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ON CERTAIN BIHYPERNOMIALS RELATED TO PELL AND PELL-LUCAS NUMBERS

Anetta SZYNAL-LIANA[1], Iwona WŁOCH[1], and Mirosław LIANA[2]

[1]Rzeszow University of Technology, The Faculty of Mathematics and Applied Physics, al. Powstańców Warszawy 12, 35-959 Rzeszów, POLAND
[2]Rzeszow University of Technology, The Faculty of Management, al. Powstańców Warszawy 10, 35-959 Rzeszów, POLAND

Abstract. The bihyperbolic numbers are extension of hyperbolic numbers to four dimensions. In this paper we introduce the concept of Pell and Pell-Lucas bihypernomials as a generalization of bihyperbolic Pell and Pell-Lucas numbers, respectively.

1. Introduction

Let consider Pell and Pell-Lucas numbers which belong to the family of the Fibonacci type numbers, for details see [14]. We recall that the nth Pell number P_n is defined by $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$. The nth Pell-Lucas number Q_n is defined by $Q_n = 2Q_{n-1} + Q_{n-2}$ for $n \geq 2$ with $Q_0 = Q_1 = 2$.

For the nth Pell number and the nth Pell-Lucas number the explicit formulas have the form

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$$

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

The above equations are named as Binet type formulas for Pell and Pell-Lucas numbers, respectively. For other properties of P_n and Q_n see [5,6,9]. In [7] Horadam and Mahon introduced Pell and Pell-Lucas polynomials and next their properties were studied among others in [4].

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Corresponding author: aszynal@prz.edu.pl; 0000-0001-5508-0640
iwloch@prz.edu.pl; 0000-0002-9969-0827
mliana@prz.edu.pl; 0000-0001-5801-1755

Let x be any variable quantity. Polynomials $P_n(x)$ and $Q_n(x)$ defined as follows

$$P_n(x) = 2x \cdot P_{n-1}(x) + P_{n-2}(x) \text{ for } n \geq 2 \text{ with } P_0(x) = 0, P_1(x) = 1$$

$$Q_n(x) = 2x \cdot Q_{n-1}(x) + Q_{n-2}(x) \text{ for } n \geq 2 \text{ with } Q_0(x) = 2, Q_1(x) = 2x$$

generalize Pell and Pell-Lucas numbers and they are called as Pell polynomials and Pell-Lucas polynomials, respectively. Clearly $P_n(1) = P_n$ and $Q_n(1) = Q_n$.

Let

$$\alpha(x) = x + \sqrt{x^2+1}, \quad \beta(x) = x - \sqrt{x^2+1} \tag{1}$$

be roots of the characteristic equation for the Pell and Pell-Lucas polynomials. Then solving this equation we have

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{2}$$

and

$$Q_n(x) = \alpha^n(x) + \beta^n(x), \tag{3}$$

respectively.

We recall selected identities for Pell and Pell-Lucas polynomials, which will be used in the next part of this paper.

Theorem 1. [7] *Let n be an integer. Then*

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x) = 2x \cdot P_n(x) + 2P_{n-1}(x), \text{ for } n \geq 1, \tag{4}$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2+1)P_n(x), \text{ for } n \geq 1, \tag{5}$$

$$\sum_{l=1}^{n-1} P_l(x) = \frac{P_n(x) + P_{n-1}(x) - 1}{2x}, \text{ for } n \geq 2, \tag{6}$$

$$\sum_{l=1}^{n-1} Q_l(x) = \frac{Q_n(x) + Q_{n-1}(x) - 2 - 2x}{2x}, \text{ for } n \geq 2. \tag{7}$$

For Pell numbers and Pell polynomials we can find different generalizations given by the kth order linear recurrence relations, $k \geq 2$. One of the fundamental generalization of Pell polynomials is Horadam polynomials which describe a wide family of polynomials defined by linear recurrence relations of order two. Some properties of the Horadam polynomials can be found in [8]. Horadam polynomials play an important role in the theory of hypercomplex numbers, for details see [12–14]. In this paper we will use Pell and Pell-Lucas polynomials in the theory of bihyperbolic numbers.

