of ith sample standard deviation is formulated by a 100(1-α)% confidence interval for σ which is given in 2.13.

$$\left[\tilde{s}_i^l[\alpha], \tilde{s}_i^u[\alpha]\right] = \left[\sqrt{\frac{(n-1)s_i^2}{\chi^2_{\alpha/2,(n-1)}}}, \sqrt{\frac{(n-1)s_i^2}{\chi^2_{1-\alpha/2,(n-1)}}}\right] \tag{2.13}$$

If data consists of *m* samples each have *n* measurements, then the random variable $\frac{(n-1)\sum s_i^2}{\sigma^2}$ is distributed with chi-square distribution with *m*(*n*-1) degrees of freedom which leads to a 100(1-α)% confidence interval given in 2.14.

$$\sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{\alpha/2,m(n-1)}}} \leq \sigma \leq \sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{1-\alpha/2,m(n-1)}}} \tag{2.14}$$

Fuzzy upper and lower control limits are the fuzzy estimates of B6σ and B5σ, where B5 and B6 are the Shewhart control chart constants. So, we define 3σ α - cut fuzzy center number and control limits by 2.15, 2.16 and 2.17.

$$\left[\widetilde{UCL}_s^l[\alpha], \widetilde{UCL}_s^u[\alpha]\right] = \left[B_6\sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{\alpha/2,m(n-1)}}}, B_6\sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{1-\alpha/2,m(n-1)}}}\right] \tag{2.15}$$

$$\left[\widetilde{CL}_s^l[\alpha], \widetilde{CL}_s^u[\alpha]\right] = \left[\sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{\alpha/2,m(n-1)}}}, \sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{1-\alpha/2,m(n-1)}}}\right] \tag{2.16}$$

$$\left[\widetilde{LCL}_s^l[\alpha], \widetilde{LCL}_s^u[\alpha]\right] = \left[B_5\sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{\alpha/2,m(n-1)}}}, B_5\sqrt{\frac{(n-1)\sum s_i^2}{\chi^2_{1-\alpha/2,m(n-1)}}}\right] \tag{2.17}$$

We develop the decision and membership functions by the same idea. If $f_{i_{s,1}}[\alpha]\, f_{i_{s,2}}[\alpha] < 0$, then the process variability is higher than the specified limits and the process is out of control. Similarly, the membership function which is denoted by $\varphi_{i_s}[\alpha]$, shows deviation from the in control state, and the process is out of control, if $\varphi_{i_s}[\alpha] \geq 1$. The decision and the membership functions of fuzzy S control chart are given in 2.18 and 2.19.

$$\text{Process} = \begin{cases} \text{in control,} & \text{if } f_{i_{s,1}}[\alpha]f_{i_{s,2}}[\alpha] \geq 0 \\ \text{out of control,} & \text{otherwise} \end{cases} \tag{2.18}$$

$$\varphi_{i_s}[\alpha] = \frac{\sum_{k=1}^{4}\left|f_{i_{s,k}}[\alpha]\right| + 2\left(s_i^u[\alpha] - s_i^l[\alpha]\right) - \left(\widetilde{\text{UCL}}_s^u[\alpha] + \widetilde{\text{UCL}}_s^l[\alpha] - \widetilde{\text{LCL}}_s^u[\alpha] - \widetilde{\text{LCL}}_s^l[\alpha]\right)}{2\min\left\{\sum_{k=0}^{1}\left|f_{i_{s,2k+1}}[\alpha]\right|, \sum_{k=1}^{2}\left|f_{i_{s,2k}}[\alpha]\right|\right\}} \tag{2.19}$$

where $f_{i_{s,1}}[\alpha] = \widetilde{\text{UCL}}_s^u[\alpha] - s_i^u[\alpha]$, $f_{i_{s,2}}[\alpha] = s_i^l[\alpha] - \widetilde{\text{LCL}}_s^l[\alpha]$, $f_{i_{s,3}}[\alpha] = \widetilde{\text{UCL}}_s^l[\alpha] - s_i^u[\alpha]$ and $f_{i_{s,4}}[\alpha] = s_i^l[\alpha] - \widetilde{\text{LCL}}_s^u[\alpha]$.

