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Some Fixed Point Theorems on b-θ-metric spaces via b-simulation Functions

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Abstract

We introduce the concept of b-θ-metric space as a generalization of θ-metric space and investigate some of its properties. Then, we establish a fixed point theorem in b-θ-metric spaces via b-simulation functions. Thus, we deduce Banach type fixed point in such spaces. Also, we discuss some fixed point results in relation to existing ones.

1. Introduction

Fixed point theory plays a fundamental role in various fields of mathematics, engineering and applied science. A basic result in fixed point theory is the Banach contraction principle which is an important tool for solving nonlinear analysis' problems. This result has been generalized and extended in various generalized metric spaces.

Many authors have generalized metric spaces in several ways. Bakhtin [1] introduced the concept of b-metric space, which is a generalized form of metric space (see also [2]). Since then, several authors have many fixed point results for single- valued and multi- valued operators in b- metric spaces (see [2]-[4]).

Khojasteh et al. [5] introduced θ-metric space by using a more generalized inequality instead of triangle inequality. They are inspired by fuzzy metric spaces, which are generalizations of metric spaces. Then they proved Banach and Caristi type fixed point in θ-metric spaces.

Khojasteh et al. [6] introduced $\mathscr{Z}$-contraction as a new type of nonlinear contractions via simulation function which is useful to express a family of contractivity conditions. After then Chanda and Dey [7] obtained some fixed point results on θ-metric spaces by using simulation functions. Also, Demma et al.[8] deduced several related results in fixed point theory in b-metric space via b-simulation functions.

In this paper, we defined b-θ-metric space as a generalization of b-metric space with the help of $\mathscr{B}$-action and studied its fundamental properties. Also, we compare it to both b-metric and θ-metric space. Then we obtain a fixed point result in b-θ-metric spaces by using b-simulation functions. So we get the Banach contraction principle in such spaces. Finally, we give some fixed point results regarding existing ones in b-metric spaces and θ-metric spaces.

2. Preliminaries

Definition 2.1. *[1, 2] Let W be a nonempty set and $b \geq 1$ be a given real number. A function $d : W \times W \to [0,\infty)$ is a b-metric on W iff it satisfies the following conditions for all $\omega, \varpi, \rho \in W$.*

$(b1)$ $d(\omega,\varpi) = 0$ *iff* $\omega = \varpi$.

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$(b2)$ $d(\omega,\varpi)=d(\varpi,\omega)$.

$(b3)$ $d(\omega,\varpi)\leq b[d(\omega,\rho)+d(\rho,\varpi)]$.
Then, the pair (W,d) is called a b-metric space.

Definition 2.2. *[5] Let $\theta:[0,\infty)\times[0,\infty)\to[0,\infty)$ be a continuous mapping with respect to each variable. θ is called an $\mathscr{B}$-action iff it satisfies the following conditions:*

$(B1)$ $\theta(0,0)=0$ *and* $\theta(\omega,\varpi)=\theta(\varpi,\omega)$ *for all* $\omega,\varpi\geq 0$,

$(B2)$ $\theta(\omega,\varpi)<\theta(\rho,\nu)$ *if* $\omega<\rho$ *and* $\varpi\leq\nu$ *or* $\omega\leq\rho$ *and* $\varpi<\nu$.

$(B3)$ *For each* $r\in Im(\theta)-\{0\}$ *and for each* $\omega\in(0,r]$, *there exists* $\varpi\in(0,r]$ *such that* $\theta(\omega,\varpi)=r$, *where* $Im(\theta)=\{\theta(\omega,\varpi):\omega>0,\varpi\geq 0\}$.

$(B4)$ $\theta(\omega,0)\leq\omega$ *for all* $\omega>0$.
The set of all $\mathscr{B}$-actions is denoted by $\mathscr{M}$.

