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Hedronometric Proportions for Rectangular Tetrahedra

Baki Karliga

Abstract.
The main aim of this paper is to obtain hedronometric proportions, which are three dimensional versions of trigonometric proportions for right triangle, as well as for dihedral and vertex angles of rectangular tetrahedron in terms of hypotenuse and perpendicular face area.

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1. INTRODUCTION

"Hedronometry" is an extension of trigonometry toward three dimensional Euclidean space, focusing on the relationships between the angles and the faces of a tetrahedron. Tetrahedra have more degrees of freedom than triangles, and consequently are not uniquely determined by the areas of their faces in the way that triangles are uniquely determined by the length of their edges. Yet there do exist a number of Hedronometric analogues to trigonometric formula that provide vaguely familiar relationships between the faces and angles.

Let T be a right triangle with right vertex A_i generated by the points $A_i,\ A_j,\ A_k$, and let $\phi_i = \frac{\pi}{2},\ \phi_j,\ \phi_k$ be internal angles of T. Then it follows from the elementary geometry in that the trigonometric proportions for T is given by

$$cos\phi_p = \frac{\|\overrightarrow{A_iA_p}\|}{\|\overrightarrow{A_pA_m}\|}$$
$$sin\phi_p = \frac{\|\overrightarrow{A_iA_m}\|}{\|\overrightarrow{A_pA_m}\|} \tag{1.1}$$

where $(p,\ m)$ is a permutation $\{j,\ k\}$.

In this paper we give two new hedronometric analogues to trigonometric proportions for right triangles. That is, we derive three dimensional versions of formula

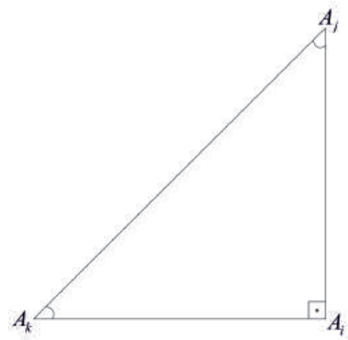

FIGURE 1. *Right Triangle*

(1.1) for dihedral and vertex angles of a rectangular tetrahedron in terms of the hypotenuse and perpendicular face areas [Theorem 2.2, Theorem 3.3].

A characterization which relates to vertex angles of a rectangular tetrahedron is also given [Corollary 3.2].

2. Dihedral Angles and Hedronometric Proportions

Let $A_i,\ A_j,\ A_k,\ A_\ell$ be independent points in a three dimensional Euclidean space R^3. A tetrahedron T is given by the convex hull of A_i and linear independent set $\{\overrightarrow{A_iA_j},\ \overrightarrow{A_iA_k},\ \overrightarrow{A_iA_\ell}\}$ ([1],[6],[2],[7]). If linear independent set is orthogonal then tetrahedron is called a rectangular tetrahedron with right vertex A_i.

Let S_m be a face of rectangular tetrahedron which is not containing A_m, and let $|S_m|$ be area of S_m for each $m \in \{i,j,k,l\}$. Then, S_i and S_m $(m \neq i)$ are called hypotenuse and perpendicular face of rectangular tetrahedron T.

If (p,q,m,n) is a permutation of $\{i,j,k,l\}$, then the dihedral angle θ_{mn} at common edge $S_m \cap S_n = a_{pq}$ is the angle formed at any normal plane to a_{pq}. We call a_{is} perpendicular edge of T for each $s \in \{j,k,l\}$, and also call perpendicular face S_m of T for each $m \in \{j,k,l\}$. S_p and S_m, S_n are opposite perpendicular face and adjacent perpendicular face at edge a_{ip} for each $p \in \{j,k,l\}$.

Lemma 2.1. *Let T be a tetrahedron with vertices $A_i,\ A_j,\ A_k,\ A_\ell$. Let $S_p, p \in \{i,\ j,\ k,\ \ell\}$ denotes a face of T. Then it follows*

$$| S_p |=| S_q | \cos\theta_{pq}+ | S_m | \cos\theta_{pm}+ | S_n | \cos\theta_{pn} \tag{2.1}$$

where $(p,\ q,\ m,\ n)$ is a permutation of $\{i,\ j,\ k,\ \ell\}$.

