Turkish **J**ournal of **M**athematics and **C**omputer **S**cience
TJMCS

http://tjmcs.matder.org.tr/

@TJMCS

http://www.matder.org.tr/

Matrix Representation on Quaternion Algebra

Gülay Koru Yücekaya

Abstract. The quaternions, denoted by $\mathbb{H}$, were first defined by W.R. Hamilton in 1843 as an extension of the four dimensions complex numbers. Hamilton has included a new multiplication process to vector algebra by defining quaternions for two vectors where the division process is available. In this paper, basic operations on $\mathbb{H}/\mathbb{Z}_p$quaternion and the matrix form which belong to $\mathbb{H}/\mathbb{Z}_p$ quaternion algebra are given.

2010 AMS Classification: 15A03, 15A33, 15A30, 20H25, 11R52

Keywords: Ring, Field, Quaternions, Quaternion Algebra.

1. INTRODUCTION

In this section, basic definitions and theorems are given for our study.

Definition 1.1. *Let $\mathbb{N}$ be the set of natural numbers, and .$(a,b),(c,d)\in\mathbb{N}\times\mathbb{N}$. $\overline{(a,b)}$ equivanlence class which includes as (a,b) element is called an integer according $\sim$ to equivanlence relation in $\mathbb{N}\times\mathbb{N}$ which is defined as*

$$(a,b)\sim(c,d)\Leftrightarrow a+d=b+c$$

and it is denoted by $\mathbb{Z}$.

Theorem 1.1. $(\mathbb{Z},+,.)$ *is a ring*

Theorem 1.2. *To be equal relation is an equivalence relation among the elements in $\mathbb{Z}$ module p.*

Thus, according to module p, if the equivalence classes set

$$\{(x,y)\mid x\in\mathbb{Z},\ y\in\mathbb{Z}x\equiv y\pmod p\}.$$

Which is separated from equivalence relation by $\mathbb{Z}$.is denoted $\mathbb{Z}_p$, that is

$$\mathbb{Z}_p=\left\{\overline{0},\overline{1},\overline{2},...,\overline{p-1}\right\}.$$

Theorem 1.3. $(\mathbb{Z}_p,+,.)$ *is an unit and commutative ring* [2].

Theorem 1.4. *If p.is a prime, then* $(\mathbb{Z}_p, +, .)$ *is a field* [2].

Definition 1.2. *The set of*

$$q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$$

is called real quaternions. Such that ordered a_0, a_1, a_2, a_3 *four real numbers accompany to* $e_0 = 1, e_1, e_2, e_3$ *units which enable*

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_1 \times e_2 &= e_3,\ e_2 \times e_3 = e_1,\ e_3 \times e_1 = e_2 \\ e_2 \times e_1 &= -e_3,\ e_3 \times e_2 = -e_1,\ e_1 \times e_3 = -e_2 \end{aligned} \tag{1.1}$$

properties. Here, a_0, a_1, a_2, a_3 real numbers are components of q quaternion and it is written as $\{\mathbb{H}, \oplus, \mathbb{R}, +, ., \odot, \times\}$ an associative algebra where quaternions set is $\mathbb{H}$.

This algebra is called quaternion algebra and shortly denoted by $\mathbb{H}$. One basis of this algebra is $\{1, e_1, e_2, e_3\}$ and the dimension is four [4].

2. $\mathbb{H}/\mathbb{Z}_p$ Quaternion Algebra

In this study, let p be a prime $e_0 = 1, e_1^2 = p-1 = -1$ and $a, b \in \mathbb{Z}_p$. The elements of the form $ae_0 + be_1$ will be denoted by the set $\mathbb{Z}_p[e_1]$.