Let $\mathbb{H}_2$ be the set of bihyperbolic numbers ζ of the form

$$\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1, j_2, j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1, \; j_1j_2 = j_2j_1 = j_3, \; j_1j_3 = j_3j_1 = j_2, \; j_2j_3 = j_3j_2 = j_1. \tag{8}$$

From the definition of bihyperbolic numbers follows that their multiplication can be made analogously to the multiplication of algebraic expressions. The addition and the subtraction of bihyperbolic numbers is done by adding and subtracting corresponding terms and hence their coefficients.

Since the addition and multiplication on $\mathbb{H}_2$ are commutative and associative, so $(\mathbb{H}_2, +, \cdot)$ is a commutative ring.

Note that bihyperbolic numbers are a generalization of hyperbolic numbers. For the definition of hyperbolic numbers and their properties see [10, 11]. For the algebraic properties of bihyperbolic numbers see [1].

A special kind of bihyperbolic numbers, namely bihyperbolic Pell numbers, were introduced in [2] in the following way.

The nth bihyperbolic Pell number BhP_n is defined as

$$BhP_n = P_n + j_1 P_{n+1} + j_2 P_{n+2} + j_3 P_{n+3}. \tag{9}$$

By analogy

$$BhQ_n = Q_n + j_1 Q_{n+1} + j_2 Q_{n+2} + j_3 Q_{n+3} \tag{10}$$

is the nth bihyperbolic Pell-Lucas number. Note that some combinatorial properties of bihyperbolic Pell numbers we can find in [3].

Based on definitions of BhP_n and BhQ_n we introduce Pell and Pell-Lucas bihypernomials.

For $n \geq 0$ Pell and Pell-Lucas bihypernomials are defined by

$$BhP_n(x) = P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x) \tag{11}$$

and

$$BhQ_n(x) = Q_n(x) + j_1 Q_{n+1}(x) + j_2 Q_{n+2}(x) + j_3 Q_{n+3}(x), \tag{12}$$

respectively. Note that $BhP_n(1) = BhP_n$ and $BhQ_n(1) = BhQ_n$.

2. Main results

In this section we will give some identities for Pell bihypernomials and Pell-Lucas bihypernomials.

Theorem 2. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$BhP_n(x) = 2x \cdot BhP_{n-1}(x) + BhP_{n-2}(x) \text{ for } n \geq 2 \tag{13}$$

with $BhP_0(x) = j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2 + 1)$
and $BhP_1(x) = 1 + j_1 \cdot 2x + j_2 \cdot (4x^2 + 1) + j_3 \cdot (8x^3 + 4x)$.

Proof. If $n = 2$ we have

$$\begin{aligned} BhP_2(x) &= P_2(x) + j_1P_3(x) + j_2P_4(x) + j_3P_5(x) \\ &= 2x + j_1 \cdot (4x^2 + 1) + j_2 \cdot (8x^3 + 4x) + j_3 \cdot (16x^4 + 12x^2 + 1) \\ &= 2x \cdot (1 + j_1 \cdot 2x + j_2 \cdot (4x^2 + 1) + j_3 \cdot (8x^3 + 4x)) \\ &\quad + j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2 + 1) \\ &= 2x \cdot BhP_1(x) + BhP_0(x). \end{aligned}$$