2.4. A Numerical Example

In this section, we demonstrate the fuzzy control chart approach on a well-known example from Montgomery, 2019. The data present the inside diameter measurements of forged automobile engine piston rings of 25 samples. Each consists of five piston rings and data follow normal distribution.

First, triangular fuzzy number estimates of mean and standard deviation are calculated for each sample. α - cut of the fuzzy numbers are given in Table 1. Second, triangular fuzzy control limits and center numbers are computed by 2.8, 2.9 and 2.10, and 2.15, 2.16 and 2.17: $\left[\widetilde{\text{UCL}}_{\bar{x}}^l[0.6], \widetilde{\text{UCL}}_{\bar{x}}^u[0.6]\right] = [74.0126, 74.0171]$, $\left[\widetilde{\text{CL}}_{\bar{x}}^l[\alpha], \widetilde{\text{CL}}_{\bar{x}}^u[\alpha]\right] = [74.0007, 74.0016]$, $\left[\widetilde{\text{LCL}}_{\bar{x}}^l[0.6], \widetilde{\text{LCL}}_{\bar{x}}^u[0.6]\right] = [73.9852, 73.9898]$, $\left[\widetilde{\text{UCL}}_s^l[0.6], \widetilde{\text{UCL}}_s^u[0.6]\right] = [0.0190, 0.0205]$, $\left[\widetilde{\text{CL}}_s^l[\alpha], \widetilde{\text{CL}}_s^u[\alpha]\right] = [0.0099, 0.0102]$, $\left[\widetilde{\text{LCL}}_s^l[0.6], \widetilde{\text{LCL}}_s^u[0.6]\right] = [0.0000, 0.0000]$. Then, decision function and membership degrees of the fuzzy numbers are calculated for both of the control charts (see Table 1). Finally, the fuzzy control charts are plotted: Figures 4 and 5 show fuzzy $\overline{X}$ control chart and S chart, respectively.

Table 1

Fuzzy $\bar{X}$ and S control chart example summary statistics.