Theorem 2.1. *The set*

$$\begin{aligned} \mathbb{H}/\mathbb{Z}_p = \{ & q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 \mid a_i \in \mathbb{Z}_p, 0 \le i \le 3, p = 4k+3 \text{ prime}, \\ & e_0 = 1, e_1^2 = e_2^2 = e_3^2 = p-1 = -1 \} \end{aligned}$$

is a vector space over $(\mathbb{Z}_p, +, .)$ *field.*

Proof. Let $\forall q_1 = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3, q_2 = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}/\mathbb{Z}_p$ and $a_i, b_i \in \mathbb{Z}_p$, $i = 0, 1, 2, 3$. $\mathbb{H}/\mathbb{Z}_p$ under the addition is defined

$$\begin{aligned} \oplus : \mathbb{H}/\mathbb{Z}_p \times \mathbb{H}/\mathbb{Z}_p &\to \mathbb{H}/\mathbb{Z}_p \\ (q_1, q_2) &\to q_1 \oplus q_2 \end{aligned}$$

That is,

$$q_1 \oplus q_2 = (a_0 + b_0)\, e_0 + (a_1 + b_1)\, e_1 + (a_2 + b_2)\, e_2 + (a_3 + b_3)\, e_3.$$

So, $(\mathbb{H}/\mathbb{Z}_p, \oplus)$ is an Abelian group.Let be the set $\mathbb{H}/\mathbb{Z}_p$ under the multiplication

$$\begin{aligned} \odot : \mathbb{Z}_p \times \mathbb{H}/\mathbb{Z}_p &\to \mathbb{H}/\mathbb{Z}_p \\ (a, q) &\to a \odot q. \end{aligned}$$

That is defined by (1.1)

$$\begin{aligned} a \odot q &= a \odot (a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3) \\ &= (aa_0)\, e_0 + (aa_1)\, e_1 + (aa_2)\, e_2 + (aa_3)\, e_3 \end{aligned}$$

which has the properties indicated below.

V1) For $\forall a \in \mathbb{Z}_p$, $\forall q_1, q_2 \in \mathbb{H}/\mathbb{Z}_p$,

$$a \odot (q_1 \oplus q_2) = (a \odot q_1) \oplus (a \odot q_2)\,,$$

V2) For $\forall a, b \in \mathbb{Z}_p$, $\forall q \in \mathbb{H}/\mathbb{Z}_p$,

$$(a+b) \odot q = (a \odot q) \oplus (b \odot q),$$

V3) For $\forall a, b \in \mathbb{Z}_p$, $\forall q \in \mathbb{H}/\mathbb{Z}_p$,

$$(a.b) \odot q = a \odot (b \odot q),$$

V4) For $\forall q \in \mathbb{H}/\mathbb{Z}_p, 1 \in \mathbb{Z}_p$

$$1 \odot q = q.$$

Therefore, $\{\mathbb{H}/\mathbb{Z}_p, \oplus, \mathbb{Z}_p, +, ., \odot\}$ is a vector space. This vector space will be denoted by $\mathbb{H}/\mathbb{Z}_p$ shortly. □

Definition 2.1. *Let be $\mathbb{H}/\mathbb{Z}_p$ a vector space.A multiplication on this vector space is defined*

$$\begin{array}{cccc} \times & : & \mathbb{H}/\mathbb{Z}_p \times \mathbb{H}/\mathbb{Z}_p & \to & \mathbb{H}/\mathbb{Z}_p \\ & & (q_1, q_2) & \to & q_1 \times q_2 \end{array}$$

That is

$$\begin{aligned} q_1 \times q_2 &= (a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3) \times (b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3) \\ &\quad + (a_0b_0 + (p-1)\, a_1b_1 + (p-1)\, a_2b_2 + (p-1)\, a_3b_3)\, e_0 \\ &\quad + (a_0b_1 + a_1b_0 + a_2b_3 + (p-1)\, a_3b_2)\, e_1 \\ &\quad + (a_0b_2 + (p-1)\, a_1b_3 + a_2b_0 + a_3b_1)\, e_2 \\ &\quad + (a_0b_3 + a_1b_2 + (p-1)\, a_2b_1 + a_3b_0)\, e_3. \end{aligned}$$

This multiplication is called quaternion multiplication [3].

