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On Morgan-Voyce Polynomials Approximation For Linear Differential Equations

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Abstract. In this paper, a matrix method for approximately solving certain linear differential equations is presented. This method is called Morgan-Voyce matrix method and converts a linear differential equation into a matrix equation. Then, the equation reduces to a matrix equation corresponding to a system of linear algebraic equations with unknown Morgan-Voyce coefficients. The examples are included to demonstrate the applicability of the technique.

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1. INTRODUCTION

In this study, we consider the high order linear differential equations with variable coefficients in the form

$$\sum_{k=0}^{m} f_k(x)y^{(k)} = g(x), \quad a \leq x \leq b \tag{1.1}$$

under the conditions

$$\sum_{k=0}^{m-1} [a_{jk}y^{(k)}(a) + b_{jk}y^{(k)}(b)] = \lambda_j, \quad j = 0, 1, 2, ..., m-1. \tag{1.2}$$

Our aim is to find an approximate solution of (1.1) expressed in the truncated Morgan-Voyce series form

$$y(x) = \sum_{n=0}^{N} a_n B_n(x) \tag{1.3}$$

where a_n, $n = 0, 1, ..., N$ are the unknown Morgan-Voyce coefficients and $B_n(x)$, $n = 0, 1, ..., N$ are the Morgan-Voyce polynomials formed

$$B_n(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n+k+1 \\ n-k \end{array} \right), \qquad n \in \mathbb{N} \tag{1.4}$$

2. Fundamental Matrix Relations

We first write (1.4)

$$\mathbf{B^T}(x) = \mathbf{R}\mathbf{X^T}(x) \Leftrightarrow \mathbf{B}(x) = \mathbf{X}(x)\mathbf{R^T} \tag{2.1}$$

where

$$\mathbf{B}(x) = [\ B_0(x) \quad B_1(x) \quad B_2(x) \quad \ldots \quad B_n(x)\ ], \quad \mathbf{X}(x) = [\ x^0 \quad x^1 \quad x^2 \quad \ldots \quad x^n\ ]$$

and

$$\mathbf{R} = \begin{pmatrix} \left(\begin{array}{c} 1 \\ 0 \end{array} \right) & 0 & 0 & \ldots & 0 \\ \left(\begin{array}{c} 2 \\ 1 \end{array} \right) & \left(\begin{array}{c} 3 \\ 0 \end{array} \right) & 0 & \ldots & 0 \\ \left(\begin{array}{c} 3 \\ 2 \end{array} \right) & \begin{array}{c} 4 \\ 1 \end{array}) & \left(\begin{array}{c} 5 \\ 0 \end{array} \right) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ \left(\begin{array}{c} n+1 \\ n \end{array} \right) & \left(\begin{array}{c} n+2 \\ n-1 \end{array} \right) & \left(\begin{array}{c} n+3 \\ n-2 \end{array} \right) & \ldots & \left(\begin{array}{c} 2n+1 \\ 0 \end{array} \right) \end{pmatrix}$$

Then, we write the solution expressed by (1.3)

$$[\mathbf{y}(x)] = \mathbf{B}(x)\mathbf{A}, \quad \mathbf{A} = [\ a_0 \quad a_1 \quad a_2 \quad \ldots \quad a_N\ ]^T$$

or using (2.1) we can write

$$\mathbf{y}(x) = \mathbf{X}(x)\mathbf{R^T}\mathbf{A} \tag{2.2}$$

and the relation between the matrix $\mathbf{X}(x)$ and its derivative $\mathbf{X}^{(1)}(x)$ is

$$\mathbf{X}^{(1)}(x) = \mathbf{X}(x)\mathbf{T^T}, \quad \mathbf{X}^{(0)}(x) = \mathbf{X}(x) \tag{2.3}$$

where

$$\mathbf{T^T} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & N \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