Let $n \geq 3$. By the definition of $P_n(x)$ we obtain

$$\begin{aligned} BhP_n(x) &= P_n(x) + j_1P_{n+1}(x) + j_2P_{n+2}(x) + j_3P_{n+3}(x) \\ &= (2x \cdot P_{n-1}(x) + P_{n-2}(x)) + j_1(2x \cdot P_n(x) + P_{n-1}(x)) \\ &\quad + j_2(2x \cdot P_{n+1}(x) + P_n(x)) + j_3(2x \cdot P_{n+2}(x) + P_{n+1}(x)) \\ &= 2x\,(P_{n-1}(x) + j_1P_n(x) + j_2P_{n+1}(x) + j_3P_{n+2}(x)) \\ &\quad + P_{n-2}(x) + j_1P_{n-1}(x) + j_2P_n(x) + j_3P_{n+1}(x) \\ &= 2x \cdot BhP_{n-1}(x) + BhP_{n-2}(x), \end{aligned}$$

which ends the proof. □

Using the same method we can prove the next result.

Theorem 3. *Let $n \geq 0$ be an integer. For any variable quantity x, we have*

$$BhQ_n(x) = 2x \cdot BhQ_{n-1}(x) + BhQ_{n-2}(x) \text{ for } n \geq 2$$

with $BhQ_0(x) = 2 + j_1 \cdot 2x + j_2 \cdot (4x^2 + 2) + j_3 \cdot (8x^3 + 6x)$
and $BhQ_1(x) = 2x + j_1 \cdot (4x^2 + 2) + j_2 \cdot (8x^3 + 6x) + j_3 \cdot (16x^4 + 16x^2 + 2)$.

Note that some identities for $BhP_n(x)$ and $BhQ_n(x)$ can be found based on identities for Pell and Pell-Lucas polynomials mentioned in the introduction of this paper.

Theorem 4. *Let $n \geq 1$ be an integer. Then*

$$BhP_{n+1}(x) + BhP_{n-1}(x) = BhQ_n(x) = 2x \cdot BhP_n(x) + 2BhP_{n-1}(x).$$

Proof. Using (4) we have

$$\begin{aligned} &BhP_{n+1}(x) + BhP_{n-1}(x) \\ &= P_{n+1}(x) + j_1P_{n+2}(x) + j_2P_{n+3}(x) + j_3P_{n+4}(x) \\ &\quad + P_{n-1}(x) + j_1P_n(x) + j_2P_{n+1}(x) + j_3P_{n+2}(x) \\ &= (P_{n+1}(x) + P_{n-1}(x)) + j_1(P_{n+2}(x) + P_n(x)) \\ &\quad + j_2(P_{n+3}(x) + P_{n+1}(x)) + j_3(P_{n+4}(x) + P_{n+2}(x)) \\ &= Q_n(x) + j_1Q_{n+1}(x) + j_2Q_{n+2}(x) + j_3Q_{n+3}(x) \\ &= BhQ_n(x). \end{aligned}$$

On the other hand

$$\begin{aligned}
&2x \cdot BhP_n(x) + 2BhP_{n-1}(x)\\
&= 2x \cdot (P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x))\\
&\ + 2\,(P_{n-1}(x) + j_1 P_n(x) + j_2 P_{n+1}(x) + j_3 P_{n+2}(x))\\
&= (2x \cdot P_n(x) + 2P_{n-1}(x)) + j_1(2x \cdot P_{n+1}(x) + 2P_n(x))\\
&\ + j_2(2x \cdot P_{n+2}(x) + 2P_{n+1}(x)) + j_3(2x \cdot P_{n+3}(x) + 2P_{n+2}(x))\\
&= Q_n(x) + j_1 Q_{n+1}(x) + j_2 Q_{n+2}(x) + j_3 Q_{n+3}(x)\\
&= BhQ_n(x).
\end{aligned}$$

□

Theorem 5. *Let $n \geq 1$ be an integer. Then*

$$BhQ_{n+1}(x) + BhQ_{n-1}(x) = 4(x^2 + 1)BhP_n(x).$$

Theorem 6. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} BhP_l(x) = \frac{BhP_n(x) + BhP_{n-1}(x) - BhP_0(x) - BhP_1(x)}{2x}.$$

