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Araştırma Makalesi

On Combining with Fourier Transform and Adomian Methods to solve the Riccati Equations

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Abstract

In this paper, we apply the Fourier transform method with the Adomian decomposition method to solve Riccati equations . Proposed method is based on the Fourier transform and Adomian decomposition methods. The solutions obtained using FADMs are compared with the numerical solutions obtained using the Rung Kutta2 and Euler method.

Keywords: *. Fourier transform method, Adomian decomposition method, numerical solution of ODE.*

Özet

Bu makalede, Riccati denklemlerini çözmek için Fourier dönüşüm yöntemini ile Adomian ayrıştırma yöntemi uyguluyoruz. Önerilen yöntem, Fourier dönüşümü ve adomian ayrıştırma yöntemlerine dayanmaktadır. FADM'ler kullanılarak elde edilen çözümler, Runge Kutta2 ve Euler yöntemi kullanılarak elde edilen sayısal çözümlerle karşılaştırılmıştır. Ayrıca çözümlerin hata grafikleri sunulmuştur.

Anahtar Kelimeler: *Fourier dönüşüm yöntemi, Adomian ayrıştırma yöntemi, Diferansiyel deklemin sayısal çözümü.*

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1. Introduction

Nonlinear differential equations have had a considerable sum of interest due to its wide applications. Nonlinear ordinary differential equations play an important role in many branches as applied and pure mathematics and their applications in applied science, applied mechanics, quantum physics, analytical chemistry, astronomy and biology. There are many analytical and numerical methods developed for the solution of nonlinear differential equations. Some of these methods are Runge Kutta Method, Euler Method, Adomian Method, Homotopy Perturbation Method, Variational Iteration Method, Tanh Method, Kudryashov Method, etc.

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The Riccati differential equation is a well-known nonlinear differential equation and has different applications in engineering and science domains, such as robust stabilization, stochastic realization theory, network synthesis, optimal control and in financial mathematics[1]. For example, a one-dimensional static Schrödinger equation [2] is closely related to the Riccati differential equation. The Riccati differential equation is named after the Italian mathematician Francesco Riccati (1676–1754) [3]. The applications of this equation may be found not only in random processes, optimal control, and diffusion problems, but also in stochastic realization theory, optimal control, network synthesis and financial mathematics. Papers associated with Riccati equation has been seen in [1-7].

In this study, Riccati equation has been solved by combining Fourier transform and Adomian method (FADM). We compared the solutions obtained using FADM with the numerical solutions obtained using the Rung Kutta2 and Euler method. We have organized this paper as follows: In Section 2, we have given Fourier transform, Adomian decomposition method, FADM, Euler method, Runge Kutta2(RK2) method. In Section 3, we have given some examples Riccati differential equations. We solved these samples with FADM and compared them with numerical methods. Finally, a conclusion is presented.

2. Basic Definitions and Theorems

2.1. Fourier Transform

One of solution methods of linear differential equations is integral transforms. The best two known integral transforms are the Laplace transform and Fourier transform. The Fourier Transform, one of the gifts of Jean-Baptiste Joseph Fourier to the world of science, is an integral transform used in many areas of engineering such as it has been very useful for analyzing harmonic signals or signals for which there is known need for local information. Then the Fourier transform analysis has also been very useful in many other areas such as quantum mechanics, wave motion, turbulence [9,10,11]. Furthermore, it has been useful in mathematics. For example, generalized integrals, integral equations, linear differential equations can be solved by using the Fourier transform. Another example of its applications could be that. Voice of every human can be expressed as the sum of sine and cosine. Since the electro-magnetic spectrum of the frequency of each voice is different, the frequency of each sine and cosine sum will be different. In this way, a voice record can be found belongs to whom using the Fourier transform. In fact, our ear automatically runs this process instead of us. But The Fourier transforms can not been used for nonlinear equations. Nonlinear equations was solved using together with Elzaki transform and Differential transform method, Elzaki transform and Homotopy Perturbation , Laplace Transform and Adomian Decomposition Method in [12,13,14].

Definition 2.1. Fourier transform of function $f(t)$ is defined as

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t).\, e^{-iwt} dt \tag{1}$$

Since integral (2) is a function of w,

$$\mathcal{F}[f(t)] = F(w)$$

can be written.

Definition 2.2. If $\mathcal{F}[f(t)] = F(w)$, then $f(t)$ is called inverse Fourier transform of $F(w)$; where

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w).e^{iwt}dt \tag{2}$$

and it is showed by $f(t) = \mathcal{F}^{-1}[F(w)]$

Theorem2.1.[10] The Fourier Transform is linear. Let $c_1, c_2 \in R$. Then

$$\mathcal{F}[c_1.f_1(t) + c_2.f_2(t)] = c_1.\mathcal{F}[f_1(t)] + c_2\mathcal{F}[f_2(t)] \tag{3}$$