| Sample	Mean	Stdev	$\bar{x}_i^l[0.6]$	$\bar{x}_i^u[0.6]$	$f_{i_{\bar{x},1}}[0.6]f_{i_{\bar{x},2}}[0.6]$	$\varphi_{i_{\bar{x}}}[0.6]$	$s_i^l[0.6]$	$s_i^u[0.6]$	$f_{i_{s,1}}[0.6]f_{i_{s,2}}[0.6]$	$\varphi_{i_s}[0.6]$
1	74.0100	0.0148	74.0064	74.0140	0.0001	0.3032	0.0134	0.0199	0.0000	0.6107
2	74.0010	0.0075	73.9987	74.0025	0.0002	0.0000	0.0068	0.0101	0.0001	0.0000
3	74.0080	0.0147	74.0042	74.0118	0.0001	0.0000	0.0134	0.0199	0.0000	0.5889
4	74.0030	0.0091	74.0007	74.0053	0.0002	0.0000	0.0082	0.0123	0.0001	0.0000
5	74.0030	0.0122	74.0003	74.0065	0.0002	0.0000	0.0111	0.0165	0.0000	0.0000
6	73.9960	0.0087	73.9934	73.9978	0.0002	0.0000	0.0079	0.0118	0.0001	0.0000
7	74.0000	0.0055	73.9986	74.0014	0.0002	0.0000	0.0050	0.0075	0.0001	0.0000
8	73.9970	0.0123	73.9937	73.9999	0.0001	0.0000	0.0111	0.0165	0.0000	0.0000
9	74.0040	0.0055	74.0028	74.0056	0.0002	0.0000	0.0050	0.0075	0.0001	0.0000
10	73.9980	0.0063	73.9964	73.9996	0.0002	0.0000	0.0057	0.0085	0.0001	0.0000
11	73.9940	0.0029	73.9935	73.9949	0.0002	0.0000	0.0026	0.0039	0.0000	0.0000
12	74.0010	0.0042	74.0003	74.0025	0.0002	0.0000	0.0038	0.0057	0.0001	0.0000
13	73.9980	0.0105	73.9957	74.0011	0.0002	0.0000	0.0095	0.0141	0.0001	0.0000
14	73.9900	0.0153	73.9863	73.9941	0.0000	0.7645	0.0139	0.0207	0.0000	1.0000
15	74.0060	0.0073	74.0041	74.0079	0.0002	0.0000	0.0066	0.0099	0.0001	0.0000
16	73.9970	0.0078	73.9946	73.9986	0.0002	0.0000	0.0071	0.0105	0.0001	0.0000
17	74.0010	0.0106	73.9981	74.0035	0.0002	0.0000	0.0096	0.0143	0.0001	0.0000
18	74.0070	0.0070	74.0056	74.0092	0.0002	0.0000	0.0063	0.0094	0.0001	0.0000
19	73.9980	0.0085	73.9960	74.0004	0.0002	0.0000	0.0077	0.0114	0.0001	0.0000
20	74.0090	0.0080	74.0072	74.0112	0.0001	0.0000	0.0072	0.0108	0.0001	0.0000
21	74.0000	0.0122	73.9967	74.0029	0.0002	0.0000	0.0110	0.0164	0.0000	0.0000
22	74.0020	0.0074	73.9997	74.0035	0.0002	0.0000	0.0067	0.0100	0.0001	0.0000
23	74.0020	0.0119	73.9994	74.0054	0.0002	0.0000	0.0108	0.0161	0.0000	0.0000
24	74.0050	0.0087	74.0030	74.0074	0.0002	0.0000	0.0079	0.0117	0.0001	0.0000
25	73.9980	0.0162	73.9941	74.0023	0.0001	0.0000	0.0146	0.0218	0.0000	1.0000
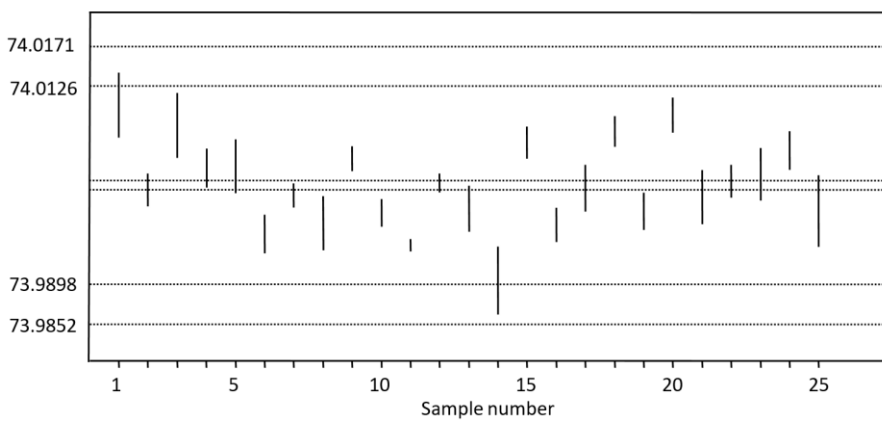
Figure 4. Fuzzy $\bar{X}$ control chart
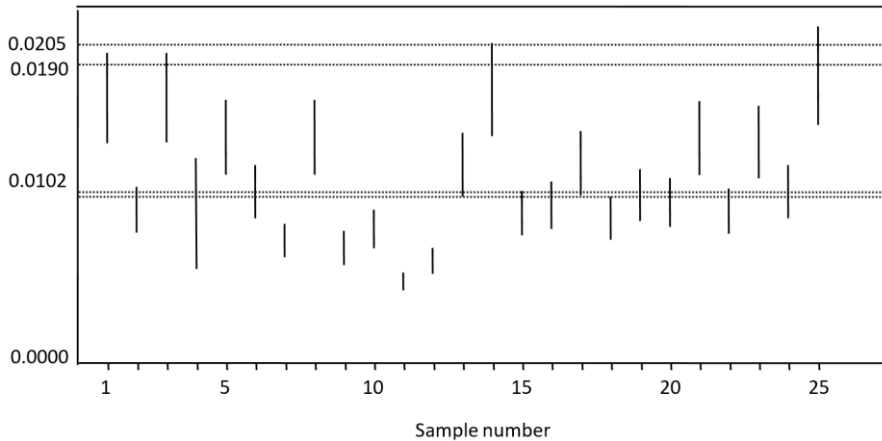
Figure 5. Fuzzy *S* control chart

Figure 4 shows that except the 1st and the 14th samples all the α - cut of fuzzy numbers are completely between the fuzzy limits. Accordingly, we conclude that based on the $\bar{X}$ chart there is no indication that the process is out of control. The membership degrees denote the same decision since, all function values are less than 1. In addition, 1st sample which has $\varphi_{1_{\bar{x}}}[\alpha] = 0.3032$ and the 14th sample with $\varphi_{14_{\bar{x}}}[\alpha] = 0.7645$ indicate warning state which means the process may not be operating properly.