So, using (2.3) the relation between the matrix $\mathbf{X}(x)$ and its derivatives is

$$\begin{aligned}\mathbf{X}^{(1)}(x) &= \mathbf{X}(x)\mathbf{T}^{\mathbf{T}}\\ \mathbf{X}^{(2)}(x) &= \mathbf{X}^{(1)}(x)\mathbf{T}^{\mathbf{T}} = \mathbf{X}(x)\mathbf{T}^{\mathbf{T}}\mathbf{T}^{\mathbf{T}} = \mathbf{X}(x)(\mathbf{T}^{\mathbf{T}})^2\\ &\vdots\\ \mathbf{X}^{(k)}(x) &= \mathbf{X}^{(k-1)}(x)\mathbf{T}^{\mathbf{T}} = \mathbf{X}(x)(\mathbf{T}^{\mathbf{T}})^{k-1}\mathbf{T}^{\mathbf{T}} = \mathbf{X}(x)(\mathbf{T}^{\mathbf{T}})^k\end{aligned} \tag{2.4}$$

We have from (2.1) and (2.4)

$$\mathbf{y}^{(k)}(x) = \mathbf{X}(x)(\mathbf{T}^{\mathbf{T}})^k\mathbf{R}^{\mathbf{T}}\mathbf{A}, \quad k = 0, 1, 2, ...m \tag{2.5}$$

3. Method of Solution

To construct the fundamental matrix equation defined in (1.1), we substitute the matrix formula (2.5) into (1.1). Thus, we obtain the matrix equation

$$\sum_{k=0}^{m} f_k(x)\mathbf{X(x)}((\mathbf{T})^T)^k\mathbf{R}^T\mathbf{A} = g(x) \tag{3.1}$$

We define the collocation points as

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, ..., N. \tag{3.2}$$

Substituting (3.2) into (3.1) we get

$$\sum_{k=0}^{m} f_k(x_i)\mathbf{X}(x_i)(\mathbf{T}^T)^k\mathbf{R}^T\mathbf{A} = g(x_i), \quad i = 0, 1, 2, ..., N. \tag{3.3}$$

So we have the system of the matrix equations

$$\sum_{k=0}^{m} \mathbf{M}_k\mathbf{Y}^k = \mathbf{G} \tag{3.4}$$

In this equations, we can write

$$\mathbf{M}_k = \begin{pmatrix} f_k(x_0) & 0 & \dots & 0\\ 0 & f_k(x_1) & \dots & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & \dots & f_k(x_N)\end{pmatrix}$$

$$\mathbf{Y}^k = \begin{pmatrix} y^k(x_0)\\ y^k(x_1)\\ \vdots\\ y^k(x_N)\end{pmatrix} = \begin{pmatrix} X(x_0)\\ X(x_1)\\ \vdots\\ X(x_N)\end{pmatrix}((\mathbf{T}^T)^k\mathbf{R}^{\mathbf{T}}\mathbf{A}), \quad G = \begin{pmatrix} g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{pmatrix}$$

where

$$\mathbf{X} = \begin{pmatrix} X(x_0) \\ X(x_1) \\ \vdots \\ X(x_N) \end{pmatrix} = \begin{pmatrix} 1 & x_0 & \dots & x_0^N \\ 1 & x_1 & \dots & x_1^N \\ \vdots & \vdots & & \vdots \\ 1 & x_N & \dots & x_N^N \end{pmatrix}$$

Therefore, the fundamental matrix relation corresponding to equation (1.1) can be written in the matrix form

$$\mathbf{WA} = \mathbf{G} \ \ or \ \ [\mathbf{W}; \mathbf{G}] \tag{3.5}$$

(3.5) can be written

$$\mathbf{W} = [w_{ij}] = \sum_{k=0}^{m} \mathbf{M}_k \mathbf{X} (\mathbf{T}^T)^k \mathbf{R}^T, \quad i, j = 0, 1, 2, ..., N \tag{3.6}$$

So, (3.4) corresponds to a linear system of $(N + 1)$ algebric equation with $(N + 1)$ unknown Morgan-Voyce coefficients.
For the condition (1.2), the condition matrix can be obtained

$$\mathbf{U}_j = \sum_{k=0}^{m} (a_{jk}\mathbf{X}(a)+b_{jk}\mathbf{X}(b))(\mathbf{T}^T)^k \mathbf{R}^T = \begin{pmatrix} u_{j0} & u_{j1} & u_{j2} & \dots & u_{jN} \end{pmatrix}, \quad j = 0, 1, 2, ..., m-1.$$

The matrix form of the condition is then,

$$\mathbf{U}_j \mathbf{A} = [\lambda_j], \quad j = 0, 1, 2, ..., m - 1$$

or the augmented matrix for the conditions is

$$\widetilde{\mathbf{U}}_j = [\mathbf{U}_j; \lambda_j], \quad j = 0, 1, 2, ..., m - 1. \tag{3.7}$$