Theorem2.2.[9] Let $f(t)$ be continuous or partly continuous in the interval$(-\infty, \infty)$

and $f(t), f'(t), f''(t), \dots, f^{(n-1)}(t) \to 0$ for$|t| \to \infty$. If $f(t), f'(t), f''(t), \dots, f^{(n-1)}(t)$ are absolutely integrable in then terval$(-\infty, \infty)$, then

$$\mathcal{F}\left[f^{(n)}(t)\right] = (iw)^n\mathcal{F}[f(t)] \tag{4}$$

Definition 2.3. The Dirac delta function can be rigorously thought of as a function on real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

The Dirac delta function has properties, that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t).\delta(t - t_0)dt = f(t_0) \tag{5}$$

$$\int_{-\infty}^{\infty} f(t).\delta^{(n)}(t - t_0)dt = (-1)^n.f^{(n)}(t_0) \tag{6}$$

$$t.\delta'(t) = -\delta(t) \tag{7}$$

$$(w - w_o)^n.\delta^{(n)}(w - w_0) = (-1)^n.n!.\delta(w - w_0) \tag{8}$$

$$\int_{-\infty}^{\infty} \frac{\delta(w-w_0)f(w)}{(w-w_0)^n} dw = \frac{1}{n!}\frac{d^n f(w)}{dw^n}(w = w_0) \tag{9}$$

Where $\delta(w - w_0)$ is defined as following

$$\delta(w - w_0) = \begin{cases} 0, & w \neq w_0 \\ \infty, & w = w_0 \end{cases}$$

Theorem2.3. [9] The Fourier transform of the Dirac Delta function is 1.

That is $\mathcal{F}[\delta(t)] = 1$.

Theorem2.4.[10] Fourier transforms of some functions are following

$i)\mathcal{F}[1] = 2\pi.\delta(w)$

$ii)\ \mathcal{F}[t^n] = 2\pi.i^n.\delta^{(n)}(w)$

$iii)\mathcal{F}[t^n.f(t)] = i^n \dfrac{d^n\mathcal{F}[f(t)]}{dw^n}$

$iv)\mathcal{F}[e^{iw_0t}] = 2\pi\delta(w - w_0)$

$v) If\ \mathcal{F}[f(t)] = F(w), then\ \mathcal{F}[e^{iw_0t}.f(t)] = F(w - w_0)$

$vi)\mathcal{F}[e^{at}] = 2\pi\delta(w + ia)$

$vii) If\ \mathcal{F}[f(t)] = F(w), then\ \mathcal{F}[e^{a.t}.f(t)] = F(w + ia)$

2.2. Adomian Decomposion Method

The Adomian Decomposition Method (ADM) is a method which is used in several areas of mathematics. Recently a great deal of interest has been focused on the application of Adomian's decomposition method to solve a wide variety of linear and nonlinear problems. This method has been introduced by Adomian and it can be used in the linear and nonlinear differential equations, in the differential equations systems, in the integral equations, in the difference equations, in the differential-difference equations, and in the algebraic equations. This method generates a solution in the form of a series whose terms are determined by a recursive relationship using the Adomian polynomials.

If the nonlinear term is $f(u)$ in the equation, Adomian polynomials are as follows.

$A_0 = f(u_0)$

$A_1 = u_1 \frac{df(u_0)}{du_0}$

$A_2 = u_2 \frac{df(u_0)}{du_0} + \frac{u_1^2}{2!}\frac{d^2f(u_0)}{du_0^2}$

$A_3 = u_3 \frac{df(u_0)}{du_0} + u_1u_2 \frac{d^2f(u_0)}{du_0^2} + \frac{u_1^3}{3!}\frac{d^3f(u_0)}{du_0^3}$

$$\vdots$$

As you can see, A_0 is only depends on u_0, A_1 is depends on u_0 and u_1, A_2 is depends on u_0, u_1 and u_2, A_n is depends on $u_0, u_1, u_2, \dots u_n$.

2.3. Application to Riccati Equation of Fourier Transform and Adomain Method

Let consider following general Riccati equation.

$y' + P(x)y + Q(x)y^2 = R(x), y(0) = c$

We let use Fourier transform for above Riccati equation. Thus we get that:

$\mathcal{F}(y') + \mathcal{F}(P(x)y) + \mathcal{F}(Q(x)y^2) = \mathcal{F}\big(R(x)\big)$

$iwY = \mathcal{F}\big(R(x)\big) - \mathcal{F}(P(x)y) - \mathcal{F}(Q(x)y^2)$

$Y = \frac{\mathcal{F}(R(x)) - \mathcal{F}(P(x)y) - \mathcal{F}(Q(x)y^2)}{iw}$

Now, we let use inverse Fourier transform.

$\mathcal{F}^{-1}(Y) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}(R(x)) - \mathcal{F}(P(x)y) - \mathcal{F}(Q(x)y^2)}{iw}\right)$

$y_{n+1} = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(P(x)y_n) + \mathcal{F}(Q(x)A_n)}{iw}\right)$

$y_0 = c + \mathcal{F}^{-1}\left(\frac{\mathcal{F}(R(x))}{iw}\right), A_0 = {y_0}^2$

$y_1 = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(P(x)y_0) + \mathcal{F}(Q(x)A_0)}{iw}\right)$

$A_1 = 2y_0.y_1$

$y_2 = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(P(x)y_1) + \mathcal{F}(Q(x)A_1)}{iw}\right)$

$A_2 = 2y_0.y_2 + {y_1}^2$

$y_3 = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(P(x)y_2) + \mathcal{F}(Q(x)A_2)}{iw}\right)$

$A_3 = 2y_3.y_0 + 2y_1.y_2$

$y_4 = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}(P(x)y_3) + \mathcal{F}(Q(x)A_3)}{iw}\right)$

$A_4 = 2y_4.y_0 + {y_2}^2 + 2y_1y_3$

$\vdots$

$y \cong \sum_{k=0}^{n} y_k$.