Theorem 2.2. *The quaternion multiplication have these properties shown below.*
K1) For $\forall q_1, q_2 \in \mathbb{H}/\mathbb{Z}_p$,

$$q_1 \times q_2 \in \mathbb{H}/\mathbb{Z}_p,$$

K2) For $\forall a \in \mathbb{Z}_p$, $\forall q_{1,} q_2 \in \mathbb{H}/\mathbb{Z}_p$,

$$a \odot (q_1 \times q_2) = (a \odot q_1) \times q_2 = q_1 \times (a \odot q_2),$$

K3) For $\forall q_{1,} q_2, q_3 \in \mathbb{H}/\mathbb{Z}_p$,

$$(q_1 \oplus q_2) \times q_3 = (q_1 \times q_2) \oplus (q_2 \times q_3)$$

$$q_1 \times (q_2 \oplus q_3) = (q_1 \times q_2) \oplus (q_1 \times q_3),$$

K4) For $\forall q_{1,} q_2, q_3 \in \mathbb{H}/\mathbb{Z}_p$,

$$(q_1 \times q_2) \times q_3 = q_1 \times (q_2 \times q_3).$$

Thus, $\{\mathbb{H}/\mathbb{Z}_p, \oplus, \mathbb{Z}_p, +, ., \odot, \times\}$ is an algebra [5]*. This algebra over $\mathbb{Z}_p$ field is called quaternion algebra and it is denoted by $\mathbb{H}/\mathbb{Z}_p$*

Conclusion 2.1. *Quaternion multiplication has no commutative property. That is, for $\forall q_1, q_2 \in \mathbb{H}/\mathbb{Z}_p$,*

$$q_1 \times q_2 \neq q_2 \times q_1.$$

Specially, for $\forall q_1 = a_0e_0, q_2 = b_0e_0 \in \mathbb{H}/\mathbb{Z}_p$,there exists commutative property.

3. Matris Representation of $\mathbb{H}/\mathbb{Z}_p$ Quaternion Algebra

Theorem 3.1. *For* $\forall q_1 = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$, $q_2 = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}/\mathbb{Z}_p$,
this multiplication can be expressed with the help of a linear operatör.

Proof.

$$\begin{array}{rcccccc} L_{q_1} & : & \mathbb{H}/\mathbb{Z}_p & \overset{linear}{\to} & \mathbb{H}/\mathbb{Z}_p & & \\ & & q_2 & \to & L_{q_1}(q_2) & = & q_1 \times q_2 \end{array}$$

so we obtain

$$\begin{aligned} L_{q_1}(e_0) &= q_1 \times e_0 \\ &= a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3. \\ L_{q_1}(e_1) &= q_1 \times e_1 \\ &= (p-1)\,a_1e_0 + a_0e_1 + a_3e_2 + (p-1)\,a_2e_3, \\ L_{q_1}(e_2) &= q_1 \times e_2 \\ &= (p-1)\,a_2e_0 + (p-1)\,a_3e_1 + a_0e_2 + a_1e_3, \\ L_{q_1}(e_3) &= q_1 \times e_3 \\ &= (p-1)\,a_3e_0 + a_2e_1 + (p-1)\,a_1e_2 + a_0e_3. \end{aligned}$$

L_{q_1} corresponds to the lineer operator represented by the matrix $H^+(q_1)$

$$H^+(q_1) = \begin{bmatrix} a_0 & (p-1)\,a_1 & (p-1)\,a_2 & (p-1)\,a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & (p-1)\,a_1 \\ a_3 & (p-1)\,a_2 & a_1 & a_0 \end{bmatrix}.$$

So that $q_1 \times q_2$ quaternion multiplication, $H^+(q_1)\,q_2$ can be expressed in the form of matrix multiplication.Actually

$$\begin{aligned} H^+(q_1)\,q_2 &= \begin{bmatrix} a_0 & (p-1)\,a_1 & (p-1)\,a_1 & (p-1)\,a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & (p-1)\,a_1 \\ a_3 & (p-1)\,a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= \begin{bmatrix} a_0b_0 + (p-1)\,a_1b_1 + (p-1)\,a_1b_2 + (p-1)\,a_3b_3 \\ a_1b_0 + a_0b_1 + (p-1)\,a_3b_2 + a_2b_3 \\ a_2b_0 + a_3b_1 + a_0b_2 + (p-1)\,a_1b_3 \\ a_3b_0 + (p-1)\,a_2b_1 + a_1b_2 + a_0b_3 \end{bmatrix} \\ &= q_1 \times q_2. \end{aligned}$$