Proof. By proof of Theorem 3 in [7] (or [3]), we know that

$$| S_p | N_p+ | S_q | N_q+ | S_m | N_m+ | S_n | N_n = 0$$

where N_t is unit outer normal vector of $S_t,\ t \in \{i,\ j,\ k,\ \ell\}$. By taking the inner product of the sides with N_p and by using

$$\langle N_t,\ N_s\rangle = - \| N_t \| \| N_s \| \cos\theta_{ts}; t,\ s \in \{i,\ j,\ k,\ \ell\}$$

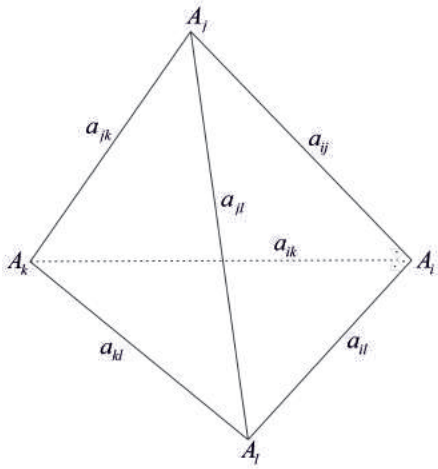

FIGURE 2. Edges of Rectangular Tetrahedron

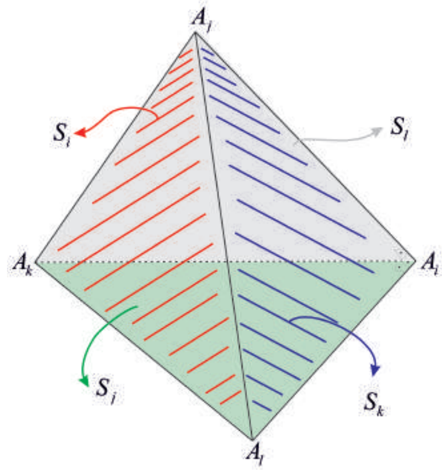

FIGURE 3. Faces of Rectangular Tetrahedron

we obtain equation (2.1). □

Theorem 2.2. *Let T be a rectangular tetrahedron with right vertex A_i generated by points $A_i,\ A_j,\ A_k,\ A_\ell$. Then the hedronometric proportions of T at S_p is given by*

$$\cos\theta_{ip} = \frac{|\ S_p\ |}{|\ S_i\ |}$$

$$\sin\theta_{ip} = \frac{\sqrt{|\ S_m\ |^2 + |\ S_n\ |^2}}{|\ S_i\ |}$$

where $S_m,\ S_n$ adajacent perpendicular faces and S_p opposite perpendicular face at edge a_{ip} of T.

Proof. By using $\theta_{pq} = \frac{\pi}{2},\ p,\ q \in \{j,\ k,\ \ell\}$ in Lemma 2.1, we see that

$$|\ S_p\ | = |\ S_i\ | \cos\theta_{ip} \tag{2.2}$$

or

$$\cos\theta_{ip} = \frac{|\ S_p\ |}{|\ S_i\ |}.$$

By substituting equation (2.2) at De Gua formula,

$$|\ S_i\ |^2 = |\ S_j\ |^2 + |\ S_k\ |^2 + |\ S_\ell\ |^2.$$

Then, we obtain

$$\sin\theta_{ip} = \frac{\sqrt{|\ S_m\ |^2 + |\ S_n\ |^2}}{|\ S_i\ |}$$

for the permutation $(p,\ m,\ n)$ of $\{j,\ k,\ \ell\}$. □

3. Vertex Angles and Hedronometric Proportions

A vertex angle θ_p is defined by the area of spherical triangle which is determined by unit sphere centered at A_p and three edges joining A_p.

Theorem 3.1. *The measure of the vertex angle θ_p and dihedral angles $\theta_{pq},\ \theta_{pm},\ \theta_{pn}$ of a tetrahedron satisfies the following relationships*

$$\theta_p = \theta_{pq} + \theta_{pm} + \theta_{pn} - \pi \tag{3.1}$$

where $(p,\ q,\ m,\ n)$ is a permutation of $\{i,\ j,\ k,\ \ell\}$.