Definition 2.3. *[5] Let W be a nonempty set. A mapping $d_\theta:W\times W\to[0,\infty)$ is called a θ-metric on W with respect to $\mathscr{B}$-action $\theta\in\mathscr{M}$ if d_θ satisfies the following conditions:*

$(\theta 1)$ $d_\theta(\omega,\varpi)=0$ *iff* $\omega=\varpi$,

$(\theta 2)$ $d_\theta(\omega,\varpi)=d_\theta(\varpi,\omega)$,

$(\theta 3)$ $d_\theta(\omega,\varpi)\leq\theta(d_\theta(\omega,\rho),d_\theta(\rho,\varpi))$ *for all* $\omega,\varpi,\rho\in W$.
Then, the pair (W,d_θ) is called a θ-metric space.

Definition 2.4. *[8] Let (W,d) be a b-metric space. A b-simulation function is a function $\varsigma:[0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfying the following conditions:*

$(\varsigma 1)$ $\varsigma(\omega,\varpi)<\varpi-\omega$ *for all* $\omega,\varpi>0$.

$(\varsigma 2)$ *If* $\{\omega_n\},\{\varpi_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$0<\lim_{n\to\infty}\omega_n\leq\lim_{n\to\infty}\inf\varpi_n\leq\lim_{n\to\infty}\sup\varpi_n\leq b\lim_{n\to\infty}\omega_n<\infty$$

then

$$\lim_{n\to\infty}\sup\varsigma(b\omega_n,\varpi_n)<0.$$

3. Main results

Definition 3.1. *Let W be a nonempty set and $b\geq 1$ be a given real number. A mapping $d_\theta^b:W\times W\to[0,\infty)$ is called a b-θ-metric on W with respect to $\mathscr{B}$-action $\theta\in\mathscr{M}$ if it satisfies the following properties for each $\omega,\varpi,\rho\in W$.*

$(b\theta 1)$ $d_\theta^b(\omega,\varpi)=0$ *iff* $\omega=\varpi$.

$(b\theta 2)$ $d_\theta^b(\omega,\varpi)=d_\theta^b(\varpi,\omega)$.

$(b\theta 3)$ $d_\theta^b(\omega,\varpi)\leq b\theta(d_\theta^b(\omega,\rho),d_\theta^b(\rho,\varpi))$.
Then, the pair (W,d_θ^b) is called b-θ-metric space.

Remark 3.2. *Every θ-metric space is b-θ-metric space and the concept of b-θ-metric space coincides with the concept of θ-metric space when $b=1$.*

Example 3.3. *Let $W=\{\omega,\varpi,\nu\}$ and $d_\theta^b:W\times W\to[0,\infty)$ be defined by*

$$d_\theta^b(\omega,\varpi)=d_\theta^b(\varpi,\omega)=d_\theta^b(\omega,\nu)=d_\theta^b(\nu,\omega)=1$$
$$d_\theta^b(\varpi,\nu)=d_\theta^b(\nu,\varpi)=2,\ d_\theta^b(\omega,\omega)=d_\theta^b(\varpi,\varpi)=d_\theta^b(\nu,\nu)=0.$$

Suppose that $\theta(u,\rho)=\frac{1}{2}(u+\rho)$. Then, (W,d_θ^b) is b-θ-metric space with $b=2$ but it is not θ-metric space since $d_\theta^b(\varpi,\nu)>\theta(d_\theta^b(\varpi,\omega),d_\theta^b(\omega,\nu))$.

Remark 3.4. *The concept of b-θ-metric space coincides with the concept of b-metric space when $\theta(u,\rho)=u+\rho$. Every b-θ-metric space is b-metric space when $\theta(u,\rho)=k(u+\rho),k\in(0,1]$.*

Example 3.5. *Let $W=\{\omega,\varpi,\nu\}$ and $d_\theta^b:W\times W\to[0,\infty)$ be defined by*

$$d_\theta^b(\nu,\omega)=d_\theta^b(\omega,\nu)=d_\theta^b(\varpi,\nu)=d_\theta^b(\nu,\varpi)=1$$
$$d_\theta^b(\omega,\varpi)=d_\theta^b(\varpi,\omega)=3,\ d_\theta^b(\omega,\omega)=d_\theta^b(\varpi,\varpi)=d_\theta^b(\nu,\nu)=0.$$

Suppose that $\theta(u,\rho)=\frac{u\rho}{1+u\rho}$. Then, (W,d_θ^b) is b-metric space with $b=\frac{3}{2}$ but it is not b-θ-metric space.