2.4. Euler Method

In sometimes, Euler method is numerical methods which is used to solving differential equations. The differential equations from first order with initial value problem is defined as following;

$$\frac{dy}{dx} = f(x, y), \; y(x_0) = y_0 \tag{10}$$

The Euler method which is the first order Runge Kutta mehod is given as follow. To find the desired solution we chop the interval into small subdivisions of length h. Using initial condition sultion is generated by using following the iterative relation.

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{11}$$

The iterative process is terminated that is reached the end of the interval

2.5. Runge Kutta-2 Method

Runge Kutta methods are a numerical method used in approximate solution of nonlinear differential equations. In this study, Runge Kutta 2 method has been used to compare with the FADM. Runge-Kutta method of order second is given the following formulas[16].

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + h, y_n + k_1) \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2) \end{aligned} \tag{12}$$

3. Related Examples

In this subsection, some examples which are examined in the [1,2,4,7,8] are solved by proposed method in section 3.

Example 3.1. [4]. Consider the following Riccati equation:
$y' + y^2 = 1 + x^2$ subject to initial condition $y(0) = 1$.

Solution: The exact solution of the above equation with initial condition is

$$y = x + \frac{e^{-x^2}}{1+\int_0^x e^{-t^2} dt} \tag{13}$$

When we solve this differential equation by FADM method, the equation x is given below.

$P(x) = 0, Q(x) = 1, R(x) = 1 + x^2$

$$y_0 = 1 + \mathcal{F}^{-1}\left(\frac{\mathcal{F}(1+x^2)}{iw}\right) = 1 + \mathcal{F}^{-1}\left(\frac{2\pi\delta-2\pi\delta''}{iw}\right) = 1 + x + \frac{x^3}{3}$$

$$y_1 = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}({y_0}^2)}{iw}\right) = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(1+x^2+\frac{x^6}{9}+2x+\frac{2x^3}{3}+\frac{2x^4}{3}\right)}{iw}\right)$$

$$y_1 = -2\pi\mathcal{F}^{-1}\left(\frac{\delta-\delta''-\frac{\delta^{(6)}}{9}+2i\delta'-\frac{2i}{3}\delta'''+\frac{2}{3}\delta^{(4)}}{iw}\right)$$

$$y_1 = -x - x^2 - \frac{x^3}{3} - \frac{x^4}{6} - \frac{2x^5}{15} - \frac{x^7}{63}$$

$$y_2 = -\mathcal{F}^{-1}\left(\frac{2\mathcal{F}\left(\left(1+x+\frac{x^3}{3}\right)\left(-x-x^2-\frac{x^3}{3}-\frac{x^4}{6}-\frac{2x^5}{15}-\frac{x^7}{63}\right)\right)}{iw}\right)$$

$$y_2 = -2\mathcal{F}^{-1}\left(\frac{\mathcal{F}\left(x+2x^2+\frac{4x^3}{3}+\frac{5x^4}{6}+\frac{19x^5}{30}+\frac{11x^6}{45}+\frac{x^7}{14}+\frac{19x^8}{315}+\frac{x^{10}}{189}\right)}{iw}\right)$$

$$y_2 = -4\pi\mathcal{F}^{-1}\left(\frac{i\delta'+2\delta''-\frac{4i}{3}\delta'''+\frac{5}{6}\delta^{(4)}+\frac{19i}{30}\delta^{(5)}-\frac{11}{45}\delta^{(6)}-\frac{i}{14}\delta^{(7)}+\frac{19}{315}\delta^{(8)}-\frac{\delta^{(10)}}{189}}{iw}\right)$$

$$y_2 = x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \frac{x^5}{3} + \frac{19x^6}{60} + \frac{22x^7}{315} + \frac{x^8}{56} + \frac{38x^9}{2835} + \frac{2x^{11}}{2079}$$

$$g2 = 1 - x^2 - \frac{x^4}{6} - \frac{2x^5}{15} - \frac{x^7}{63}$$

$$g3 = 1 + \frac{4}{3}x^3 + \frac{x^4}{2} + \frac{x^5}{5} + \frac{19x^6}{60} + \frac{17x^7}{315} - \frac{x^8}{56} + \frac{38x^9}{2835} + \frac{2x^{11}}{2079}$$

