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Dihedral Angles and Edge Lengths of Spacelike $n-$simplex in S_1^n

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Abstract.
In this paper, we give necessary and sufficent conditions for $\frac{n(n+1)}{2}$ positive real numbers ϕ_{ij} and θ_{ij} to be dihedral angles and edge lengths of an $n-$simplex having spacelike faces with codimensions 1 and 2 in de Sitter space.

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1. INTRODUCTION

Let ${R_1}^{n+1}$ be the $(n+1)-$Minkowski space in which the scalar product of two vectors $x = (x_0, x_1, ..., x_n)$ and $y = (y_0, y_1, ..., y_n)$ is given by

$$\langle x, y \rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i.$$

In what follows we will take $n-$dimensional hyperbolic space $H^n = \{x \in {R_1}^{n+1} \mid \langle x, x \rangle = -1\, x_0 > 0\}$. The hyperbolic distance between any two points v_i and v_j is given by the real number ϕ_{ij} =arccosh $(-< v_i, v_j >)$ in H^n. (see for detail [5], [12], [6], [7], [1],[2]).

Suppose that $\triangle$ is an $n-$simplex in the hyperbolic space H^n with vertices $v_1, v_2, ..., v_{n+1}$. The face of $\triangle$ opposite to v_i is the intersection of the timelike hyperplane ${u_i}^{\perp}$ in ${R_1}^{n+1}$ with H^n . We call the codimension 2 face $({u_i}^{\perp} \cap {u_j}^{\perp}) \cap H^n$ opposite to $v_i, v_j, i \neq j$ the $ij-$face of $\triangle$. We denote the dihedral angle at the $ij-$face of $\triangle$ in H^n by θ_{ij} and we set $\theta_{ii} = \pi$. The symmetric matrix $G = [-\cos\theta_{ij}]$ is called the **Gram Matrix** of $\triangle$. The hyperbolic distance ϕ_{ij} = arccosh$(-< v_i, v_j >)$ between any two vertices $v_i, v_j, i \neq j$ is called the edge length of $\triangle$. $M = [< v_i, v_j >] = [-\cosh\phi_{ij}]$ is called **Edge Matrix** of $\triangle$ in H^n(see for detail [4]). In what follows we will restrict our attention to the class of $n-$ simplices $\triangle^n$ having spacelike faces with codimensions 1 and 2 in ${S_1}^n$.

Consider the 2−dimensional vector subspace of ${R_1}^{n+1}$ spanned by the unit timelike vectors v_i and v_j. Since v_i, v_j are timelike, this plane is timelike (see [6],p.141), and so the codimension 2 subspace ${v_i}^{\perp} \cap {v_j}^{\perp}$ is spacelike. Thus the codimension 1 spacelike hyperplanes ${v_i}^{\perp} \cap {S_1}^n$ and ${v_j}^{\perp} \cap {S_1}^n$ intersect at the codimension 2 spacelike hyperplane $({v_i}^{\perp} \cap {v_j}^{\perp}) \cap {S_1}^n$ spanned by spacelike vectors, $u_1, ..., \hat{u}_i, ..., \hat{u}_j, ..., u_{n+1}$. Hence, it is also natural to call the matrices $M^* = [< u_i, u_j >]$ and $G^* = [< v_i, v_j >]$ edge matrix and Gram matrix of $\triangle^*$ which has spacelike faces ${v_i}^{\perp} \cap {S_1}^n$ $i = 1, ..., n$.

Given a qxq matrix A, we denote the cofactor of A corresponding to the $(i, j)-$ entry and the determinant of A by $A_{ij}$and $\mid A \mid$.

Let a spacelike $n-$simplex $\triangle^*$ in S_1^n and hyperbolic $n-$simplex $\triangle$, and let M^*, M and G^*, G be edge matrix and Gram matrix of $\triangle^*$ and $\triangle$, respectively.

Two questions were raised by Fenchel in his book [3](p.170-174). The first question is "What are the conditions six numbers have to satisfy in order that they are the dihedral angles of a hyperbolic tetrahedron?" and the other is "What are the conditions six positive numbers have to satisfy in order that they are the edge lengths of a hyperbolic tetrahedron?"

The first question of Fenchel solved by Luo in [5] and the other by Karlığa in [4], for $n-$simplex in hyperbolic space. Recently, the definition of dihedral angle in semi-riemannian geometry appeared in [11],[10]. The duality between polyhedra in the hyperbolic space and the de Sitter space appeared in [11], [8], [9], [10].

The main aim of this paper is to give hyperbolic and de Sitter duality exchanging dihedral angles and edge lengths of $\triangle^*$ and $\triangle$.

2. Main Theorems

Theorem 2.1. *Let $\triangle^*$ be a spacelike $n-$simplex with faces outer normals $v_1, v_2, ...v_{n+1}$. Then, Gram matrix $G^* = [\langle v_i, v_j \rangle]$ of $\triangle^*$ is equal to edge matrix M of $\triangle$ if and only if there exist $\frac{n(n+1)}{2}$ real positive numbers $\phi_{ij} = \phi_{ji}$ such that $G^* = [-\cosh \phi_{ij}]$ which satisfies*
i) $\mid G^* \mid < 0$
ii) all principal submatrices of ${G^*}^{-1}$ are positive definite
iii $G^*_{ij} > 0$
where $i \neq j$ and $i, j = 1, ..., n+1$.

Proof. Suppose that all conditions are sufficient. By Theorem 2.1 in [4] and (i), (ii) and (iii), there exist $\frac{n(n+1)}{2}$ real positive numbers $\phi_{ij} = \phi_{ji}$ which are edge lengths of $\triangle$ in H^n. Namely, $G^* = M$. Conversely, suppose we are given a spacelike $n-$simplex $\triangle^*$ whose face unit outer normals are $v_1, v_2, ...v_{n+1}$. Then, we have hyperbolic simplex $\triangle$ with vertices $v_1, v_2, ...v_{n+1}$ such that $M = [\langle v_i, v_j \rangle] = G^*$. By Theorem 2.1 in [4], there exist $\frac{n(n+1)}{2}$ real positive numbers $\phi_{ij} = \phi_{ji}$ such that $G^* = [-\cosh \phi_{ij}]$ satisfies (i), (ii) and (iii).

□

Theorem 2.2. *Let $\triangle^*$ be a spacelike $n-$simplex with vertices $u_1, u_2, ...u_{n+1}$. Then, edge matrix $M^* = [\langle u_i,\ u_j \rangle]$ of $\triangle^*$ is equal to Gram matrix G of $\triangle$ if and only if*

there exist $\frac{n(n+1)}{2}$ *real positive numbers* $\theta_{ij} = \theta_{ji} \in (0, \pi)$ *such that* $M^* = [-\cos\theta_{ij}]$ *which satisfies*
i) $\mid M^* \mid < 0$
ii) all principal submatrices of M^* *are positive definite*
iii $M^*_{ij} > 0$
where $i \neq j, i, j = 1, ..., n+1$.

Proof. Suppose that all conditions are sufficient. By main theorem of [5], there exists hyperbolic $n-$simplex $\triangle$ with Gram matrix $G = [-\cos\theta_{ij}] = M^*$.

Conversely, suppose we are given a spacelike $n-$simplex $\triangle^*$ with vertices $u_1, u_2, ...u_{n+1}$. By main theorem of [5], we have hyperbolic $n-$simplex $\triangle$ whose Gram matrix $G = [\langle u_i, u_j \rangle] = M^*$ satisfies (i), (ii) and (iii). □

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