The pattern of fuzzy $\bar{X}$ control chart also gives signals about the result of the fuzzy *S* control chart since, α - cut set of a high variation sample is greater than the set of the low variation. Figure 4 shows greater variation for the 1st, 3rd, 14th and 25th samples. The details are presented in fuzzy *S* control chart. Figure 5 shows that upper limit of 14th and 25th α - cut fuzzy numbers are out of the upper fuzzy control limit with the membership degree 1.000. Consequently, the process is out of control by the 14th and 25th samples.

In this example, we use $\alpha = 0.6$. If we want to have more sensitive control charts, the value should be decreased. That is the main difference between the proposed fuzzy charts and Shewhart control charts. As we increase α, we would have the same results with Shewhart charts.

3. Results and Discussion

Throughout the section, all ARL statistics are provided from 100,000 replications of computer simulation and the random variables are generated from standard normal distribution. The simulations are performed for various combinations of in control ARL, α - cut level and sample size. ARL$_0$ is fixed at 370 and 500, and three values of α: 0.4, 0.6 and 0.8, and two values of sample size: 10 and 15 are considered. Process mean shifts with magnitudes of $\delta_1 \in \{0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00, 3.00\}$ and changes in variability with magnitudes of $\delta_2 \in \{0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 2.00\}$ are investigated. The samples are generated from N($\mu+\delta_1\sigma$, σ) and N(μ, $\sigma+\delta_2\sigma$). Fuzzy $\bar{X}$ control chart and Shewhart $\bar{X}$ control chart simulation results and are given in Tables 2 and 3, and Figures 6, 7, 8 and 9.

Table 2

ARL of fuzzy $\bar{X}$ chart and Shewhart $\bar{X}$ chart for the shift of mean when $\alpha = 0.4, 0.6, 0.8$, ARL$_0$=370, 500 and $n = 10$.

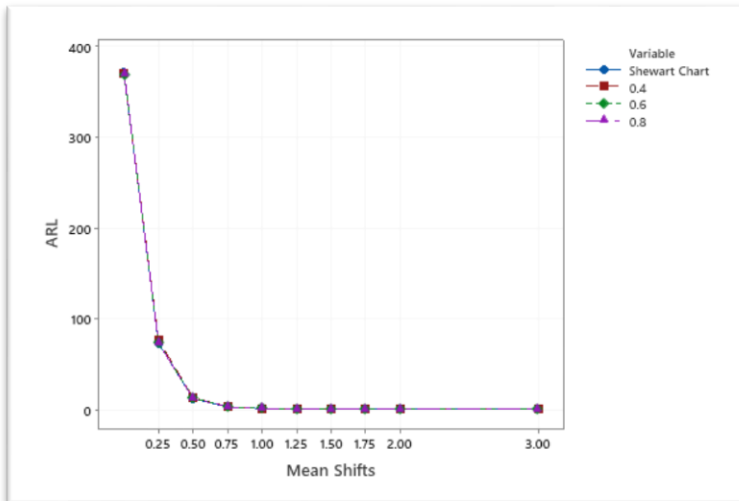| Shifts (δ_1)	Shewhart $\bar{X}$ chart	Fuzzy $\bar{X}$ chart			Shewhart $\bar{X}$ chart	Fuzzy $\bar{X}$ chart		
		$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$		$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
0	370.3983	369.2675	369.3401	370.5864	499.6091	500.9779	499.0698	500.0169
0.25	73.2735	76.5974	74.1125	73.4780	92.6582	96.7288	94.0668	93.0119
0.50	12.8251	13.6806	13.1365	12.8663	15.2280	16.2006	15.5751	15.2960
0.75	3.7749	4.0319	3.8593	3.7992	4.2321	4.5206	4.3360	4.25100
1.00	1.7716	1.8590	1.7987	1.7761	1.8910	1.9959	1.9259	1.8982
1.25	1.2053	1.2392	1.2169	1.2079	1.2409	1.2790	1.2547	1.2436
1.50	1.0424	1.0525	1.0458	1.0433	1.0517	1.0634	1.0559	1.0526
1.75	1.0057	1.0078	1.0063	1.0059	1.0073	1.0099	1.0083	1.0076
2.00	1.0004	1.0007	1.0005	1.0005	1.0006	1.0009	1.0007	1.0007
3.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Figure 6. Plot of ARL versus mean shifts for Shewhart $\bar{X}$ chart and fuzzy $\bar{X}$ charts, ARL$_0$=370
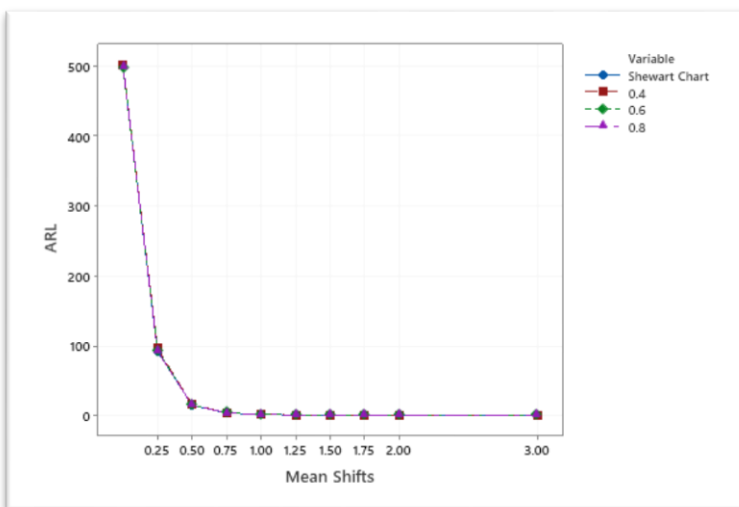