Table 1. The results of the exact solution and the aproximation solutions (obtained by FADMs and the numerical methods)

xi	yi exact	yEuler	yRK2	g2apr	g3apr
0	1.0000	1	1	1	1
0.1	1.0003	1	1.0005	0.99	1.0014
0.2	1.0024	1.001	1.002	0.9597	1.0116
0.3	1.0078	1.0098	1.0051	0.9083	1.0408
0.4	1.0177	1.0438	1.0053	0.8343	1.1016
0.5	1.0330	1.1349	9.66E-05	0.7353	1.2095
0.6	1.0545	1.3311	7.21E-05	0.6076	1.3845
0.7	1.0827	1.6949	-3.2E-05	0.4463	1.6522
0.8	1.1181	2.2916	-2.9567	0.2447	2.0462
0.9	1.1607	3.1625	-2.217	-0.0057	2.6101
1	1.446	4.2874	-10.2583	-0.3159	3.4005

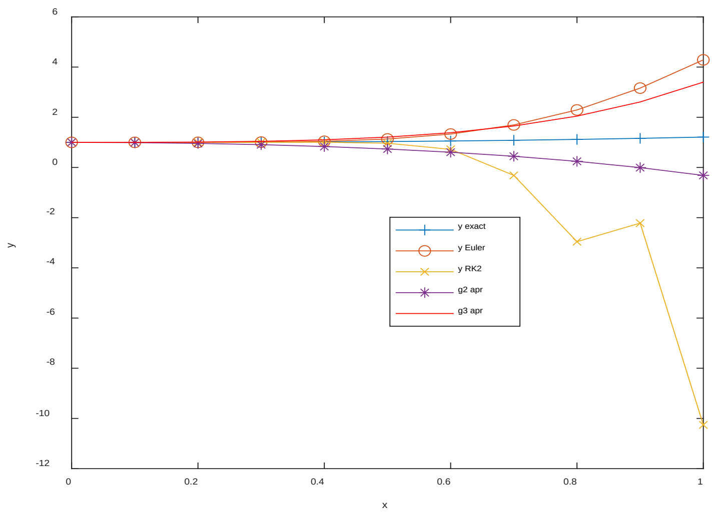

Figure 1. The graphics of exact solution and the aproximation solutions

Table 2. The errors of the aproximation solutions

xi	Er_Euler	Er_RK2	Er_g2apr	Er_g3apr
0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.0003	0.0002	0.0103	0.0011
0.2000	0.0014	0.0004	0.0427	0.0091
0.3000	0.0020	0.0027	0.0995	0.0330
0.4000	0.0262	0.0123	0.1833	0.0839
0.5000	0.1019	0.0666	0.2977	0.1765
0.6000	0.2766	0.3331	0.4469	0.3300
0.7000	0.6122	1.3990	0.6365	0.5695
0.8000	1.1735	4.0748	0.8734	0.9281
0.9000	2.0018	3.3777	1.1664	1.4493
1.0000	3.0768	1.1469	1.5265	2.1899

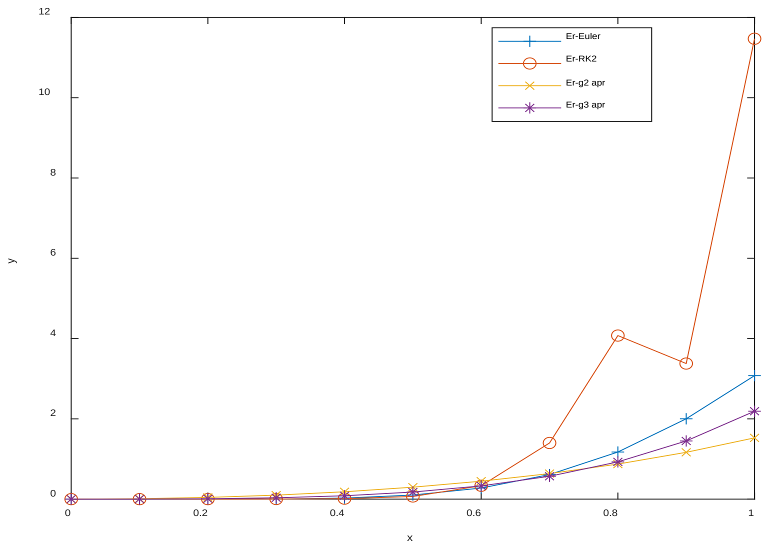

Figure 2. The graphics of errors of the aproximation solutions

Example 3.2.. [1,4,7] Consider the following example
$y' = 1 - y^2,\ y(0) = 0.$

Solution: Coefficients of the equation are $P(x) = 0, Q(x) = 1, R(x) = 1.$
We let's study to find terms of solution series.

$$y_0 = \mathcal{F}^{-1}\left[\frac{\mathcal{F}(1)}{iw}\right] = x$$
$$y_1 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}(x^2)}{iw}\right] = -\frac{x^3}{3}$$
$$y_2 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(\frac{-2x^4}{3}\right)}{iw}\right] = \frac{2x^5}{15}$$

$$y_3 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(\frac{17x^6}{45}\right)}{iw}\right] = -\frac{17x^7}{315}$$

$$y_4 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(-\frac{22x^8}{105}\right)}{iw}\right] = \frac{62x^9}{2835}$$

$$y_5 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(\frac{1382}{14175}x^{10}\right)}{iw}\right] = -\frac{1382x^{11}}{155925}$$

$$\vdots$$

$$y \cong g_k = \sum_{i=0}^{k-1} y_i$$

These components give the first four components of exact solution of the equation.

