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## Finite Groups Having Monolithic Characters of Prime Degree


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### ABSTRACT

Let $G$ be a finite group. An irreducible character $\chi$ is called monolithic when the factor group $G/\ker(\chi)$ has unique minimal normal subgroup. In this paper, we prove that for the smallest prime $q$ dividing the order of $G$ if $G$ has a faithful imprimitive monolithic character of degree $q$, then $G$ becomes a nonabelian $q$-group or a Frobenius group with cyclic Frobenius complement whose order is $q$. Under certain conditions, we also classify finite groups in which their nonlinear irreducible characters are monolithic.


**Keywords:** *Finite groups, Monolithic characters, Primitive characters*

## Asal Dereceli Monolitik Karakterlere Sahip Sonlu Gruplar


### ÖZ

$G$ bir sonlu grup olsun. Bir $\chi$ indirgenemez karakterine, eğer $G/\ker(\chi)$ bölüm grubunun yalnız bir tane minimal normal alt grubu varsa monolitik karakter denir. Bu çalışmada, $G$ grubunun mertebesini bölen en küçük asal sayı $q$ olmak üzere, $G$ derecesi $q$ olan bir sadık imprimitif monolitik karaktere sahipse ya bir abelyen olmayan $q$-grubu ya da $q$ mertebeli Frobenius tümleyeni olan bir Frobenius grubu olduğunu ispat ediyoruz. Lineer olmayan tüm indirgenemez karakterleri monolitik olan sonlu grupları da bazı koşullar altında sınıflandırıyoruz.


**Anahtar Kelimeler:** *Sonlu gruplar, Monolitik karakterler, Primitif karakterler*



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# I. INTRODUCTION

It is known that there exists a very strong interplay between the group structure and its characters. In the present paper, we give some results on the relations between the group structure and its monolithic characters. Before mentioning our results, we present a brief introduction to character theory of finite groups for the reader's convenience. Our notations are standard and taken mainly from [4]. During the paper, we are concerned with only finite groups, so we just say the group instead of the finite group.
The $\mathbb{C}$-representation of a group $G$ is a homomorphism $\psi: G \longrightarrow GL(n, \mathbb{C})$ for some integer $n$, where $\mathbb{C}$ denotes complex numbers and $GL(n, \mathbb{C})$ is the general linear group of non-singular $n \times n$ matrices over $\mathbb{C}$. The $\mathbb{C}$-character $\chi$ of $G$ afforded by the representation $\psi$ is the function from $G$ to $\mathbb{C}$ with $\chi(m) = tr(\psi(m))$ for $m \in G$, where $tr(\psi(m))$ is trace of the matrix $\psi(m)$. We know that every character is a class function, which means that it takes same value on the conjugacy classes of $G$. $\chi(1)$ is called the degree of $\chi$ and we say that $\chi$ is a linear character when $\chi(1) = 1$. A character afforded by an irreducible representation of $G$ is said to be irreducible. The notation $Irr(G)$ is general used to denote the set of irreducible characters of $G$. The fundamental formula

$$|G| = |G:G'| + \sum_{\substack{\chi \epsilon Irr(G) \\ \chi(1)>1}} \chi(1)^2$$

is valid and the number of linear irredubile characters of $G$ is equal to $|G:G'|$, where $G'$ is the commutator subgroup of $G$.
Let $\chi \in Irr(G)$. The kernel of $\chi$ is given by $\ker(\chi) = \{ g \in G \mid \chi(g) = \chi(1) \}$ and if $\ker(\chi) = 1$, we say that $\chi$ is a faithful character of $G$. By Theorem 2.32 of [4], we know that if $G$ has a faithful irreducible character, then $Z(G)$, the center of $G$, is cyclic, and also if $G$ is a $p$-group having cyclic center, then $G$ has a faithful irreducible character. By $\mathrm{Z}(\chi) = \{ g \in G \mid |\chi(g)| = \chi(1) \}$, we mean the center of $\chi$. Also, it is easy to see that $N = \cap \{ker(\chi) | \chi \in Irr(G) \ and \ N \leq \ker(\chi) \}$ and $Irr(G/N) = \{ \chi \in Irr(G) \mid N \leq \ker(\chi) \}$ when $N \trianglelefteq G$. On the other hand, the restriction character $\chi$ to the subgroup $K$ of $G$ is a character of $K$, which is denoted by $\chi_K$. Conversely, an irreducible character $\mu$ of $K$ determines the character $\mu^G$ of $G$, which is given by Definition 5.1 of [4]. Now, we may present some definitions as follows:.