Under the conditions (1.2) to obtain the solution of equation (1.1), we replace the last m rows of the matrix (3.5) by the rows matrices (3.7) and we get the new augmented matrix,

(3.8)

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = \begin{pmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & \dots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1-m)0} & w_{(N-1-m)1} & \dots & w_{(N-1-m)N} & ; & g(x_{N-1-m}) \\ w_{(N-m)0} & w_{(N-m)1} & \dots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & \dots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \dots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(m-1)1} & u_{(m-1)2} & \dots & u_{(m-1)N} & ; & \lambda_{m-1} \end{pmatrix}$$

This augmented matrix system can be written

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{G}}$$

If $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = N+1,$ we can write
$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}$ Thereby, we uniquely determine the coefficients $a_n(n = 0, 1, ..., N)$ by means of the equation (3.8) and the coefficients matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} a_0 & a_1 & \dots & a_N \end{pmatrix}^T$$

So Eq. (1.1) with the conditions (1.2) has a unique solution and this solution is given by Morgan-Voyce series solution

$$y(x) = \sum_{n=0}^{N} a_n B_n(x)$$

4. Numerical Examples

In this section, we give several numerical examples to show the applicability of the method. We performed all calculations on a Intel PC using MATLAB.
Example 1: Let us consider nonhomogeneous fourth order linear differential equation given by

$$2y^{(4)}(x) - (x^2+1)y^{''}(x) + 12y(x) = 30x^3 - 12x^2 + 54x + 168, \quad 0 \le x \le 2 \tag{4.1}$$

under the conditions

$$y(0) = 10, \quad y^{'}(1) = 26. \tag{4.2}$$

The exact solution is $y_{exact}(x) = x^4 + 5x^3 + 7x + 10$. The approximate solution $y(x)$ by the truncated Morgan-Vyce series is

$$y(x) = \sum_{n=0}^{N} a_n B_n(x).$$

We will consider for $N = 4$ and $N = 6$.
For $N = 4$ the Morgan-Voyce collocation points are

$$\{x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_2 = 2\}$$

The functions for this example are

$$f_0(x) = 12,\ f_1(x) = 0,\ f_2(x) = -(x^2+1),\ f_3(x) = 0,\ f_4(x) = 2,\ g(x) = 30x^3-12x^2+54x+168$$

The matrix form of the differential equation is

$$\sum_{k=0}^{m} \mathbf{M}_k \mathbf{X}(\mathbf{T}^T)^k \mathbf{R}^T \mathbf{A} \tag{4.3}$$

Since the equation (4.1) is a 4th order, the formula (4.3) turns out to be

$$\{\mathbf{M}_0\mathbf{X}+\mathbf{M}_1\mathbf{X}\mathbf{T}^T+\mathbf{M}_2\mathbf{X}(\mathbf{T}^T)^2+\mathbf{M}_3\mathbf{X}(\mathbf{T}^T)^3+\mathbf{M}_4\mathbf{X}(\mathbf{T}^T)^4\}\mathbf{R}^T\mathbf{A}=\mathbf{G}$$

where

$$\mathbf{M}_0=\begin{pmatrix} 12&0&0&0&0\\ 0&12&0&0&0\\ 0&0&12&0&0\\ 0&0&0&12&0\\ 0&0&0&0&12 \end{pmatrix},\ \mathbf{M}_1=\mathbf{M}_3=\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\mathbf{M}_2=\begin{pmatrix} -1&0&0&0&0\\ 0&-\frac{5}{4}&0&0&0\\ 0&0&-2&0&0\\ 0&0&0&-\frac{13}{4}&0\\ 0&0&0&0&-5 \end{pmatrix},\mathbf{M}_4=\begin{pmatrix} 2&0&0&0&0\\ 0&2&0&0&0\\ 0&0&2&0&0\\ 0&0&0&2&0\\ 0&0&0&0&2 \end{pmatrix},$$

$$\mathbf{R}^T=\begin{pmatrix} 1&0&0&0&0\\ 2&1&0&0&0\\ 3&4&1&0&0\\ 4&10&6&1&0\\ 5&20&21&8&1 \end{pmatrix},\mathbf{T}^T=\begin{pmatrix} 0&1&0&0&0\\ 0&0&2&0&0\\ 0&0&0&3&0\\ 0&0&0&0&4\\ 0&0&0&0&0 \end{pmatrix},\ \mathbf{G}=\begin{pmatrix} 168\\ \frac{783}{4}\\ 240\\ \frac{1293}{4}\\ 468 \end{pmatrix}$$

