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ON THE MATHEMATICAL THEORY OF TIDAL WAVE PROPAGATION

SAMUEL H. MAKARIOUS

Mathematics Deparment, University of Sokoto, Nigeria

ARSTRACT

**The vertical structure of tbe tidal-wave-propagation is expressed in terms of the solutions ta a second-order linear inhomogeneous differential equation, with variable coefficient.
In tbe present paper, analytical expressions for tbese solutions are obtained, for tbe first time, through tbe simulation of the inhomogeneous term, responsible for exciting tbe observed tines.**

1. INTRODUCTION

The fundamental equation that governs the vertical structure of the tidal-wave propagation [1] is:

(1.1) $d^2y_n/dx^2 + \mu^2{}_n(x)\, y_n = Q_n(x);$

subject to the boundary conditions:

(1.2) $y'_n(0) = K_n y_n(0);\ K_n = (H(0)/h_n - \frac{1}{2})$

and

(1.3) $y_n(x^*)$ is bounded; x^* is the upper bound.

$y_n(x)$ is the wave function, defined as:

(1.4) $y_n = (X_n - kj_n/gH)\, e^{-x/2}$

wehere $X_n(x)$ is the velocity divergence, $J_n(x)$ is the non-adiabatic heating rate per unit mass, and x is the reduced height;

(1.5) $x = \int dz/H(z),\ H(z) = RT(z)/g$

The forcing function $Q_n(x)$ is defined as:

(1.6) $Q_n(x) = (k/\gamma\, gh_n)\, J_n\, e^{-x/2}$

The subscript n indicates the mode number, h_n is the equivalent depth of the mode of oscillation that characterizes its propagation, and may have negative values for certain kind of modes.

In the existing tidal theory [1], the real difficulty lies in specifying Q_n with sufficient accuracy from our knowledge of the radiating processes and temperature change. Moreover, previous investigators [2,3] have adopted simplified models for Q_n in order to render the mathematical treatment more tractable. In comparing these theoretical predictions with the observed tides, Groves [4] has found that changes as large as 300 %, in the upward energy flux of a particular mode, can result for various distributions of Q_n.

For direct comparison between observations and theory, simulation based on the theory is often powerful. This called the attention for the inverse problem of deriving the forcing function from the observed tides. With such an approach, numerical solutions to Eq. (1.1) have been recently obtained [5]. In the present paper, the deduced empirical formulae for the forcing function are utilized in deriving analytical expressions for the solutions to (1.1).

2. A model for the Variable Coefficient

The variable coefifcient μ_n^2 (x), in (1.1), is defined as:

(2.1) $\mu_n^2 (x) = -\frac{1}{4} + (kH + dH/dx)/h_n$

For a reasonable choice of H (x), Eq. (1.1) is a well-behaved, non-singular differential equation. When H, (kH + dH/dx) or (dH/dx) is constant, 1.1 has nomogeneous solutions which are exponential, sinusoidal or Bessel functions [1,2]. For problems of any complexity, no closed-form solution exists and it has to be approached numerically.

Table 1. *The Model*

Region	1	2	3	4	5
x x0	x1	x2	x3	x4	x5
0	5.3	7.0	11.9	15.6	16.8
z (km) 0	36.53	49.15	81.45	100.09	109.22
H (km) 8.47	7.22	7.51	5.16	6.44	8.70
p 1.33	0.00	-0.12	0.00	0.25	0.25

With a realistic distribution of H (x), it was found [5] that the domain is divided into five distinct regions, with respect to the characteristics of the tidal modes propagation, as in table 1.

The range x_o, $< x < x_4$ is further divided into 78 subregions of spacing $\triangle x = 0{,}2$. for each of which $(kH + dH/dx)$ satisfies a linear form, thus

(2.2) $\mu_n^2 (x) = p_n x + q_n;\ x_o \leq x \leq x_4$

p_n changes sign at x_1 and x_3.