Figure 7. Plot of ARL versus mean shifts for Shewhart $\bar{X}$ chart and fuzzy $\bar{X}$ charts, ARL$_0$=500

Table 3

ARL of fuzzy $\bar{X}$ chart and Shewhart $\bar{X}$ chart for the changes in variability when $\alpha = 0.4, 0.6, 0.8$, $ARL_0 = 370, 500$ and $n = 10$.

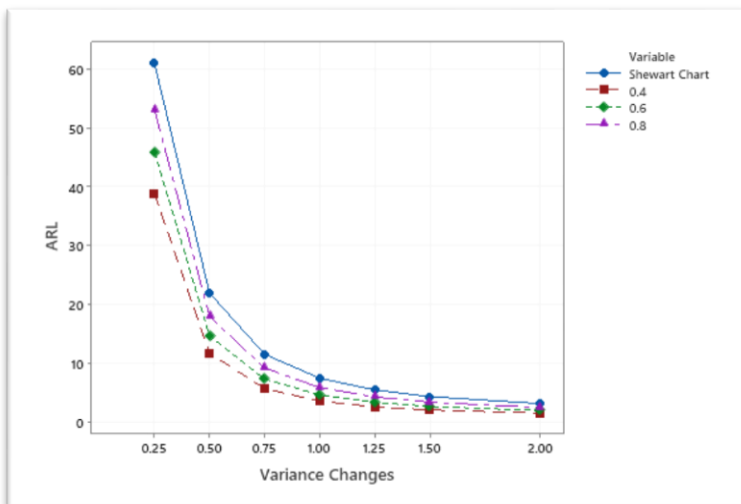|	ARL$_0$ = 370				ARL$_0$ = 500			
	Shewhart $\bar{X}$ chart	Fuzzy $\bar{X}$ chart			Shewhart $\bar{X}$ chart	Fuzzy $\bar{X}$ chart		
Changes (δ_2)		$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$		$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
0.25	60.9939	38.8686	45.9998	53.1343	74.4272	47.0730	55.7550	64.6926
0.50	21.9779	11.7383	14.6946	18.0115	25.3817	13.3060	16.8254	20.7548
0.75	11.5639	5.7343	7.3219	9.2302	12.9126	6.2558	8.0971	10.2481
1.00	7.4842	3.5648	4.6290	5.8863	8.1735	3.8262	5.0134	6.4022
1.25	5.4818	2.5890	3.3635	4.2862	5.8945	2.7451	3.5822	4.6007
1.50	4.3452	2.0600	2.6635	3.3918	4.6198	2.1642	2.7980	3.5944
2.00	3.1515	1.5177	1.9386	2.4658	3.3002	1.5660	2.0165	2.5698