**Definition 1.1.** Assume that $G$ denotes a group and $\chi \in Irr(G)$. The irreducible character $\chi$ is called imprimitive if we have $\lambda^G = \chi$ for some $K < G$ and $\lambda \in Irr(K)$. We note that $\chi(1) = |G:K|\lambda(1)$. If a character $\chi$ cannot be induced from a character of a proper subgroup of $G$, then the character $\chi$ is said to be primitive.

Also, if a group $G$ has a faithful primitive character and $A \trianglelefteq G$ is abelian, then $A \leq Z(G)$ by Corollary 6.13 of [4].

**Definition 1.2.** An irreducible character $\chi$ of a group $G$ is said to be monolithic when the factor group $G/\ker(\chi)$ has only one minimal normal subgroup.

Here it is suitable to recall that every irreducible character of a $p$-group is monolithic. We can easily observe that the irreducible characters having a maximal kernel among the kernels of nonlinear irreducible characters of a group $G$ must be monolithic. For a solvable group $G$, we also note by Lemma 2 of [2] that $D_m := \bigcap_{i=1}^{n} ker(\chi_i) \leq Z(G)$, where $\chi_i$ all nonlinear monolithic characters of $G$ and $D_m \cap G' = 1$. Thus, monolithic characters are an important part of $Irr(G)$. Details of the concepts of monolithic characters can be found in Chapter 30 of [1]. We now are ready to present our results:

**Theorem 1.** If a group $G$ has a faithful imprimitive monolithic character of degree $q$, where $q$ is the smallest prime dividing $|G|$, then the group $G$ is solvable and the following hold:

(i) $G$ is a nonabelian $q$-group having cyclic center and there is an abelian subgroup $T$ of $G$ with $|G:T| = q$, or
(ii) $G = N \rtimes Q$ is a Frobenius group, where $Q$ is a cyclic Frobenius complement of order $q$ and $N$ is the Frobenius kernel, which is an abelian $r$-group for some prime $r$.

**Theorem 2.** Let $G$ be a group in which its every nonlinear irreducible characters are monolithic. Assume also that $G/F(G)$ is cyclic. For a prime $q$ not dividing $|G/F(G)|$ , $G$ has a primitive faithful character of degree $q$ if and only if the following situation obtains:

$G = Q \rtimes C_n$, where the group $Q = F(G)$ is an extraspecial $q$-group with $|F(G)| = q^3$, $Z(G) \neq 1$ and the action of the cyclic group $C_n$ on the factor group $F(G)/Z(G)$ is irreducible and faithful. Also, $n = |G/F(G)|$ divides $q + 1$. Moreover, the exponent of $F(G)$ is $q$ when $q \neq 2$, otherwise, $F(G) \cong Q_8$ and so $G \cong SL(2,3)$.

## II. THE PROOF OF MAIN RESULTS

We remark that if a solvable group $G$ has a monolithic faithful character, then there exists unique minimal normal subgroup $M$ in $G$ which becomes an elementary abelian $r$-group for some prime $r$. In consequence of this, the Fitting subgroup denoted by $F(G)$ also becomes a $r$-group. Furthermore, we know that $F(G) > Z(G)$ when $G$ is a nonabelian group.