Substituting these matrix into (3.6) we have

$$[\mathbf{W};\mathbf{G}]=\begin{pmatrix} 12&24&34&36&18&;&168\\ 12&30&\frac{121}{2}&\frac{435}{4}&\frac{339}{2}&;&\frac{783}{4}\\ 12&36&92&216&456&;&240\\ 12&42&\frac{257}{2}&\frac{1449}{4}&\frac{1827}{2}&;&\frac{1293}{4}\\ 12&48&170&552&1578&;&468 \end{pmatrix}$$

From section 3, the condition matrix is

$$\begin{pmatrix} \mathbf{U}_0&;&\lambda_0\\ \mathbf{U}_1&;&\lambda_1 \end{pmatrix}=\begin{pmatrix} 1&2&3&4&5&;&10\\ 0&1&6&25&90&;&26 \end{pmatrix}$$

So, with the conditions, the new augmented matrix can be written

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{pmatrix} 12&24&34&36&18&;&168\\ 12&30&\frac{121}{2}&\frac{435}{4}&\frac{339}{2}&;&\frac{783}{4}\\ 12&36&92&216&456&;&240\\ 1&2&3&4&5&;&10\\ 0&1&6&25&90&;&26 \end{pmatrix} \tag{4.4}$$

Because $det\tilde{\mathbf{W}}\neq 0$ we have $\mathbf{A}=(\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}$.
So, we can obtain the coefficient matrix A as

$$\mathbf{A}=\begin{pmatrix} -32&29&-3&-3&1 \end{pmatrix}^T$$

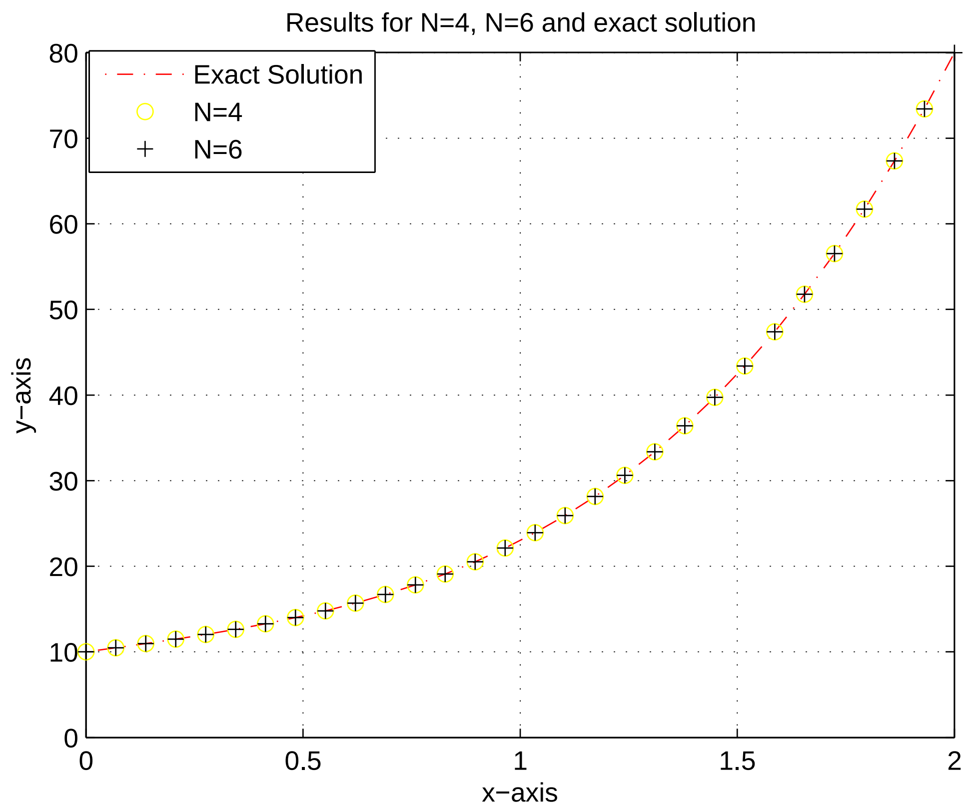

Figure 1. Comparing the exact solution and the approximate solutions

Therefore, for $N = 4$ the approximate solution $y(x)$ by the truncated Morgan-Voyce series is