Proof. For an integer $n \geq 2$ we have

$$\begin{aligned}
&\sum_{l=1}^{n-1} BhP_l(x) = BhP_1(x) + BhP_2(x) + \cdots + BhP_{n-1}(x)\\
&= P_1(x) + j_1 P_2(x) + j_2 P_3(x) + j_3 P_4(x)\\
&\ + P_2(x) + j_1 P_3(x) + j_2 P_4(x) + j_3 P_5(x) + \cdots\\
&\ + P_{n-1}(x) + j_1 P_n(x) + j_2 P_{n+1}(x) + j_3 P_{n+2}(x)\\
&= P_1(x) + P_2(x) + \cdots + P_{n-1}(x)\\
&\ + j_1(P_2(x) + P_3(x) + \cdots + P_n(x) + P_1(x) - P_1(x))\\
&\ + j_2(P_3(x) + P_4(x) + \cdots + P_{n+1}(x) + P_1(x) + P_2(x) - P_1(x) - P_2(x))\\
&\ + j_3(P_4(x) + P_5(x) + \cdots + P_{n+2}(x) + P_1(x) + P_2(x) + P_3(x)\\
&\ - P_1(x) - P_2(x) - P_3(x)).
\end{aligned}$$

Using (6) we obtain

$$\begin{aligned}
\sum_{l=1}^{n-1} BhP_l(x) &= \frac{P_n(x)+P_{n-1}(x)-1}{2x} \\
&+ j_1\left(\frac{P_{n+1}(x)+P_n(x)-1}{2x} - P_1(x)\right) \\
&+ j_2\left(\frac{P_{n+2}(x)+P_{n+1}(x)-1}{2x} - P_1(x) - P_2(x)\right) \\
&+ j_3\left(\frac{P_{n+3}(x)+P_{n+2}(x)-1}{2x} - P_1(x) - P_2(x) - P_3(x)\right) \\
&= \frac{P_n(x)+P_{n-1}(x)-1}{2x} \\
&+ j_1\frac{P_{n+1}(x)+P_n(x)-1-2x}{2x} \\
&+ j_2\frac{P_{n+2}(x)+P_{n+1}(x)-1-2x-4x^2}{2x} \\
&+ j_3\frac{P_{n+3}(x)+P_{n+2}(x)-1-2x-4x^2-2x(4x^2+1)}{2x} \\
&= \frac{P_n(x)+j_1P_{n+1}(x)+j_2P_{n+2}(x)+j_3P_{n+3}(x)}{2x} \\
&+ \frac{P_{n-1}(x)+j_1P_n(x)+j_2P_{n+1}(x)+j_3P_{n+2}(x)}{2x} \\
&+ \frac{-(0+1)-j_1(1+2x)-j_2(2x+(4x^2+1))-j_3((4x^2+1)+(8x^3+4x))}{2x} \\
&= \frac{BhP_n(x)+BhP_{n-1}(x)-BhP_0(x)-BhP_1(x)}{2x}.
\end{aligned}$$

Thus the Theorem is proved. □

Theorem 7. *Let $n \geq 2$ be an integer. Then*

$$\sum_{l=1}^{n-1} BhQ_l(x) = \frac{BhQ_n(x)+BhQ_{n-1}(x)-BhQ_0(x)-BhQ_1(x)}{2x}.$$

Now we give Binet type formulas for Pell and Pell-Lucas bihypernomials.

Theorem 8. *(Binet type formula for Pell bihypernomials) Let $n \geq 0$ be an integer. Then*

$$\begin{aligned}
BhP_n(x) &= \frac{\alpha^n(x)}{\alpha(x)-\beta(x)}\left(1+j_1\alpha(x)+j_2\alpha^2(x)+j_3\alpha^3(x)\right) \\
&- \frac{\beta^n(x)}{\alpha(x)-\beta(x)}\left(1+j_1\beta(x)+j_2\beta^2(x)+j_3\beta^3(x)\right),
\end{aligned} \tag{14}$$

where $\alpha(x)$, $\beta(x)$ *are given by* (1).