$$g2 = x - \frac{x^3}{3} + \frac{2}{15}x^5$$

$$g3 = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7$$

$$g4 = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9$$

$$g5 = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \frac{1382}{155925}x^{11}$$

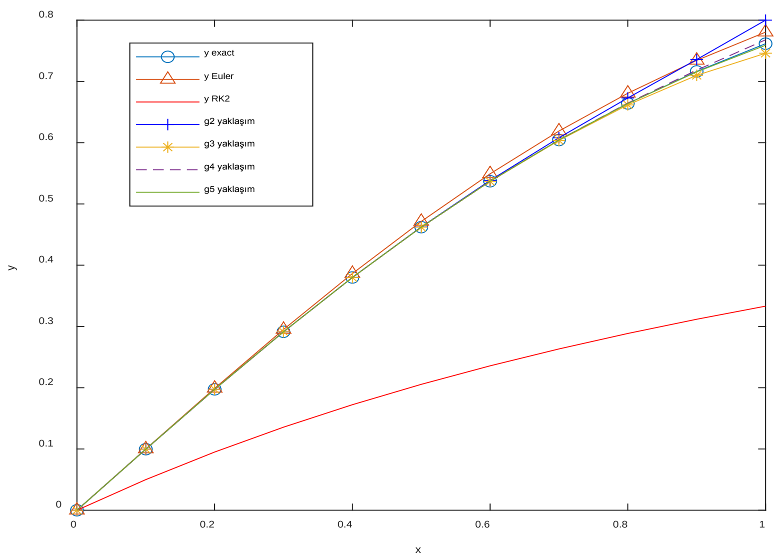

Figure 3. The graphics of exact solution and the aproximation solutions

Table 3 The results of the exact solution and the aproximation solutions (obtained by FADMs and the numerical methods)

xi	f(xi)	yEuler	yRK2	g2(xi)	g3(xi)	g4(xi)	g5(xi)
0.0000	0	0	0	0	0	0	0
0.1000	0.0997	0.1	0.05	0.0997	0.0997	0.0997	0.0997
0.2000	0.1974	0.199	0.095	0.1974	0.1974	0.1974	0.1974
0.3000	0.2913	0.295	0.1356	0.2913	0.2913	0.2913	0.2913
0.4000	0.3799	0.3863	0.1723	0.38	0.3799	0.3799	0.3799
0.5000	0.4621	0.4714	0.2055	0.4625	0.4621	0.4621	0.4621
0.6000	0.537	0.5492	0.2357	0.5384	0.5369	0.5371	0.537
0.7000	0.6044	0.619	0.2633	0.6081	0.6036	0.6045	0.6043
0.8000	0.664	0.6807	0.2886	1	0.6617	0.6646	0.6639
0.9000	0.7163	0.7344	0.3118	0.7357	0.7099	0.7184	0.7156
1.0000	0.7616	0.7804	0.3332	0.8	0.746	0.7679	0.759

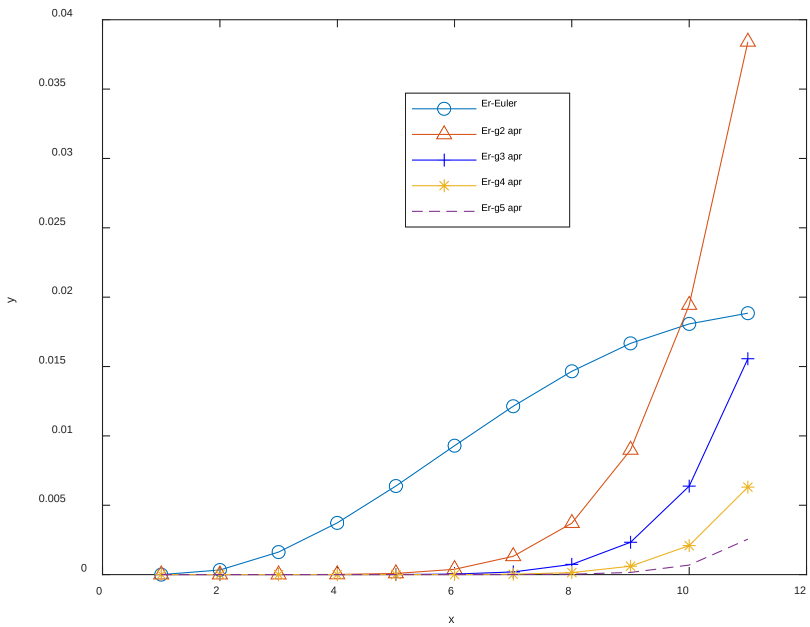

Figure 4. The graphics of errors of the aproximation solutions

Table 4. The errors of the aproximation solutions

xi	Er_Euler	Er_RK2	Er_g2(xi)	g3(xi)	g4(xi)	Er_g5(xi)
0.0000	0	0	0	0	0	0
0.1000	0.0003	0.0497	0	0	0	0
0.2000	0.0016	0.1024	0	0	0	0
0.3000	0.0037	0.1557	0	0	0	0
0.4000	0.0064	0.2077	0.0001	0	0	0
0.5000	0.0093	0.2566	0.0004	0	0	0
0.6000	0.0121	0.3013	0.0013	0.0002	0	0
0.7000	0.0147	0.3411	0.0037	0.0007	0.0001	0
0.8000	0.0167	0.3755	0.009	0.0023	0.0006	0.0002
0.9000	0.0181	0.4045	0.0194	0.0064	0.0021	0.0007
1.0000	0.0188	0.4284	0.0384	0.0156	0.0063	0.0026