Definition 3.6. *Let (W,d_θ^b) be a b-θ-metric space. We define the open ball with center ω and radius $r>0$ by*

$$B_{d_\theta^b}(\omega, r) = \{\varpi \in W : d_\theta^b(\omega, \varpi) < r\}$$

Example 3.7. *$W = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and let $d_\theta^b : W \times W \to [0,\infty)$ be defined by*

$$d_\theta^b(0,1) = d_\theta^b(1,0) = 2,$$

$$d_\theta^b(1, \tfrac{1}{n}) = d_\theta^b(\tfrac{1}{n}, 1) = \tfrac{1}{n} \text{ if } n \geq 2,$$

$$d_\theta^b(0, \tfrac{1}{n}) = d_\theta^b(\tfrac{1}{n}, 0) = 3 \text{ if } n \geq 2,$$

$$d_\theta^b(\tfrac{1}{n}, \tfrac{1}{m}) = d_\theta^b(\tfrac{1}{m}, \tfrac{1}{n}) = \tfrac{1}{n} + \tfrac{1}{m} \text{ if } m,n \geq 2,$$

$$d_\theta^b(m,n) = 0 \text{ iff } m = n.$$

Suppose that $\theta(u,\rho) = u + \rho + u\rho$. Then, (W, d_θ^b) is a b-θ-metric space with $b = 2$. $B_{d_\theta^b}(0,3) = \{0,1\}$ and there is no open ball with center 1 contained in $B_{d_\theta^b}(0,3)$. Thus, $B_{d_\theta^b}(0,3)$ is not open.

Definition 3.8. *Let (W, d_θ^b) be a b-θ-metric space. Then, a sequence $\{\varpi_n\}$ in W is said to be*

1. *convergent iff there exists $\varpi \in W$ such that $d_\theta^b(\varpi_n, \varpi) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} \varpi_n = \varpi$,*
2. *Cauchy iff $d_\theta^b(\varpi_n, \varpi_m) \to 0$ as $n,m \to \infty$.*

Definition 3.9. *The b-θ-metric space (W, d_θ^b) is complete if every Cauchy sequence in W converges to $\varpi \in W$.*

Proposition 3.10. *If (W, d_θ^b) is a b-θ-metric space, then the following hold:*

1. *The limit of a convergent sequence is unique.*
2. *Each convergent sequence is a Cauchy sequence.*

Proof.

1. Suppose that $\lim_{n\to\infty} \varpi_n = \varpi$ and $\lim_{n\to\infty} \varpi_n = \omega$. We claim that $\varpi = \omega$. Since $\lim_{n\to\infty} \varpi_n = \varpi$ and $\lim_{n\to\infty} \varpi_n = \omega$, then $d_\theta^b(\varpi_n, \varpi) \to 0$ and $d_\theta^b(\varpi_n, \omega) \to 0$ as $n \to \infty$. From $(b\theta 3)$, we have
$$0 \leq d_\theta^b(\varpi, \omega) \leq b\theta(d_\theta^b(\varpi_n, \varpi), d_\theta^b(\varpi_n, \omega)).$$
Letting $n \to \infty$ in the above inequality, using the continuity of θ, we get $d_\theta^b(\varpi, \omega) = 0$. Thus, $\varpi = \omega$.
2. It is obvious.

□

Example 3.11. *Let $W = \mathbb{N} \cup \{\infty\}$ and let $d_\theta^b : W \times W \to [0,\infty)$ be defined by*

$$d_\theta^b(\varpi, \omega) = \begin{cases} 5 & \text{if } \varpi, \omega \in \mathbb{N} (\varpi \neq \omega), \\ 2 & \text{if one of } \varpi, \omega \in \mathbb{N} \text{ and the other is } \infty, \\ 0 & \text{if } \varpi = \omega. \end{cases}$$

Suppose that $\theta(u,\rho) = \sqrt{u^2 + \rho^2}$. Then, (W, d_θ^b) is a b-θ-metric space with $b = 2$. Let $\varpi_n = 5n$ for each $n \in \mathbb{N}$. Then, $d_\theta^b(5n, 2) \to 5$ as $n \to \infty$. But $d_\theta^b(\infty, 2) \to 2$ since $\varpi_n \to \infty$. Thus, it is not continuous.