$$y(x) = 10 + 7x + 5x^3 + x^4.$$

For $N = 6$ the similar calculations show that $\mathbf{A}$ is

$$\mathbf{A} = \begin{pmatrix} -32 & 29 & -3 & -3 & 1 & -3.4e-014 & 5e-015 \end{pmatrix}^T$$

and hence the approximate solution $y(x)$ is

$$y(x) = 10-7x+6.017915496.10^{(-13)}x^2+5x^3+x^4+2.307511219.10^{(-14)}x^5+4.772254196.10^{(-15)}x^6$$

In figure 1, we compare the exact solution and the approximate solutions.

Example 2: Let us now consider a second order differential equation,

$$y^{''}(x) + y(x) = cos^2(x), \quad 0 \le x \le 1 \tag{4.5}$$

under the conditions

$$y(0) = 1, \quad y^{'}(0) = -1. \tag{4.6}$$

The exact solution is $y_{exact}(x) = sin^2(x) - sin(x) + 1$. The approximate solution $y(x)$ by the truncated Morgan-Voyce series is

$$y(x) = \sum_{n=0}^{N} a_n B_n(x).$$

We will solve this problem with Morgan-Voyce collocation method for $N = 3, \ N = 5, \ N = 8$. As in the previous example, by the Morgan-Voyce polynomials, we obtain the approximate solutions of the problem for $N = 3, \ N = 5, \ N = 8$, respectively,

$$y_3(x) = 1 - x + x^2 - 0.04857392616x^3 \tag{4.7}$$

$$y_5(x) = 1 - x + x^2 + 0.1760202929x^3 - 0.377080708x^4 + 0.06587478908x^5 \tag{4.8}$$

$$\begin{aligned} y_8(x) =& 1 - x + x^2 + 0.16665279x^3 - 0.3331996609x^4 - 0.008906179133x^5 + 0.0457548307x^6 \\ & - 0.001405858856x^7 - 0.002295849228x^8 \end{aligned} \tag{4.9}$$

In figure 2, we compare the exact solution and the approximate solutions.In Table 4.1, we illustrate the exact solutions of the differential equation (4.5) and its numerical results of the approximate solutions for $N = 3, 5$ and 8 using the present method.

Table 4. 1: Numerical solutions of Example 2

Exact Solution		Present Method					
		$N=3$	$N=3$	$N=5$	$N=5$	$N=8$	$N=8$
t_i	$y_{exact}(x_i)$	$y_3(t_i)$	$E_3(t_i)$	$y_5(t_i)$	$E_5(t_i)$	$y_8(t_i)$	$E_8(t_i)$
0.0	1	1	0	1	0	1	0
0.1	0.910133294	0.90995142	0.000181869	0.91013897	5.676537e-006	0.910133289	5.079156e-009
0.2	0.840800172	0.83961141	0.001188764	0.84082591	2.574094e-005	0.840800157	1.488156e-008
0.3	0.791811986	0.78868850	0.003123482	0.79185827	4.628403e-005	0.791811963	2.264282e-008
0.4	0.762228303	0.75689127	0.005337034	0.76228659	5.828744e-005	0.762228272	3.123148e-008
0.5	0.750423308	0.74392826	0.006495049	0.75049358	7.027106e-005	0.750423270	3.881956e-008
0.6	0.754178649	0.74950803	0.004670617	0.75427315	9.449774e-005	0.754178603	4.638895e-008
0.7	0.770798741	0.77333914	0.002540402	0.77090946	1.107170e-004	0.770798687	5.403538e-008
0.8	0.797243670	0.81513015	0.017886480	0.79725599	1.231260e-005	0.797243615	5.487016e-008
0.9	0.830274138	0.87458961	0.044315470	0.82981455	4.595925e-004	0.830273864	27.38042e-008
1.0	0.866602433	0.95142607	0.084823640	0.86481438	1.788059e-003	0.866600073	236.0883e-008