In the upper-most region, $x_4 \leq x \leq x_5$, the existing model suggests that $(kH + dH/dx)$ meets smoothly an exponential profile, thus

(2.3) $\mu_n^2 (x) = -\frac{1}{4} + \tau_n e^{x/4};\ x_4 < x < x_5$.

The parameter ρ is defined by:

(2.4) $\mu_n^2 + \frac{1}{2}) = \partial_n e^{\rho x}$

p_n and ∂_n are defined for each mode, but ρ is the same for all modes. Therefter, the subscript n is dropped, for simplicily in writing, and modes are treated independently.

3. Simulation of the Solution

In this section, the solutions to the D. Eq. (1.1) is simulated, using the observational evidences, in the five regions of the domain (table 1), and maintaining the contiuity at the separation levels in the solutions and their derivatives.

(i) Theoretically, the vertical dependence terms, V (x), in the velocity fields are usually expressed in terms of the solutions, y (x), as:

(3.1) $V(x) = (dy/dx - y/2)\, e^{x/2}$

Based on the analysis, of the observed tides, V (x) have been previously computed [6] at x = 0 (0.1) 8.4, and they have been found to be represented fairly accurately in complex exponential forms, in each of the first two regions; $0 \leq x \leq x_1$ and $x_1 \leq x \leq x_2$ as:

(3.2) $V(x) = \sum_{\nu=0}^{3} a_\upsilon e^{i \alpha_\upsilon x};\ 0 \leq x \leq x_1$

and

(3.3) $V(x) = \sum_{\nu=0}^{3} b_\upsilon e^{i \beta_\upsilon x};\ x_1 < x < x_2$

Representations in (3.2) } and (3.3) are justified by the fact that Q (x) on the right-hand side of (1.1) is a periodic force of excitation.

On equating (3.2), and (3.3) independently, to (3.1), the complex coefficients a_ν (and b_ν) and arguments α_ν (and β_ν) have been evaluated, by applying the Complex Fast Fourier Transform Technique [5] to the observed values of V (x); thus

(3.4) $(dy/dx - y/2)\, e^{x/2} = \Sigma_\nu\, a_\nu\, e^{i\,\alpha_\nu x};\ x_0 \leq x \leq x_1$

and similar expression for the region $x_1 \leq x \leq x_2$.

Differentiating (3.4) we find that the solution y (x) satisfies the linear second order differential equation:

(3.5) $d^2y/dx^2 - y/4 = r(x)$

where

(3.6) $r(x) = e^{-x/2} \sum_\nu i\alpha_\nu\, a_\nu\, e^{i\,\alpha_\nu x}$

Hence the general solution is:

(3.7) $y(x) = A\, e^{x/2} + B\, e^{-x/2} + e^{x/2} \int_0^x e^{-x/2}\, r\, dx - e^{-x/2} \int_0^x e^{x/2}\, r dx$

Substituting from (3.6), for the evaluation of the integrals, we find:

(3.8) $y(x) = e^{x/2} \left[A - \sum_\nu \frac{i\,\alpha_\nu\, a_\nu}{(i\,\alpha_\nu - 1)}\right]$

$+ e^{-x/2} \left[B + \sum_\nu \frac{a_\nu}{(i\,\alpha_\nu - 1)} \quad e^{i\alpha_\nu x} + \sum_\nu a_\nu\right]$

On differentiation:

(3.9) $y'(x) = e^{x/2} \left[A - \sum_\nu \frac{i\,\alpha_\nu\, a_\nu}{(i\,\alpha_\nu - 1)}\right] /2 - e^{-x/2}$

$\left[B + \sum_\nu \frac{(1 - 2\, i\alpha_\nu)}{(i\alpha_\nu - 1)}\, a_\nu\, e^{i\alpha_\nu x} + \sum_\nu a_\nu\right] /2$

The two constants of integration A and B are determined through using the initial values of y (o) and the lower boundary condition (1.2), which can be rewritten in the form:

(3.10) $y'(o) = K\, y(o) = \sum_\nu a_\nu + y(o)/2$

Thus we must have

$$A + B = y(o) \text{ and } A - B = 2 \sum_{\nu} a\upsilon + y(o)$$

i.e.