Example 3.3: [2,7] Consider the following example
$y' = 1 + 2y - y^2,\ y(0) = 0.$

Solution: Coefficients of the equation are $P(x) = -2, Q(x) = 1, R(x) = 1$.
We let's study to find terms of solution series.

$$y_0 = \mathcal{F}^{-1}\left[\frac{\mathcal{F}(1)}{iw}\right] = x$$

$$y_1 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}(-2x)+\mathcal{F}(x^2)}{iw}\right] = -\mathcal{F}^{-1}\left[\frac{-4\pi i\delta' - 2\pi\delta''}{iw}\right]$$

$$= \int_{-\infty}^{\infty} \frac{2.\delta' e^{iwx}}{w} dw + \int_{-\infty}^{\infty} \frac{\delta'' e^{iwx}}{iw} dw = 2\int_{-\infty}^{\infty} -\frac{\delta\, e^{iwx}}{w^2} dw + 2\int_{-\infty}^{\infty} \frac{\delta\, e^{iwx}}{iw^3} dw$$

$$= x^2 - \frac{x^3}{3}$$

$$y_2 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(-2x^2+\frac{2x^3}{3}\right)+\mathcal{F}\left(2x^3-\frac{2x^4}{3}\right)}{iw}\right]$$

$$= -\mathcal{F}^{-1}\left[\frac{4\pi\delta'' - \frac{16}{3}\pi i\delta''' - \frac{4\pi}{3}\delta^{(iv)}}{iw}\right] = \frac{2x^3}{3} - \frac{2x^4}{3} + \frac{2x^5}{15}$$

$$y_3 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(-\frac{4x^3}{3}+\frac{4x^4}{3}-\frac{4x^5}{15}\right)+\mathcal{F}\left(\frac{7x^4}{3}-2x^5+\frac{17x^6}{45}\right)}{iw}\right]$$

$$= -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(-\frac{4x^3}{3}+\frac{11x^4}{3}-\frac{34x^5}{15}+\frac{17x^6}{45}\right)}{iw}\right]$$

$$= -\frac{x^4}{3} + \frac{11x^5}{15} - \frac{17x^6}{45} + \frac{17x^7}{270}$$

⋮

$$y \cong y_0 + y_1 + y_2 + y_3$$

$$\cong \; x + x^2 + \frac{x^3}{3} - x^4 + \frac{13x^5}{15} - \frac{17x^6}{45} + \frac{17x^7}{270}.$$

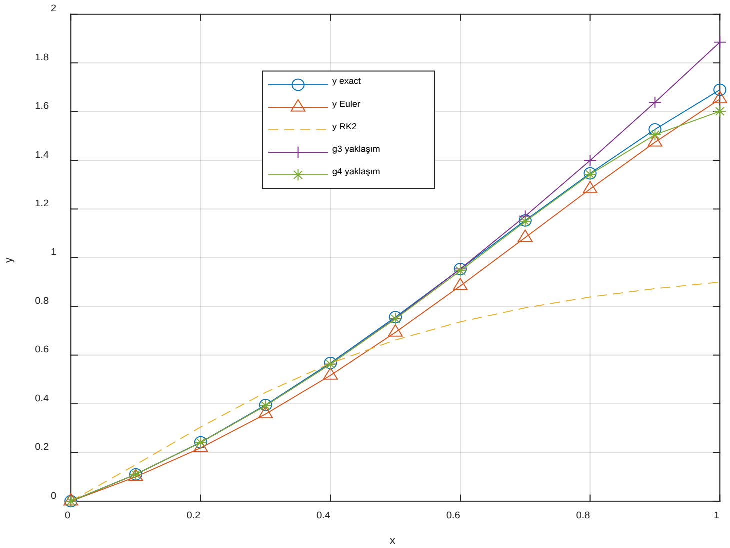

Figure 5. The graphics of exact solution and the aproximation solutions

Table 5. The results of the exact solution and the aproximation solutions which is obtained by FADMs and the numerical methods

xi	Exact Soln	yEuler	yRK2	g3(xi)	g4(xi)
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.1103	0.1000	0.1500	0.1102	0.1102
0.2000	0.2420	0.2190	0.3047	0.2413	0.2413
0.3000	0.3951	0.3580	0.4468	0.3927	0.3927
0.4000	0.5678	0.5168	0.5665	0.5632	0.5627
0.5000	0.7560	0.6934	0.6620	0.7508	0.7487
0.6000	0.9536	0.8840	0.7365	0.9539	0.9464
0.7000	1.1529	1.0827	0.7941	1.1706	1.1485
0.8000	1.3464	1.2820	0.8386	1.3992	1.3420
0.9000	1.5269	1.4741	0.8730	1.6380	1.5050
1.0000	1.6895	1.6516	0.8998	1.8852	1.6013