Proof. Using (2) and (11) we obtain

$$\begin{aligned} BhP_n(x) &= P_n(x) + j_1 P_{n+1}(x) + j_2 P_{n+2}(x) + j_3 P_{n+3}(x) \\ &= \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} + j_1 \frac{\alpha^{n+1}(x) - \beta^{n+1}(x)}{\alpha(x) - \beta(x)} \\ &\quad + j_2 \frac{\alpha^{n+2}(x) - \beta^{n+2}(x)}{\alpha(x) - \beta(x)} + j_3 \frac{\alpha^{n+3}(x) - \beta^{n+3}(x)}{\alpha(x) - \beta(x)} \end{aligned}$$

and by simple calculations the result follows. □

In the same way, using (3) and (12), we obtain the next theorem.

Theorem 9. *(Binet type formula for Pell-Lucas bihypernomials) Let* $n \geq 0$ *be an integer. Then*

$$\begin{aligned} BhQ_n(x) &= \alpha^n(x) \left(1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x)\right) \\ &\quad + \beta^n(x) \left(1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x)\right), \end{aligned} \tag{15}$$

where $\alpha(x)$, $\beta(x)$ *are given by* (1).

Using Binet type formulas for Pell and Pell-Lucas bihypernomials we can obtain Catalan type identity, Cassini type identity and d'Ocagne type identity for Pell and Pell-Lucas bihypernomials.

For simplicity of notation let

$$\begin{aligned} \hat{\alpha}(x) &= 1 + j_1\alpha(x) + j_2\alpha^2(x) + j_3\alpha^3(x), \\ \hat{\beta}(x) &= 1 + j_1\beta(x) + j_2\beta^2(x) + j_3\beta^3(x). \end{aligned}$$

Consequently we can write (14) and (15) as

$$BhP_n(x) = \frac{\alpha^n(x)\hat{\alpha}(x) - \beta^n(x)\hat{\beta}(x)}{\alpha(x) - \beta(x)} \tag{16}$$

and

$$BhQ_n(x) = \alpha^n(x)\hat{\alpha}(x) + \beta^n(x)\hat{\beta}(x). \tag{17}$$

Moreover,

$$\begin{aligned} \alpha(x) \cdot \beta(x) &= -1, \\ \alpha(x) + \beta(x) &= 2x, \\ \alpha(x) - \beta(x) &= 2\sqrt{x^2 + 1}, \\ \alpha^3(x) + \beta^3(x) &= (\alpha(x) + \beta(x))^3 - 3\alpha(x)\beta(x)(\alpha(x) + \beta(x)) = 8x^3 + 6x, \end{aligned}$$

and

$$\begin{aligned}
&\hat{\alpha}(x)\cdot\hat{\beta}(x)=\hat{\beta}(x)\cdot\hat{\alpha}(x)\\
&=1+j_1\beta(x)+j_2\beta^2(x)+j_3\beta^3(x)+j_1\alpha(x)-1-j_3\beta(x)-j_2\beta^2(x)\\
&\ +j_2\alpha^2(x)-j_3\alpha(x)+1+j_1\beta(x)+j_3\alpha^3(x)-j_2\alpha^2(x)+j_1\alpha(x)-1\\
&=j_1\left(2\alpha(x)+2\beta(x)\right)+j_3\left(\alpha^3(x)+\beta^3(x)-\alpha(x)-\beta(x)\right)\\
&=j_1\cdot 4x+j_3\left(8x^3+4x\right).
\end{aligned}$$