Figure 8. Plot of ARL versus variance changes for Shewhart $\bar{X}$ chart and fuzzy $\bar{X}$ charts, ARL$_0$=370
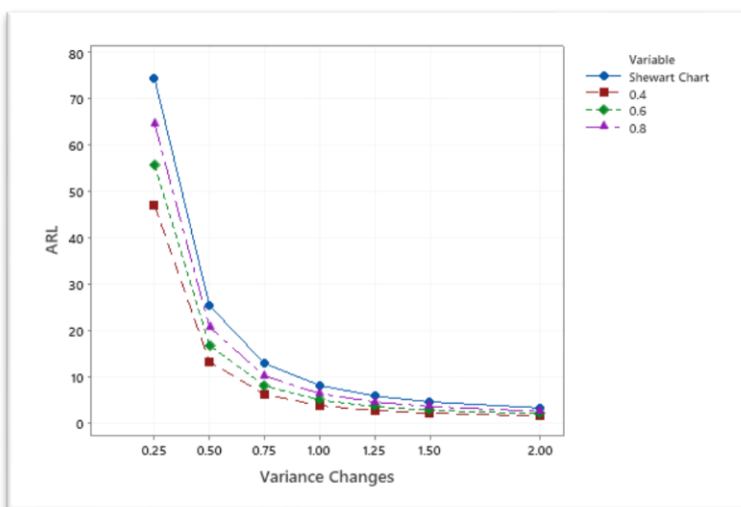

Figure 9. Plot of ARL versus variance changes for Shewhart $\bar{X}$ chart and fuzzy $\bar{X}$ charts, ARL$_0$=500

A summary of the conclusions is listed as follows:

When the process mean shifts are considered, the performance of the fuzzy chart is competitive to Shewhart control chart. Table 2 and Figures 6, 7 show that there are no significant differences between the ARL of Shewhart and fuzzy $\bar{X}$ control charts. Moreover, the fuzzy $\bar{X}$ chart performance is not affected by the change of α and ARL_0. That is, as α or ARL_0 increases the ARL of the fuzzy charts decrease slightly.

Table 2 and Figures 8, 9 show that the fuzzy chart is significantly better than the Shewhart chart for all magnitudes of variation changes. All ARL_1 of the fuzzy chart are smaller than the Shewhart chart results. The fuzzy $\bar{X}$ chart performs better for the lower values of α. The difference increases as α decreases.

The fuzzy control chart performance approaches to Shewhart control chart performance as α increases. When $\alpha = 1$, the fuzzy chart is simplified to Shewhart control chart.

The effect of sample size is also investigated for $n = 15$ by reiterating the simulations and similar tables are observed which give the same conclusions.

4. Conclusion

In this study, we propose a fuzzy control chart approach. The approach effectively combines traditional control chart concept and fuzzy set theory. Based on fuzzy estimation model of population mean and the theory of joint confidence regions for two parameters, the approach is proposed as an extension of Shewhart $\bar{X}$ and S control charts. The approach uses α - cut of fuzzy numbers and can be reduced to Shewhart control chart. A sensitivity study is conducted in which the proposed fuzzy chart and the Shewhart control chart are compared by the performance measure average run length. The results show that the proposed fuzzy control charts detect even small shifts of process mean and changes of variance effectively. Moreover, the fuzzy chart performs significantly better than Shewhart chart, when a change exists in the variance. As there exist a research gap in integrating the fuzzy statistics in quality control, there can be several extensions of the approach; the most noteworthy is to extend the approach for the other traditional control charts. Moreover, sensitizing rules may be adapted or new rules may be proposed for detecting and interpreting the systemic or nonrandom patterns of the fuzzy charts.

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Author Contributions

Nilufer Pekin Alakoc confirms sole responsibility for the study: conception and design, analysis and interpretation of results, and manuscript preparation.

Conflicts of Interest

The author declares no conflict of interest.

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