Therefore,

$$\begin{aligned} H^{+}(q_1) &= a_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} 0 & p-1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p-1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &+ a_2 \begin{bmatrix} 0 & 0 & p-1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p-1 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 0 & p-1 \\ 0 & 0 & p-1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Matrix can be written by

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & p-1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p-1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ E_2 &= \begin{bmatrix} 0 & 0 & p-1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & p-1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p-1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

$$H^{+}(q_1) = a_0 E_0 + a_1 E_1 + a_2 E_2 + a_3 E_3.$$

Here, $E_0 = I_4, E_1, E_2, E_3$.in order corresponds to $e_0 = 1, e_1, e_2, e_3$ units. There exists the properties shown below:

$$E_1^2 = E_2^2 = E_3^2 = (p-1)\, E_0 = (p-1)\, I_4,$$

$$E_1 E_2 = E_3,\ E_2 E_3 = E_1,\ E_3 E_1 = E_2$$

$$E_2 E_1 = (p-1)\, E_3,\ E_3 E_2 = (p-1)\, E_1,\ E_1 E_2 = (p-1)\, E_2.$$

By processes similar

$$\begin{aligned} R_{q_1} \ :\ \mathbb{H}/\mathbb{Z}_p &\overset{linear}{\rightarrow} \mathbb{H}/\mathbb{Z}_p \\ q_2 &\rightarrow L_{q_1}(q_2) = q_2 \times q_1 \end{aligned}$$

linear operator where

$$q_1 = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3,$$

$$q_2 = b_0 e_0 + b_1 e_1 + b_2 e_2 + b_3 e_3.$$

Matrix corresponds to R_{q_1} linear operation.

$$\begin{aligned} R_{q_1}(e_0) &= e_0 \times q_1 \\ &= a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3. \\ R_{q_1}(e_1) &= e_1 \times q_1 \\ &= (p-1)\, a_1 e_0 + a_0 e_1 + (p-1)\, a_3 e_2 + a_2 e_3, \\ R_{q_1}(e_2) &= e_2 \times q_1 \\ &= (p-1)\, a_2 e_0 + a_3 e_1 + a_0 e_1 + (p-1)\, a_1 e_3, \end{aligned}$$

$$\begin{aligned} R_{q_1}(e_3) &= e_3 \times q_1 \\ &= (p-1)\,a_3e_0 + (p-1)\,a_2e_1 + a_1e_2 + a_0e_3. \end{aligned}$$

Thus we obtain

$$H^-(q_1) = \begin{bmatrix} a_0 & (p-1)\,a_1 & (p-1)\,a_2 & (p-1)\,a_3 \\ a_1 & a_0 & a_3 & (p-1)\,a_2 \\ a_2 & (p-1)\,a_3 & a_0 & a_1 \\ a_3 & a_2 & (p-1)\,a_1 & a_0 \end{bmatrix}$$

So that $q_2 \times q_1$ quaternion multiplication, $H^-(q_1)\,q_2$ can be expressed in the form of matrix multiplication.Actually

$$\begin{aligned} H^-(q_1)\,q_2 &= \begin{bmatrix} a_0 & (p-1)\,a_1 & (p-1)\,a_1 & (p-1)\,a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & (p-1)\,a_1 \\ a_3 & (p-1)\,a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= \begin{bmatrix} a_0b_0 + (p-1)\,a_1b_1 + (p-1)\,a_1b_2 + (p-1)\,a_3b_3 \\ a_1b_0 + a_0b_1 + (p-1)\,a_3b_2 + a_2b_3 \\ a_2b_0 + a_3b_1 + a_0b_2 + (p-1)\,a_1b_3 \\ a_3b_0 + (p-1)\,a_2b_1 + a_1b_2 + a_0b_3 \end{bmatrix} \\ &= q_2 \times q_1 \end{aligned}$$