4. Fixed point results

Let $W \neq \emptyset$ and T be a self mapping on W. Let $\varpi_0 \in W$ and $\varpi_n = T\varpi_{n-1}$ for all $n \in \mathbb{N}$. Then, $\{\varpi_n\}$ is called a Picard sequence of initial point at ϖ_0 and $Fix(T) = \{\varpi \in W : \varpi = T\varpi\}$ is the set of fixed points of T.

Theorem 4.1. *Let (W, d_θ^b) be a complete b-θ-metric space and let $T : W \to W$ be a mapping. Suppose that there exists a b-simulation function ς such that*

$$\varsigma(b d_\theta^b(T\varpi, T\rho), d_\theta^b(\varpi, \rho)) \geq 0 \text{ for all } \varpi, \rho \in W.$$

Then, T has a unique fixed point.

Proof. Let $\{\varpi_n\}$ be a sequence of Picard with initial point $\varpi_0 \in W$. Suppose that $\varpi_n \neq \varpi_{n-1}$ for all $n \in \mathbb{N}$. We prove this theorem in 4 cases.

Case 1: We claim that $\lim_{n\to\infty} d_\theta^b(\varpi_{n-1},\varpi_n)=0$.
By the hypotheses and using $(\varsigma 1)$, respectively, we have

$$\begin{array}{rcl} 0 & \leq & \varsigma(bd_\theta^b(\varpi_n,\varpi_{n+1}),d_\theta^b(\varpi_{n-1},\varpi_n)) \\ & < & d_\theta^b(\varpi_{n-1},\varpi_n)-bd_\theta^b(\varpi_n,\varpi_{n+1}). \end{array}$$

Thus, for all $n\in\mathbb{N}$, we get

$$bd_\theta^b(\varpi_n,\varpi_{n+1})<d_\theta^b(\varpi_{n-1},\varpi_n).$$

That is, $\{d_\theta^b(\varpi_{n-1},\varpi_n)\}$ is a decreasing sequence of positive real numbers. Hence, there exists $r\geq 0$ such that $\lim_{n\to\infty} d_\theta^b(\varpi_{n-1},\varpi_n)=r$. Assume $r>0$. From $(\varsigma 2)$ for $\nu_n=d_\theta^b(\varpi_n,\varpi_{n+1}),\omega_n=d_\theta^b(\varpi_{n-1},\varpi_n)$, we obtain

$$0 \quad \leq \lim_{n\to\infty}\sup\varsigma(bd_\theta^b(\varpi_n,\varpi_{n+1}),d_\theta^b(\varpi_{n-1},\varpi_n))<0.$$

This is a contradiction. Thus, $r=0$. That is $\lim_{n\to\infty} d_\theta^b(\varpi_{n-1},\varpi_n)=0$.

Case 2: Our aim is to show that $\{\varpi_n\}$ is a bounded sequence.
Suppose that ϖ_n is not a bounded sequence. Then, there exists a subsequence $\{\varpi_{n(k)}\}$ of $\{\varpi_n\}$ such that $n(1)=1$ and $n(k+1)$ is the minimum integer for each $k\in\mathbb{N}$ such that

$$d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})>1 \text{ and } d_\theta^b(\varpi_m,\varpi_{n(k)})\leq 1 \text{ for } n(k)\leq m\leq n(k+1)-1.$$

Thus, by using $(b\theta 3)$, we have

$$\begin{array}{rcl} 1<d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)}) & \leq & b\theta(d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k+1)-1}),d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)})) \\ & \leq & b\theta(d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k+1)-1}),1). \end{array}$$

By taking the limit from two sides of above inequality, we get

$$1<\lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})\leq b.$$

From Case 1 and $(b\theta 3)$, we have

$$\begin{array}{rcl} bd_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)}) & \leq & d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1}) \\ & \leq & b\theta(d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)}),d_\theta^b(\varpi_{n(k)},\varpi_{n(k)-1})) \\ & \leq & b\theta(1,d_\theta^b(\varpi_{n(k)},\varpi_{n(k)-1})). \end{array}$$