***Proof of Theorem 1.*** Assume that $\theta$ is a faithful, imprimitive and monolithic character of $G$ with $\theta(1) = q$. By the reason that $\theta$ is imprimitive, there is $T < G$ and $\lambda \in Irr(T)$ with $\lambda^G = \theta$. Therefore, $|G:T| = q$ and $\lambda(1) = 1$ since $|G:T|\lambda(1) = \theta(1) = q$. Thus, we obtain $T \trianglelefteq G$ from Corollary 4.5 of [5] since $|G:T| = q$ and $q$ is the smallest prime which divides $|G|$. On the other side, we can obtain by Frobenius Reciprocity (see Lemma 5.2 of [4]) that $\lambda$ is an irreducible constituent of the restriction character $\theta_T$. By Clifford's Theorem 6.2 of [4], we have that the character $\theta_T$ can be written as the sum of some linear characters of $T$. Thus, we get that $T' \leq \ker(\theta_T) = \ker(\theta) \cap T = 1$ since $\theta$ is faithful. This implies that $T \trianglelefteq G$ is abelian, hence we get that $F(G) \geq T$. Since both $G/T$ and $T$ are solvable, then $G$ is a solvable from Corollary 8.4 of [5]. By Ito's Theorem 6.15 of [4], we conclude that $\varphi(1) = q$ for every $\varphi \in Irr(G)$ satisfying $\varphi(1) \neq 1$. Suppose that $T < F(G)$. Because the reason that $F(G) = G$ has only one minimal normal subgroup, then $G$ becomes a $q$- group. Now let $T = F(G)$. Because the fact that the minimal normal subgroup of $G$ is unique, we conclude that $F(G)$ is a $r$-group, where $r$ is a prime such that $q \neq r$. Resulting from $|G/F(G)| = q$, we obtain the semiproduct group $G = F(G) \rtimes C_q$, where $C_q$ is the Hall $q$-subgroup of $G$. It is easily seen that $Z(G)G' \leq F(G)$, which leads to fact that $Z(G) \cap G' = 1$ by Corollary 9.16 of [5]. Since $1 \neq G'$ and the group $G$ has the unique minimal normal subgroup, we get the result $Z(G) = 1$. Thus, for all $1 \neq x \in F(G)$, the centralizer $C_G(x)$ of $x$ is equal to $F(G)$, which leads to fact that $G$ is a Frobenius group. The proof is complete.

**Remark 2.1.** We know that every nilpotent group is an $M$-group from Corollary 6.14 of [4]. Thus, irreducible characters of $p$-groups must be monomial and monolithic. Now we can say that for the group $G$ as in the case (i) of Theorem 1, the set $Irr(G)$ contains a faithful imprimitive monolithic character of degree $q$ because $Z(G)$ is cyclic and $G$ has an abelian normal subgroup of index $q$. But for the group $G$ as in the case (ii) of Theorem 1, we cannot say whether $G$ has a monolithic faithful character or not. Because the Frobenius group $E(5^2) \rtimes C_3$ has a faithful monolithic character but a Frobenius group $E(7^2) \rtimes C_3$ does not, where the notation $E(p^n)$ is denoted to mean an elementary abelian $p$-group whose order is $p^n$. However, by the following corollary, we gives a complete classification when $q = 2$.

**Corollary 2.2.** A group $G$ has a faithful imprimitive monolithic character of degree 2 if and only if $G$ is a 2-group as in Theorem A or $G$ is a dihedral group with $|G| = 2r^a$, where $a$ is a positive integer and $r$ is a prime not equal to 2.