Table 6. The errors of the aproximation solutions

xi	Er_Euler	Er_RK2	Er_g3	Er_g4
0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.0103	0.0397	0.0001	0.0001
0.2000	0.0230	0.0628	0.0007	0.0007
0.3000	0.0371	0.0517	0.0024	0.0025
0.4000	0.0510	0.0014	0.0046	0.0052
0.5000	0.0626	0.0940	0.0052	0.0073
0.6000	0.0695	0.2170	0.0004	0.0071
0.7000	0.0703	0.3588	0.0177	0.0044
0.8000	0.0644	0.5078	0.0529	0.0043
0.9000	0.0529	0.6539	0.1111	0.0219
1.0000	0.0379	0.7897	0.1957	0.0882

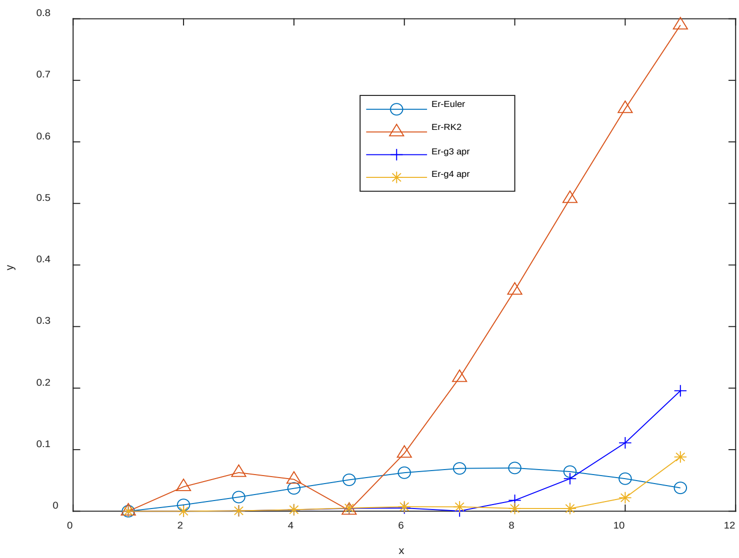

Figure 6. The graphics of errors of the aproximation solutions

Example 3.4. [8] We let find solution of following Riccati equation which has variable coefficients

$$y' = 3 + 3x^2y - xy^2$$

with intial condition

$$y(0) = 1.$$

Solution: The exact solution of the above equation with initial condition is

$$y = 3x + \frac{e^{-x^3}}{1+\int_0^x te^{-t^3}dt} \quad (14)$$

Coefficients of the equation are $P(x) = -3x^2, Q(x) = x, R(x) = 3$.

We let's study to find terms of solution series.

$$y_0 = 1 + \mathcal{F}^{-1}\left[\frac{\mathcal{F}(3)}{iw}\right] = 1 + 3x$$

$$y_1 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}(3x^2+x)}{iw}\right] = -\mathcal{F}^{-1}\left[\frac{-6\pi\delta'' + 2\pi i\delta'}{iw}\right]$$

$$= \int_{-\infty}^{\infty} \frac{6.\delta e^{iwx}}{iw^3} dw + \int_{-\infty}^{\infty} \frac{\delta e^{iwx}}{w^2} dw = \frac{6}{i}\frac{(ix)^3}{6} + \frac{(ix)^2}{2} = -x^3 - \frac{x^2}{2}$$

$$y_2 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(-3x^5 - \frac{7x^4}{2} - x^3\right)}{iw}\right]$$

$$\int_{-\infty}^{\infty}\left(\frac{-3.5!\delta}{w^6} - \frac{7i.4!\delta}{2w^5} + \frac{6.\delta}{w^4}\right)e^{iwx}dw$$

$$= \frac{x^6}{2} + \frac{7}{10}x^5 + \frac{x^4}{4}$$

$$y_3 = -\mathcal{F}^{-1}\left[\frac{\mathcal{F}\left(\frac{3x^5}{4} + \frac{63x^6}{20} + \frac{41}{10}x^7 + \frac{3x^8}{2}\right)}{iw}\right]$$

$$= -\frac{x^6}{8} - \frac{63}{140}x^7 - \frac{41}{80}x^8 - \frac{x^9}{6}$$

$\vdots$

$$y = y_0 + y_1 + y_2 + y_3 + \cdots$$

$$= 1 + 3x - \frac{x^2}{2} - x^3 + \frac{x^4}{4} + \frac{7}{10}x^5 + \frac{3x^6}{8} - \frac{63}{140}x^7 - \frac{41}{80}x^8 - \frac{x^9}{6} + \cdots$$

Solution which are obtained are compatible with in [8].