Theorem 10. *(Catalan type identity for Pell bihypernomials) Let $n\geq 0$, $r\geq 0$ be integers such that $n\geq r$. Then*

$$\begin{aligned}
&BhP_{n-r}(x)\cdot BhP_{n+r}(x)-\left(BhP_n(x)\right)^2\\
&=\frac{(-1)^{n-r+1}((x+\sqrt{x^2+1})^r-(x-\sqrt{x^2+1})^r)^2}{4x^2+4}\left(j_1\cdot 4x+j_3\left(8x^3+4x\right)\right).
\end{aligned}$$

Proof. By formula (16) we have

$$\begin{aligned}
&BhP_{n-r}(x)\cdot BhP_{n+r}(x)-\left(BhP_n(x)\right)^2\\
&=-\frac{\alpha^{n-r}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\frac{\beta^{n+r}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)-\frac{\beta^{n-r}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)\frac{\alpha^{n+r}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\\
&+\frac{\alpha^{n}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\frac{\beta^{n}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)+\frac{\beta^{n}(x)}{\alpha(x)-\beta(x)}\hat{\beta}(x)\frac{\alpha^{n}(x)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\\
&=\frac{\alpha^n(x)\beta^n(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)\frac{2\alpha^r(x)\beta^r(x)-(\beta^r(x))^2-(\alpha^r(x))^2}{(\alpha(x)\beta(x))^r}\\
&=\frac{(\alpha(x)\beta(x))^n}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x)(-1)\frac{(\alpha^r(x)-\beta^r(x))^2}{(\alpha(x)\beta(x))^r}\\
&=\frac{(-1)^{n-r+1}(\alpha^r(x)-\beta^r(x))^2}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x),
\end{aligned}$$

so the result follows. □

Theorem 11. *(Catalan type identity for Pell-Lucas bihypernomials) Let $n\geq 0$, $r\geq 0$ be integers such that $n\geq r$. Then*

$$\begin{aligned}
&BhQ_{n-r}(x)\cdot BhQ_{n+r}(x)-\left(BhQ_n(x)\right)^2\\
&=(-1)^{n-r}(\alpha^r(x)-\beta^r(x))^2\cdot\hat{\alpha}(x)\hat{\beta}(x)\\
&=(-1)^{n-r}((x+\sqrt{x^2+1})^r-(x-\sqrt{x^2+1})^r)^2\left(j_1\cdot 4x+j_3\left(8x^3+4x\right)\right).
\end{aligned}$$

Note that for $r=1$ we get the Cassini type identities for Pell and Pell-Lucas bihypernomials.

Corollary 1. *(Cassini type identity for Pell bihypernomials) Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} &BhP_{n-1}(x) \cdot BhP_{n+1}(x) - (BhP_n(x))^2 \\ &= (-1)^n \left(j_1 \cdot 4x + j_3 \left(8x^3 + 4x\right)\right). \end{aligned}$$

Corollary 2. *(Cassini type identity for Pell-Lucas bihypernomials) Let $n \geq 1$ be an integer. Then*

$$\begin{aligned} &BhQ_{n-1}(x) \cdot BhQ_{n+1}(x) - (BhQ_n(x))^2 \\ &= (-1)^{n-1}(4x^2+4) \left(j_1 \cdot 4x + j_3 \left(8x^3 + 4x\right)\right). \end{aligned}$$

Theorem 12. *(d'Ocagne type identity for Pell bihypernomials) Let $m \geq 0$, $n \geq 0$ be integers such that $m \geq n$. Then*

$$\begin{aligned} &BhP_m(x) \cdot BhP_{n+1}(x) - BhP_{m+1}(x) \cdot BhP_n(x) \\ &= \frac{(-1)^n \left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x). \end{aligned}$$