***Proof.*** By considering the proof of above theorem, we can say that $G$ has a normal and abelian subgroup $K$ whose index in $G$ is equal to 2. It is obviously seen that $G$ is a 2-group as in (i) of Theorem A when $K < F(G)$. When we consider the case $K = F(G)$, then we get a Frobenius group $G = K \rtimes C_2$, in which $C_2$ is a Hall 2-subgroup of $G$. Since $F(G)$ is an abelian $r$-group for some prime $r \neq 2$, then it can be obtained by Theorem 7.12 of [5] that there is a nontrivial and cyclic subgroup $U$ of $F(G)$ satisfying the property that $F(G) = U \times V$ for some $V \leq F(G)$. Suppose that $V > 1$. Since $V \trianglelefteq F(G)$, we have $F(G) \leq N_G(V)$, where $N_G(V)$ is the the normalizer of $V$ in $G$. Let $1 \neq \sigma \in C_2$. Then for all $1 \neq x \in V$ we get $x^\sigma \neq x$ because $Z(G) = 1$. This leads to fact that for all $x \in V$, we have $x^\sigma = x^{-1}$. Then we conclude $N_G(V) = G$, which means that $V \trianglelefteq G$. Similarly, it can be obtained that $U \trianglelefteq G$, which leads to contradiction by the reason that $G$ has only one minimal normal subgroup and $U \cap V = 1$. Therefore, we have $V = 1$. This shows us that $F(G)$ must be cyclic, as the desired result. Conversely, let $G$ be a dihedral group as in the theorem, which leads to fact that $G$ is an $M$-group. Also, the minimal normal subgroup of $G$ needs to be unique because of the fact that the cyclic group $F(G)$ contains every minimal normal subgroup of $G$. Therefore, the set $Irr(G)$ contains a faithful monomial and monolithic character of degree 2, and so we are done.

Before the proof of Theorem 2, we want to recall that for an extraspecial $r$-group $R$, we can view $R/Z(R)$ as a symplectic vector space over the prime field $GF(r)$, where $r$ is a prime. This fact has a substantial role to prove the following main theorem.