(3.11) $A = \Sigma\upsilon\, a\upsilon + y(o)$ and $B = -\sum_{\nu} a\upsilon$

The general solution is, therefore,

$$(3.12)\quad y(o) = e^{x/2}\left[y(o) - \sum_{\nu} \frac{a\upsilon}{(i\alpha\upsilon - 1)}\right]$$

$$+ e^{-x/2} \sum_{\nu} \frac{a\upsilon}{(i\alpha\upsilon - 1)} e^{i\alpha\nu} x;\ \underline{\underline{o \leq x \leq x_1}}$$

It should be noted that the numerical values for the solution (3.12) are found to be comparable, within $\pm$ 0.5 %, on the average, with those obtained by direct integration of (3.1), reflecting the validity of the approximation (3.4) and also the numerical stability of the solution.

(ii) In the region $x_1 \leq x \leq x_2$, the solution y satisfies also (3.5), but with r (x) given in the form:

$$(3.13)\quad r(x) = e^{-x/2} \sum_{\nu=0}^{3} i\beta\upsilon\, b\upsilon\, e^{i\beta\nu x}$$

Hence, the general solution is:

$$3.14\quad y(x) = e^{x/2}\left[A - \sum_{\nu} \frac{i\,\beta\upsilon\, b\upsilon\, e^{(i\beta\nu - 1)}\, x_1}{(i\,\beta\upsilon - 1)}\right]$$

$$+ e^{-x/2}\left[B + \sum_{\nu} \frac{b\upsilon\, e^{i\beta\nu}\, x}{(i\,\beta\upsilon - 1)} + \sum_{\nu} b\upsilon\, e^{i\beta\nu x_1}\right]$$

The linearized tidal theory requires that contact should be maintained at the boundaries between the regions, i.e. it requires continiuity in y(x) and y'(x). Thus the constants A and B are determined as:
(3.15) $A = y(1)\, e^{-x_1/2} + \sum_{\nu} b\upsilon\, e^{(i\beta\nu - 1)x_1}$ and $B = \sum_{\nu} b\upsilon\, e^{i\beta\nu x_1}$

Therefore, the general solution assumes the form:

$$(3.16)\quad y(x) = e^{x/2}\left[y(1)\, e^{-x_1/2} - \sum_{\nu} \frac{b\upsilon\, e^{(i\beta\nu - 1)}\, x_1}{(i\,\beta\upsilon - 1)}\right]$$

$$+ \; e^{-x/2} \sum_{\nu} \frac{b_\nu e^{i\beta\nu x}}{(i\beta_\nu - 1)} \; ; \; x_1 \leq x \leq x_2$$

(iii) In the region $x_2 \leq x \leq x_1$, the type of data, similar to those utilized in (i) and (ii) for the evaluation of V (x), is scarce. However, inferences on the existence of the tidal patterns are provided by other techniques-observation [4]. By inspection, V (x) is speculated to be expressed as:

(3.17) $V(x) = (c_1 \, e \, i \, \gamma_1 \, x + c_2 \, e \, i \, \gamma_2 \, x) \, e^{x/2}$

The harmonic coefficients are determined to meet the requirements for Q(x), namely:

a) maximum heating rate occurs in the 50 – 60 km range [4], and

b) heating rate Q vanishes at and above 80 km [2].