Again by taking the limit from two sides of above inequality, we obtain

$$b<\lim_{k\to\infty} bd_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})\leq \lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1})\leq b$$

Thus,

$$\lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1})=b \text{ and } \lim_{k\to\infty} d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})=1.$$

Now, by $(\varsigma 2)$, with $\nu_k=d_\theta^b(\varpi_{n(k+1)},\varpi_{n(k)})$ and $\omega_k=d_\theta^b(\varpi_{n(k+1)-1},\varpi_{n(k)-1})$, we get

$$0\leq \lim_{k\to\infty}\varsigma(b\nu_k,\omega_k)<0.$$

This is a contradiction. Hence, $\{\varpi_n\}$ is a bounded sequence.

Case 3: We will show that $\{\varpi_n\}$ is a Cauchy sequence.
Let $M_n=\sup\{d_\theta^b(\varpi_i,\varpi_j): i,j\geq n \text{ and } n\in\mathbb{N}\}$. From Case 2, for each $n\in\mathbb{N}$, $M_n<\infty$. Here, M_n is a positive decreasing sequence. So, there exists $M\geq 0$ such that $\lim_{n\to\infty} M_n=M$.
Assume that $M>0$. For $k\in\mathbb{N}$, there exist $n(k),m(k)\in\mathbb{N}$ such that $m(k)>n(k)\geq k$ and

$$M_k-\frac{1}{k}<d_\theta^b(\varpi_{m(k)},\varpi_{n(k)})\leq M_k.$$

After taking the limit in the above inequality, we have

$$\lim_{k\to\infty} d_\theta^b(\varpi_{m(k)},\varpi_{n(k)})=M.$$

From Case 1 and the definition of M_n, we obtain

$$bd_\theta^b(\varpi_{m(k)},\varpi_{n(k)}) \le d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1}) \le M_{k-1}.$$

Again, by taking the limit, we find

$$bM \le \lim_{k\to\infty}\inf d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1}) \le \lim_{k\to\infty}\sup d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1}) \le M.$$

If $b>1$, then $M=0$. If $b=1$, from $(\varsigma 2)$ with $\nu_k = d_\theta^b(\varpi_{m(k)},\varpi_{n(k)})$ and $\omega_k = d_\theta^b(\varpi_{m(k)-1},\varpi_{n(k)-1})$, we obtain

$$0 \le \lim_{k\to\infty}\sup \varsigma(b\nu_k,\omega_k) < 0.$$

This is a contradiction. Thus, $M=0$. This implies that $\{\varpi_n\}$ is a Cauchy sequence.

Case 4: Since (W,d_θ^b) is a complete b-θ-metric space and $\{\varpi_n\}$ is a Cauchy sequence from Case 3, there exists $\rho\in W$ such that $\lim_{n\to\infty}\varpi_n=\rho$. We must show that $\rho\in Fix(T)$. From Case 1,

$$bd_\theta^b(T\varpi_n,T\rho) \le d_\theta^b(\varpi_n,\rho) \text{ for all } n\in\mathbb{N}.$$

Thus,

$$\begin{aligned}0 \le d_\theta^b(\rho,T\rho) &\le b\theta(d_\theta^b(\rho,\varpi_{n+1}),d_\theta^b(\varpi_{n+1},T\rho)) \\ &< b\theta(d_\theta^b(\rho,\varpi_{n+1}),\frac{1}{b}d_\theta^b(\varpi_n,\rho)).\end{aligned}$$

By taking the limit from two sides of above inequality, we get $d_\theta^b(\rho,T\rho)=0$ since $\lim_{n\to\infty}\varpi_n=\rho$. Therefore, $\rho=T\rho$.

Finally, we must show that the uniqueness of fixed point. Assume that there exists $w\in W$ such that $w=Tw$ and $w\neq\rho$. By Case 1, we get

$$0 \le bd_\theta^b(Tw,T\rho) \le d_\theta^b(w,\rho).$$

This implies that $b\le 1$. This is a contradiction with our assumption. Hence, T has a unique fixed point. □

Corollary 4.2. *Let (W,d_θ^b) be a complete b-θ-metric space and $T:W\to W$ be a mapping satisfies the following inequality*

$$bd_\theta^b(T\omega,T\varpi) \le \alpha d_\theta^b(\omega,\varpi)$$

for each $\omega,\varpi\in W$, where $\alpha\in[0,1)$. Then, T has a unique fixed point.