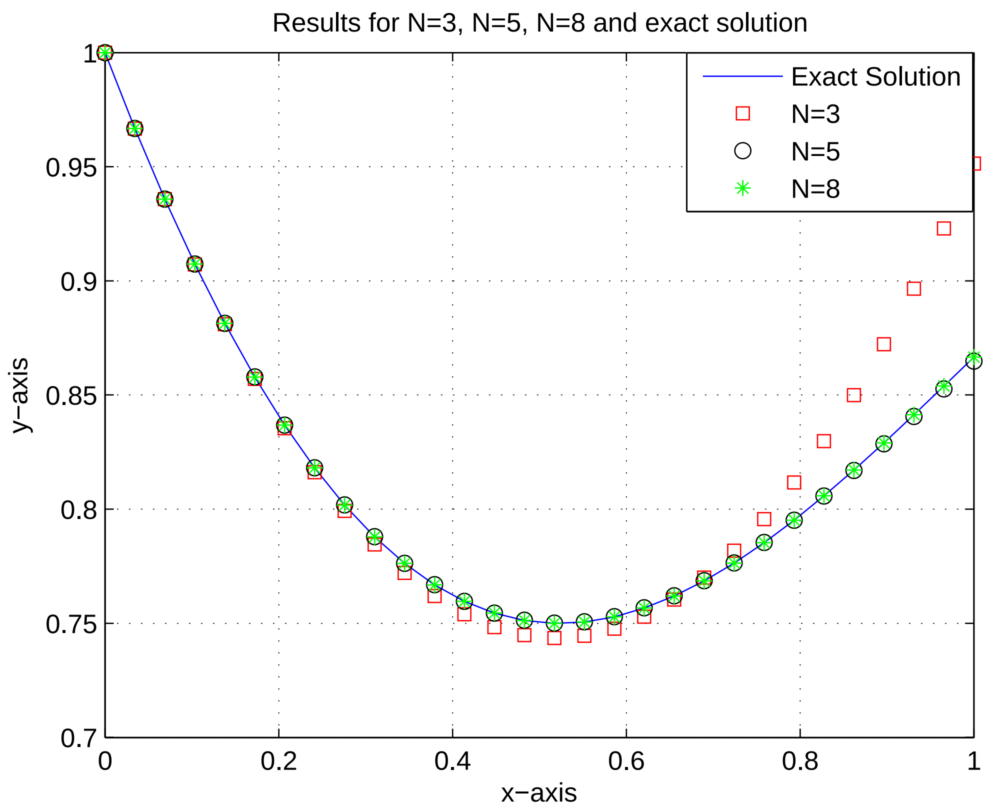

FIGURE 2. Comparing the exact solution and the approximate solutions

REFERENCES

[1] Ş. Yüzbaşı, N. Şahin, M. Sezer. Numerical solutions of systems of linear Fredholm integro-differential equations with Bessel Polynomial bases. *Computers&Mathematics with Applications*, pp. 3079-3096, 22 April 2011.

[2] M. N. S. Swamy. Further properties of Morgan-Voyce Polynomials. *Fibonacci Quarterly*, Vol. 6, No. 2, pp. 167-175, Apr. 1968.

[3] H. H. Sorkun, S. Yalçınbas. Approximate solutions of linear Volterra integral equation systems with variable coefficients. *Appl. Math. Modell.*, doi:10.1016/j.apm.2010.02.034., (2010).

[4] A. Akyüz-Daşçıoğlu, M. Sezer. Chebyshev polynomial solutions of systems of higher-order linear Fredholm-Volterra integro-differential equations. *J. Franklin Ins.*, vol. 342, pp. 688-701, (2005).

[5] M. Sezer and A. Akyüz-Daşcıoğlu. A Taylor method for numerical solution of generalized pantograph equations with linear functional argument. *J. Comput. Appl. Math.*, vol. 200, pp. 217-225, (2007).

[6] M. Sezer. A method for the approximate solution of the second order linear differential equations in terms of Taylor polynomials. *Int J Math Educ Sci Technol.*, vol. 27, pp. 821-834, (1996).

[7] M. Sezer, S. Yalçınbaş, and N. Şahin Approximate solution of multi-pantograph equation with variable coefficients *J Comput Appl Math*, vol. 214, pp. 406-416, (2008).

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