***Proof of Theorem 2.*** It is obvious that $G$ is a solvable group because the factor group $G/F(G)$ is cyclic and $F(G)$ is nilpotent. Now, let $\theta$ be a primitive faithful character of $G$ with $\theta(1) = q$ and $q \nmid |G/F(G)|$. Then $Z(G)$ becomes a cyclic group. By the reason that the group $G$ is nonabelian, we get that $F(G)$ is a nonabelian group from Corollary 6.13 of [4]. Since $\theta$ is faithful and monolithic, then $G$ has a unique minimal normal subgroup $N$, where $N$ is cyclic of order $r$ which is prime. Therefore, $F(G)$ is a $r$-group. By the reason that $F(G)$ is not abelian and $\theta$ is faithful, we have $\theta_{F(G)} \in Irr(F(G))$ by Clifford's Theorem. Thus, $r = q$ because of the fact that $\theta(1) = q$. Also, it follows from Corollary 16.3(d) of [6] that $|F(G)/Z(G)| = q^2$. Since $q \nmid |G/F(G)|$, it is obviously seen that that $F(G)$ becomes the Sylow subgroup of $G$, and hence $G = F(G) \rtimes H$, where $H$ is a Hall $q'$-subgroup of $G$. Now, assume that $N < Z(G)$. By Corollary 1.4(i) of [6], there is $S \leq F(G)$ sayisfying the property that $Z(G)S = F(G)$ and $S \cap Z(G) = N$. By Theorem 1.9 of [6] we see that $S \trianglelefteq G$. Furthermore, we have from Corollary 1.10 (ii) of [6] that $S$ is an extraspecial $q$-group whose exponent is $q$ or 4 and $|S| = q^3$. Since $S \cap Z(G) = N$, it is obtain the direct product group $F(G)/N = S/N \times Z(G)/N$, which becomes an abelian $q$-group. By the considering that $(|G/N: F(G)/N|, |Z(G)/N|) = 1$ and that the factor group $F(G)/N$ splits over the abelian group $Z(G)/N$, we get by Gaschütz's Lemma that G/N also splits over $Z(G)/N$, that is, we get the direct product group $G/N = SH/N \times Z(G)/N$. It follows from Corollary 16.3 of [6] that $S/N \cong F(G)/Z(G)$ becomes a faithful and irreducible $G/F(G)$-module. Then the group $S/N$ becomes the unique minimal normal subgroup of the semiproduct group $SH/N = S/N \rtimes HN/N$. This leads to fact that $Irr(SH/N)$ contains a nonlinear and faithful character $\alpha$. Also, we observe that $Z(SH/N)$ is trivial. Furthermore, $Irr(Z(G)/N)$ contains a faithful character $\beta$ because the factor group $Z(G)/N$ is cyclic. By Lemma 2.27(f) of [4], $(|Z(\alpha)/N|, |Z(\beta)/N|) = 1$, and so we obtain that $\alpha \times \beta$ is a faithful and nonlinear irreducible character of the factor group $G/N$. Since from our assumption all nonlinear irreducible characters of $G$ are monolithic, then $\alpha \times \beta$ is a faithful monolithic character of the factor group $G/N$. This gives a contradiction because the group $G/N$ does not have a unique minimal normal subgroup. Thus, we get the conclusion $N = Z(G)$. Hence $F(G) = S$ is an extraspecial $q$-group whose order equals $q^3$ and of exponent $q$ or 4. By Satz II. 9.23 of [3], we know that the order of the factor group $G/F(G)$ divides $q + 1$ since $F(G)/Z(G)$ is a symplectic vector space of order $q^2$. We observe that if $q = 2$, then $G \cong SL(2,3)$ and if $q \neq 2$, the exponent of the group $F(G)$ is $q$. Conversely, let $G = F(G) \rtimes C_n$ be a group as in the theorem. It is seen easily that $Z(G)$ is the unique minimal normal subgroup of $G$ and $Z(G) = Z(F(G))$. If the group $F(G)$ is isomorphic to □$_8$ , then $G \cong SL(2,3)$. It is well-known that all irreducible characters of $SL(2,3)$ are monolithic. Also, $SL(2,3)$ has 3 faithful primitive characters whose degree is 2. From now on, it can be assumed that $F(G)$ is an extraspecial $q$-group of exponent $q \neq 2$ and $|F(G)| = q^3$. We can consider that all nonlinear irreducible characters of in $Irr(G)$ are monolithic because $Z(G)$ is the unique and also minimal normal subgroup and observing the action of $C_n$ on the factor group $F(G)/Z(G)$ is irreducible and faithful. Therefore, it is seen that the set $Irr(F(G))$ has exactly $q$-1 faithful irreducible characters of degree $q$, which are all nonlinear irreducible characters of $F(G)$. Let $\varphi \in Irr(F(G))$ with $\varphi(1) \neq 1$. We know from Corollary 11.22 of [4] that $\varphi$ is extendible to $G$ because $G/F(G)$ is cyclic and $\varphi$ is invariant in $G$. Now let $\theta \in Irr(G)$ and $\theta_{F(G)} = \varphi$. Then $\theta$ becomes a primitive and faithful character of $G$ since the group $G$ does not have any subgroups of index $q$.

Finally, we give examples of groups in Thoerem 2. For instance, the groups $He_3 \rtimes C_4$, and $He_5 \rtimes C_3$ are the examples satisfying Theorem 2, where $He_p$ is the Heisenberg group of order $p^3$ and the action of $C_n$ on the group $He_p/Z(He_p)$ is irreducible and faithful and also, $Z(He_p \rtimes C_n)$=Z($He_p$).

# III. CONCLUSION

The purpose of this study is to point out the relation between the structure of finite groups and the set of their monolithic characters. To do this, we started by giving some fundamental definitons and theorems in the character theory of finite groups. Then, we have presented main theorems of this article, by which a classification of finite groups under some conditions related to their monolithic characters is obtained.

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