Proof. By formula (16) we have

$$\begin{aligned} &BhP_m(x) \cdot BhP_{n+1}(x) - BhP_{m+1}(x) \cdot BhP_n(x) \\ &= \frac{\alpha^{m+n+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}^2(x) - \frac{\alpha^m(x)\beta^{n+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x) \\ &- \frac{\alpha^{n+1}(x)\beta^m(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}(x)\hat{\alpha}(x) + \frac{\beta^{m+n+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}^2(x) \\ &- \frac{\alpha^{m+1+n}(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}^2(x) + \frac{\alpha^{m+1}(x)\beta^n(x)}{(\alpha(x)-\beta(x))^2}\hat{\alpha}(x)\hat{\beta}(x) \\ &+ \frac{\alpha^n(x)\beta^{m+1}(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^{m+1+n}(x)}{(\alpha(x)-\beta(x))^2}\hat{\beta}^2(x) \\ &= \left(\frac{\alpha^m(x)\beta^n(x)(\alpha(x)-\beta(x))}{(\alpha(x)-\beta(x))^2} - \frac{\alpha^n(x)\beta^m(x)(\alpha(x)-\beta(x))}{(\alpha(x)-\beta(x))^2}\right)\hat{\alpha}(x)\hat{\beta}(x) \\ &= \frac{(\alpha(x)\beta(x))^n}{\alpha(x)-\beta(x)}\left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)\hat{\alpha}(x)\hat{\beta}(x) \\ &= \frac{(-1)^n \left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right)}{\alpha(x)-\beta(x)}\hat{\alpha}(x)\hat{\beta}(x), \end{aligned}$$

so the result follows. □

Theorem 13. *(d'Ocagne type identity for Pell-Lucas bihypernomials) Let* $m \geq 0$, $n \geq 0$ *be integers such that* $m \geq n$. *Then*

$$\begin{aligned} &BhQ_m(x) \cdot BhQ_{n+1}(x) - BhQ_{m+1}(x) \cdot BhQ_n(x) \\ &= (-1)^{n+1}(\alpha(x) - \beta(x)) \left(\alpha^{m-n}(x) - \beta^{m-n}(x)\right) \hat{\alpha}(x)\hat{\beta}(x). \end{aligned}$$

Theorem 14. *Let* $m \geq 0$, $n \geq 0$ *be integers. Then*

$$\begin{aligned} &BhP_m(x) \cdot BhQ_n(x) - BhQ_m(x) \cdot BhP_n(x) \\ &= \frac{2(-1)^n(\alpha^{m-n}(x) - \beta^{m-n}(x))}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x). \end{aligned}$$

Proof. By formulas (16) and (17) we have

$$\begin{aligned} &BhP_m(x) \cdot BhQ_n(x) - BhQ_m(x) \cdot BhP_n(x) \\ &= \frac{\alpha^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}^2(x) + \frac{\alpha^m(x)\beta^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x) \\ &- \frac{\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\beta}(x)\hat{\alpha}(x) - \frac{\beta^m(x)\beta^n(x)}{\alpha(x) - \beta(x)} \hat{\beta}^2(x) \\ &- \frac{\alpha^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}^2(x) + \frac{\alpha^m(x)\beta^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x) \\ &- \frac{\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\beta}(x)\hat{\alpha}(x) + \frac{\beta^n(x)\beta^m(x)}{\alpha(x) - \beta(x)} \hat{\beta}^2(x) \\ &= \frac{2\alpha^m(x)\beta^n(x) - 2\beta^m(x)\alpha^n(x)}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x) \\ &= \frac{2(-1)^n(\alpha^{m-n}(x) - \beta^{m-n}(x))}{\alpha(x) - \beta(x)} \hat{\alpha}(x)\hat{\beta}(x), \end{aligned}$$

so the result follows. □

Theorem 15. *The generating function for the Pell bihypernomial sequence* $\{BhP_n(x)\}$ *is*

$$G(t) = \frac{j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2 + 1) + (1 + j_2 + j_3 \cdot 2x)t}{1 - 2xt - t^2}.$$