Therefore,

$$\begin{aligned} H^-(q_1) &= a_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + a_1 \begin{bmatrix} 0 & p-1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p-1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ &\quad + a_2 \begin{bmatrix} 0 & 0 & p-1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p-1 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 0 & p-1 \\ 0 & 0 & p-1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Matrix can be written by

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & p-1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p-1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ E_2 &= \begin{bmatrix} 0 & 0 & p-1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & p-1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & p-1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

$$H^-(q_1) = a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3.$$

Here, $E_0 = I_4, E_1, E_2, E_3$.in order corresponds to $e_0 = 1, e_1, e_2, e_3$ units. There exists the properties shown below:

$$E_1^2 = E_2^2 = E_3^2 = (p-1)\,E_0 = (p-1)\,I_4,$$

$$E_1E_2 = E_3,\ E_2E_3 = E_1,\ E_3E_1 = E_2$$
$$E_2E_1 = (p-1)\,E_3,\ E_3E_2 = (p-1)\,E_1,\ E_1E_2 = (p-1)\,E_2.$$

Homomorphism where H^+ was not a homomorphism H^-.Thus,

$$\begin{array}{ll} i) & H^+(q_1+q_2) = H^+(q_1) + H^+(q_2) \\ ii) & H^+(q_1 x q_2) = H^+(q_1)\ H^+(q_2) \\ iii) & H^-(q_1+q_2) = H^-(q_2) + H^-(q_1) \\ iv) & H^-(q_1 x q_2) = H^-(q_2)\ H^-(q_1) \neq H^-(q_1)\ H^-(q_2)\,. \end{array}$$

H^+ and H^-.operators similar to Hamilton operators[2]. Thus $\forall q_1, q_2, q_3, q_4 \in \mathbb{H}/\mathbb{Z}_p$ following properties are provided.

$$\begin{array}{ll} i) & q_1 x q_2 = H^+(q_1)\, q_2 = H^-(q_2)\, q_1 \\ ii) & H^+(q_1 x q_2) = H^+(H^+(q_1)\, q_2) = H^+(q_1)\ H^+(q_2) \\ iii) & H^-(q_1 x q_2) = H^-(H^-(q_2)\, q_1) = H^-(q_2)\ H^-(q_1) \\ iv) & H^+(q_1 x q_2 + q_3 x q_4) = H^+(q_1)\ H^+(q_2) + H^+(q_3)\ H^+(q_4) \\ v) & H^-(q_1 x q_2 + q_3 x q_4) = H^-(q_2)\ H^-(q_1) + H^-(q_4)\ H^-(q_3) \\ vi) & H^+(H^-(q_1)\, q_2) = H^+(q_2)\ H^+(q_1) \\ vii) & H^-(H^+(q_1)\, q_2) = H^-(q_2)\ H^-(q_1) \\ viii) & H^+(q_1)\ H^-(q_2) = H^-(q_2)\ H^+(q_1) \end{array}$$

□

References

[1] A. Adler, I.E. Coury. The Theory of Number, Jones and Barlett Puplishers, Boston,(1995).
[2] O. P. Agrawal. Mechanizm and Machine Theory, Vol.22, Issue 6, p.569-675, (1987).
[3] M Aristidou. A Note on Quaternion Rings, International Journal of Algebra, Vol.3, No.15, p.725-728, (2009).
[4] H.H. Hacısalihoğlu. Motion Geometry and Quaternions Theory, Gazi University Faculty of Arts and Science Publications, Math. No.2, Ankara, (1983).
[5] I.N. Herstein. Topics in Algebra, 2nd Ed., Wiley, (1975).

Gülay Koru Yücekaya(gkoru@gazi.edu.tr) –Gazi University, Gazi Education Faculty,Mathematics Education Department, Teknikokullar, 06500, Ankara, Turkey