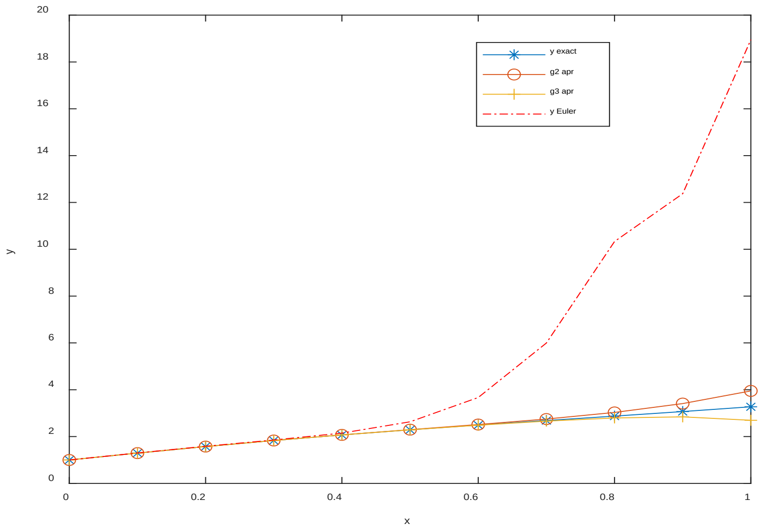

Figure 7. The graphics of exact solution and the aproximation solutions

Table 7. The results of the exact solution and the aproximation solutions (obtained by FADMs and the numerical methods)

xi	Exact Soln	g2 apr	g3 apr	yEuler	yRK2
0.0	1.0000	1.0000	1.0000	1.0000	0
0.1	1.2940	1.2940	1.2940	1.3000	0.2595
0.2	1.5726	1.5727	1.5726	1.5870	0.4728
0.3	1.8319	1.8321	1.8319	1.8543	0.6109
0.4	2.0701	2.0716	2.0700	2.1483	0.6570
0.5	2.2887	2.2953	2.2875	2.6312	0.4920
0.6	2.4914	2.5142	2.4854	3.6688	0.0886
0.7	2.6842	2.7485	2.6605	5.9960	1.5462
0.8	2.8742	3.0308	2.7954	10.3320	-123.36
0.9	3.0683	3.4091	2.8422	12.3727	-658056597.6
1.0	3.2725	3.9500	2.6958	18.9493	

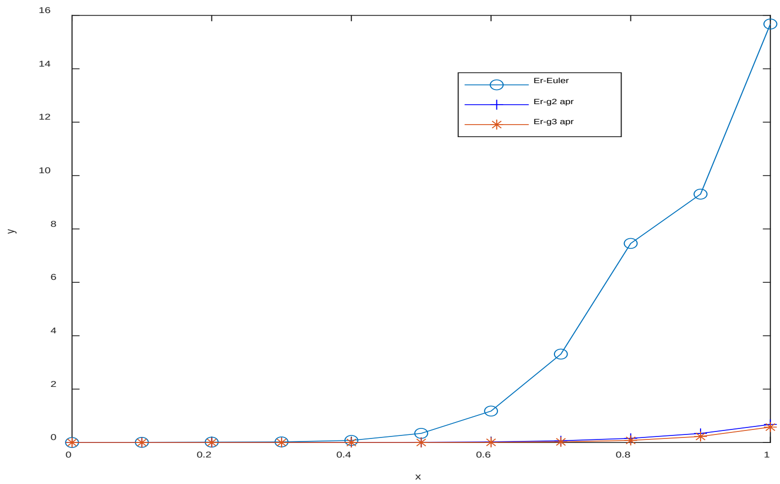

Figure 8. The graphics of errors of the aproximation solutions

Table 8. The errors of the aproximation solutions

xi	Er_Euler	Er-g2 apr	Er-g3 apr
0.0000	0.0000	0.0000	0.0000
0.1000	0.0060	0.0000	0.0000
0.2000	0.0144	0.0000	0.0000
0.3000	0.0224	0.0002	0.0000
0.4000	0.0781	0.0015	0.0002
0.5000	0.3426	0.0067	0.0011
0.6000	1.1774	0.0228	0.0059
0.7000	3.3118	0.0643	0.0237
0.8000	7.4578	0.1566	0.0789
0.9000	9.3043	0.3408	0.2261
1.0000	15.6768	0.6775	0.5767

4. Conclusion

In this study, the Riccati differential equation is solved by FADM, this solution is compared with Euler and Runge Kutta2 numerical methods and it is shown that the FADM solution is better. Approximate solutions obtained with the exact solution for a given range are compared and graphically illustrated. In order to better illustrate the performance of the methods, the absolute errors of the results are shown numerically and graphically.

The term number of the solution obtained by FADM approximates the exact solution either from below or from above, depending on whether it is odd or even. For example, in the second example we examine, while odd terms approach from above, even terms approach the real solution from below.

In the special Riccati equations solved, while RK2 method is expected to give better results than Euler method, Euler method gives better results in examples 1 and 4. In these two examples, it was seen that RK2 diverges from the exact solution in the selected solution range. Therefore, since the divergence in 4th sample is very large, RK2 is not given in the graph and table.

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