Proof. It follows from Theorem 4.1 if we take b-simulation function as $\varsigma(\nu,\rho)=\alpha\rho-\nu$ for all $\nu,\rho\ge 0$. □

Remark 4.3. *Let (W,d_θ^b) be a complete b-θ-metric space.*

1. *Theorem 3.4 in [8] is obtained from Theorem 4.1 by taking $\theta(\nu,\rho)=\nu+\rho$.*
2. *Theorem 3.3 in [7] is obtained from Theorem 4.1 by taking $b=1$.*

Now, we illustrate the validity of fixed point result in Theorem 4.1 by the following examples.

Example 4.4. *Let $W=[0,\infty)$ and $d_\theta^b: W\times W\to[0,\infty)$ be defined by $d_\theta^b(\omega,\varpi))=|\omega-\varpi|^3$. Also, we take $\theta(\nu,\rho)=\nu+\rho+\nu\rho$. Then, (W,d_θ^b) is a complete b-θ-metric space with $b=4$. Define a mapping $T:W\to W$ by $T\omega=\frac{\omega}{a}$ for all $\omega\in W$ and $a>0, a\neq 1$. From Theorem 4.1, T has a unique fixed point $u=0$ for b-simulation function $\varsigma(\nu,\rho)=\lambda\rho-\nu$ where $\lambda\ge\frac{4}{a^3}$ for all $\nu,\rho\in[0,\infty)$, since*

$$\begin{aligned}\varsigma(4d_\theta^b(T\omega,T\varpi),d_\theta^b(\omega,\varpi)) &= \lambda d_\theta^b(\omega,\varpi)-4d_\theta^b(T\omega,T\varpi) \\ &= \lambda(|\omega-\varpi|^3)-4(|T\omega-T\varpi|^3) \\ &= \lambda(|\omega-\varpi|^3)-4(|\frac{\omega}{a}-\frac{\varpi}{a}|^3) \\ &= (\lambda-\frac{4}{a^3})(|\omega-\varpi|^3) \\ &\ge 0.\end{aligned}$$

Example 4.5. *Let $W=[0,1]$ and $d_\theta^b: W\times W\to[0,\infty)$ be defined by $d_\theta^b(\omega,\varpi)=|\ \omega-\varpi\ |^2$. Also, we take $\theta(\nu,\rho)=\sqrt{\nu^2+\rho^2}$. Then, (W,d_θ^b) is a complete b-θ-metric space with $b=2\sqrt{2}$. Define a mapping $T:[0,1]\to[0,1]$ by $T\omega=\frac{\omega}{\sqrt{2}}+a$ for all $\omega\in W$ and $a<\frac{\sqrt{2}-1}{\sqrt{2}}$. From Theorem 4.1, T has a unique fixed point $u=\frac{\sqrt{2}a}{\sqrt{2}-1}$ for b-simulation function $\varsigma(\nu,\rho)=\lambda\rho-\nu$ where $\lambda\geq\sqrt{2}$ for all $\nu,\rho\in[0,\infty)$, since*

$$\begin{aligned}
\varsigma(2\sqrt{2}d_\theta^b(T\omega,T\varpi),d_\theta^b(\omega,\varpi)) &= \lambda d_\theta^b(\omega,\varpi)-2\sqrt{2}d_\theta^b(T\omega,T\varpi)\\
&= \lambda(|\ \omega-\varpi\ |^2)-2\sqrt{2}(|\ T\omega-T\varpi\ |^2)\\
&= \lambda(|\ \omega-\varpi\ |^2)-2\sqrt{2}(|\ \frac{\omega}{\sqrt{2}}-\frac{\varpi}{\sqrt{2}}\ |^2)\\
&= (\lambda-\frac{2\sqrt{2}}{2})(|\ \omega-\varpi\ |^2)\\
&\geq 0.
\end{aligned}$$

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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