Proof. Suppose that the generating function of the Pell bihypernomials sequence $\{BhP_n(x)\}$ has the form $G(t) = \sum_{n=0}^{\infty} BhP_n(x)t^n$. Then

$$G(t) = BhP_0(x) + BhP_1(x)t + BhP_2(x)t^2 + \cdots$$

Multiply the above equality on both sides by $-2xt$ and then by $-t^2$ we obtain

$$-G(t) \cdot (2x)t = -BhP_0(x) \cdot (2x)t - BhP_1(x) \cdot (2x)t^2 - BhP_2(x) \cdot (2x)t^3 - \cdots$$

$$-G(t)t^2 = -BhP_0(x)t^2 - BhP_1(x)t^3 - BhP_2(x)t^4 - \cdots$$

By adding these three equalities above, we will get

$$G(t)(1-2xt-t^2) = BhP_0(x) + (BhP_1(x) - BhP_0(x)\cdot(2x))t$$

since $BhP_n(x) = 2x \cdot BhP_{n-1}(x) + BhP_{n-2}(x)$ (see (13)) and the coefficient of t^n, for $n \geq 2$, are equal to zero. Moreover, $BhP_0(x) = j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2+1)$, $BhP_1(x) - BhP_0(x) \cdot (2x) = 1 + j_2 + j_3 \cdot 2x$. □

Using the same method we give the generating function $g(t)$ for Pell-Lucas bihypernomials.

Theorem 16. *The generating function for the Pell-Lucas bihypernomials sequence* $\{BhQ_n(x)\}$ *is*

$$g(t) = \frac{2 + j_1 \cdot 2x + j_2 \cdot (4x^2+2) + j_3 \cdot (8x^3+6x)}{1-2xt-t^2} + \frac{(-2x + 2j_1 + j_2 \cdot 2x + j_3 \cdot (4x^2+2))t}{1-2xt-t^2}.$$

At the end we will give the matrix generator of Pell and Pell-Lucas bihypernomials.

Theorem 17. *Let* $n \geq 0$ *be an integer. Then*

$$\begin{bmatrix} BhP_{n+2}(x) & BhP_{n+1}(x) \\ BhP_{n+1}(x) & BhP_n(x) \end{bmatrix} = \begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

Proof. (by induction on n)
If $n = 0$ then assuming that the matrix to the power 0 is the identity matrix the result is obvious. Now suppose that for any $n \geq 0$ holds

$$\begin{bmatrix} BhP_{n+2}(x) & BhP_{n+1}(x) \\ BhP_{n+1}(x) & BhP_n(x) \end{bmatrix} = \begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n.$$

We shall show that

$$\begin{bmatrix} BhP_{n+3}(x) & BhP_{n+2}(x) \\ BhP_{n+2}(x) & BhP_{n+1}(x) \end{bmatrix} = \begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^{n+1}.$$

Using induction's hypothesis we have

$$\begin{aligned} &\begin{bmatrix} BhP_2(x) & BhP_1(x) \\ BhP_1(x) & BhP_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} BhP_{n+2}(x) & BhP_{n+1}(x) \\ BhP_{n+1}(x) & BhP_n(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x \cdot BhP_{n+2}(x) + BhP_{n+1}(x) & BhP_{n+2}(x) \\ 2x \cdot BhP_{n+1}(x) + BhP_n(x) & BhP_{n+1}(x) \end{bmatrix} \\ &= \begin{bmatrix} BhP_{n+3}(x) & BhP_{n+2}(x) \\ BhP_{n+2}(x) & BhP_{n+1}(x) \end{bmatrix}, \end{aligned}$$

which ends the proof. □

Theorem 18. *Let $n \geq 0$ be an integer. Then*

$$\begin{bmatrix} BhQ_{n+2}(x) & BhQ_{n+1}(x) \\ BhQ_{n+1}(x) & BhQ_n(x) \end{bmatrix} = \begin{bmatrix} BhQ_2(x) & BhQ_1(x) \\ BhQ_1(x) & BhQ_0(x) \end{bmatrix} \cdot \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Using algebraic operations and matrix algebra, properties of these bihypernomials can be found.

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