An expression for Q(x) is obtained by differentiation of (3.1) and substitution into (1.1), this gives:

(3.18) $(x) = (\mu^2 \, (x) + \frac{1}{4} \,)y + e^{-x/2} V'$

By further differentiation, we find

(3.19) $Q'(x) = Q \, (\rho + \frac{1}{2}) + e^{-x/2} \, (V'' - V' \, (1 + \rho) + (\mu^2 + \frac{1}{4}) \, V)$

In the present model, it is assumed that Q (and identically Q') → 0 at $x = x_3$, corresponding to height aorund 82 km; this level is characterised by $\rho = 0$. In this case, V satisfies:

(3.20) $V'' - V' + (\mu_3{}^2 + \frac{1}{4}) \, V = 0; \; \mu_3{}^2 = \mu^2 \, (x_3)$

whose solution is:

(3.21) $V(x) = (c_1 \, e^{i} \, 3^{x} + C_2 e^{-i} \, 3^{x}) \, e^{x/2}$

On the other hand, the level at which maximum heating occurs (i.e. $Q' = 0$) corresponds, in the present model, to $x = x_2$, and is characterized by the minimum value of $\rho = \rho_2$ Therefore, 3.19) reduces to:

(3.22) $V'' - V' \, (1 + \rho_2) + \mu_2{}^2 + \frac{1}{4}) \, V = Q_2 \, (\rho_2 + \frac{1}{2}) \, e^{x_2/2}$, at $x = x_2$

Substituting for V as given by (3.21), together with the requirement of continuity in V_2 at $x = x_2$, the coefficients c_1 and c_2 are determined.

(3.23) $c_1 = e^{-i\mu_3 x_2} \, (q_2 + V_2 \, e^{-x_2/2} \, /2)$ and $c_2 = - \, e^{i\mu_3 x_2} \, (q_2 - V_2 \, e^{-x_2/2} \, /2)$

where

(3.24) $q_2 = (V_2 \, e^{-x_2/2} \, (\mu_2{}^2 - \mu_{\_}{}^2 - \rho_2/2) + Q_2 \, (\rho_2 + \frac{1}{2})) \, /2 \, \mu_3 \rho_2$

Therefore, the vertical dependence terms V (x) are obtained in the range $x_2 \leq x \leq x_3$, by substituting for c_1 and c_2 (3.23) in (3.21). Following similar procedures, as for the previous two regions, the general solution to (3.5) is:

$$(3.25)\ y(x) = Ae^{x/2} + Be^{-x/2} + e^{x/2} \int_{x_2}^{x} e^{-x/2} \{(V_2 + i\mu_3)\, c_1\, e^{i\mu_3 x}$$

$$+ (\tfrac{1}{2} - i\mu_1)\, c_2\, e^{-i\mu_3 x}\}\, dx$$

$$- e^{-x/2}\ {}_{x_1}\!\int^{x} e^{x/2}\ \{(\tfrac{1}{2} + i\mu_3)\, c_1\, e^{i\mu_3 x} + (1/2 - i\mu_3)\, c_2\, e^{-i\mu_3 x}\}dx$$

The constants of integration A and B satisfy the initial values of $y(x_2)$ and $y'(x_2)$, thus:

$A = y(2)\, e^{-x_2/2} + V(2)\, e^{-x_2}$ and $B = - V(2)$

Hence the general solution is:

$$(3.26)\ y(x) = e\ (x - x_2)/2 \left\{ y(2) - \frac{c_1\, e^{i\mu_3 x_2}}{(i\mu_1 - 1/2)} + \frac{c_2\, e^{-i\mu_3 x_2}}{(i\mu_1 + 1/2)} \right\}$$

$$+ \left\{ \frac{c_1\, e^{i\mu_3 x}}{(i\mu_1 - 1/2)} - \frac{c_2\, e^{-i\mu_3 x}}{(i\mu_1 - 1/2)} \right\};\ x_2 \leq x \leq x_1$$

(iv) In the region $x_3 \leq x \leq x_4$ and above, the forcing function Q(x) vanishes and, therefore, (1.1) reduces to the homogeneous form, with the variable coefficient as approximated in (2.2).