Proof. It can be seen in [4], [8]. □

Corollary 3.2. *Let T be a rectangular tetrahedron with right vertex A_i generated by points $A_i,\ A_j,\ A_k,\ A_\ell$. Then*

$$\theta_p = \theta_{pi},\ p \in \{j,\ k,\ \ell\} \tag{3.2}$$

and

$$\theta_i = \theta_{ij} + \theta_{ik} + \theta_{il} - \pi \tag{3.3}$$

Proof. Since $\theta_{mn} = \frac{\pi}{2}$ for $m \neq n; m,\ n \in \{j,\ k,\ \ell\}$, equation (3.1) reduces to $\theta_p = \theta_{pi}$ for $p \neq i$ and $\theta_i = \theta_{iq} + \theta_{im} + \theta_{in} - \pi$ for $p = i$. □

The proof of Corollary (3.2) can be given by the way of Theorem 2 in [5] (or [10]). For the sake of its clearness, we preferred the above method rather than the other.

Now we are ready to give the following theorem which is another version of equation (1.1) for vertex angles of a rectangular tetrahedron in terms of hypotenuse and perpendicular face areas.

Theorem 3.3. *Let T be a rectangular tetrahedron with right vertex A_i generated by points $A_i,\ A_j,\ A_k,\ A_\ell$. Then the hedronometric proportions of T in terms of hypotenuse and perpendicular face areas for the vertex angle θ_p is given by*

$$\cos\theta_p = \frac{|\ S_p\ |}{|\ S_i\ |}$$

$$\sin\theta_p = \frac{\sqrt{|\ S_m\ |^2 + |\ S_n\ |^2}}{|\ S_i\ |}$$

where $(p,\ m,\ n)$ is a permutation of $\{j,\ k,\ \ell\}$.

Proof. It is evident from the Theorem 2.2 and Corallary 3.2. By substituting equation (3.2) in equation (3.3), we obtain the following equations for a rectangular tetrahedron with right vertex A_i. □

$$\theta_j + \theta_k + \theta_\ell = \frac{3\pi}{2}$$

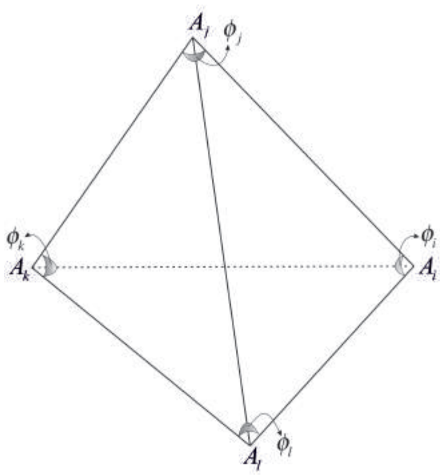

FIGURE 4. Vertex Angle

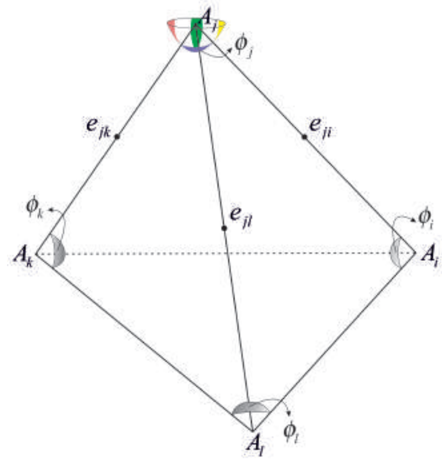

FIGURE 5. Measure of Vertex Angle

$$\theta_i + \theta_j + \theta_k + \theta_\ell = 2\pi \tag{3.4}$$

Vertex angles sum of rectengular tetrahedron is constant (i.e. 2π) by equation (3.4).

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BAKI KARLIGA –Department of Mathematics, Faculty of Sciences, Gazi University, 06500 Ankara, Turkey