On using the transformation:

$$(3.27)\ s(x) = (2/3p)\, \mu^3 (x),\ \mu^2 < 0$$

the homogeneous equation $d^2 y/dx^2 + \mu^2 (x)\, y = 0$ reduces to

$$(3.28)\ d^2y/ds^2 + (1/3s)\, dy/ds + y = 0$$

Writing

$$(3.29)\ s = (2i/3)\, t^{3/2}$$

eq. (3.28) assumes the form:

$$(3.30)\ d^2y/dt^2 - ty = 0.$$

This equation is Airy's equation [7], whose general solution is given in terms of Bessel functions of the first kind $J \pm 1/3$ in the form:

$$(3.31)\ y = t^{1/2}\ [a\, J_{1}/3\ (2it^{3/2}/3) + bj - 1/3\ (2it^{3/2}/3)],$$

or in terms of s, the solution is:

(3.32) $y = \mu \cdot [A\ J1/3\ (s) + B\ J - 1/3\ (s)]$; $x_1 \leq x \leq x_1$

In case of $\mu^2 < 0\ (= - \lambda^2)$, the transformation is:

(3.33) $\sigma(x) = (2/3p)\ \lambda^3(x)$,

and the solutions are expressed in terms of the modified Bessel functions of the first kind as

(3.34) $y = \lambda\ [-A\ I_{1/3}(\sigma) + B\ I_{-1/3}(\sigma)]$; $x_3 \leq x \leq x_4$

The constants of integration A and B are determined to satisfy the requirement of continuity in y and y' at $x = x_1$.

(v) In the uppermost region $x_4 \leq x \leq x_5$, the variable coefficient $\mu^2(x)$, in the homogeneous-form equation of (1.1), has been adjusted to meet smoothly the model given by (2.3).

Writing

(3.35) $s(x) = 8\ \sqrt{\tau\ e^{x/8}},\ \mu^2 > 0$

the homogeneous equation reduces to Bessel's differential equation:

(3.36) $d^2y/ds^2 + (1/s)dy/ds + (1 - 16/s^2)\ y = 0$,

whose solutions are Bessel functions J_4 and Y_4 of the the first and second kinds, respectively, of order 4. Hence the general solution is:

(3.37) $y(x) = A\ J_4(s) + B\ Y_4(s)$; $x_4 \leq x \leq x_5$

If $\mu^2 < 0\ (= - \lambda^2)$, $(i\sigma) = s$, we get the modified Bessel equation whose solutions are the modified Bessel functions I_4 and K_4:

(3.38) $y(x) = A\ I_4(\sigma) + B\ K_4(\sigma)$; $x_4 \leq x \leq x_5$

In this case,

(3.39) $A = 0$,

in order to comply with the upper boundary condition of bounded y(x) at $x^* = x_5$. Therefore,

(3.40) $y(x) = B\ K_4(\sigma)$; $x_4 \leq x \leq x_5$

The constants A and B are determined to satisfy the reduirements of continuity in the solution at $x = x_4$.

This completes the method of obtaining the solutions of the linear second order equation (1.1) in the five regions of the model as given in table 1.

REFERENCES:

S. CHAPMAN R. LİNDZEN, *Atmospheric Tides*, D. Rediel, Amsterdam (1970)

S.T. BUTLER K. A. SMALL, Proc. Roy. Soc. A274, 91 (1963)

M. SİEBERT, "Atmospheric Tides", *Adv. Geophys.* 7, Acad. Press, N.Y. (1961)

G. V. GROVES, J. Birt. Interp. Space, 30, 32, (1977)

S. H. MAKARİOUS, Comm. Fac. Sci. Univ. Ankara, Math. 31, 95, (1982)

S. H. MAKARİOUS, Archiv. Mechanics, 31 (1), 125, (1979)

C. J. TRANTER, "Bessel Functions Applications", English Univ. Press, London (1968)