

## Pseudo-Slant Submanifolds of a Nearly Cosymplectic Manifold

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### ABSTRACT.

In this paper, the geometry of pseudo-slant submanifolds of a nearly Cosymplectic manifold is studied. We obtain the necessary and sufficient conditions on a totally umbilical proper-slant submanifold and show that it is totally geodesic if the mean curvature vector  $H \in \mu$ . well as, we research the integrability conditions of the distributions of pseudo-slant submanifolds and prove some characterizations.

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### 1. INTRODUCTION

The differential geometry of slant submanifolds has shown an increasing development since B-Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both holomorphic and totally real submanifolds [8]. Then many research articles have been appeared on the existence of these submanifolds in different known spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta [13]. After, such submanifolds were studied by J.L. Cabrerizo et.al in the setting of Sasakian manifolds [6].

The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papaghiuc [14]. Hemi-slant submanifolds first were introduced by A. Carrizo [6, 7] and he called them pseudo-slant submanifolds. Recently, in [15] B. Sahin studied warped product submanifolds in a Kaehler manifold. In [10, 11], authors studied the pseudo-slant submanifold in trans- Sasakian manifolds.

In this paper, we study the pseudo-slant submanifolds of a nearly Cosymplectic manifold. In section 2, we review basic formulas and definitions for a nearly Cosymplectic manifold and their submanifolds, which will be used later. In section 3, we recall the definitions and some basic results of a pseudo-slant submanifold of almost contact metric manifold.

We deal with the integrability of a the distributions on the pseudo-slant submanifolds of nearly Cosymplectic manifold and then we obtain some results for these submanifolds in the setting of nearly Cosymplectic manifold.

## 2. PRELIMINARIES

In this section, we give some notations used throughout this paper. We recall some necessary fact and formulas from the theory of nearly Cosymplectic manifolds and their submanifolds.

Let  $\tilde{M}$  be a  $(2n + 1)$ -dimensional almost contact metric manifold together with a metric tensor  $g$ , a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$  and a 1-form  $\eta$  on  $\tilde{M}$  which satisfy

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi,$$

$$(2.2) \quad \phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1, \eta(X) = g(X, \xi)$$

and

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) + g(X, \phi Y) = 0$$

for any vector fields  $X, Y$  on  $\tilde{M}$ . If in addition to above relations

$$(2.4) \quad (\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = 0,$$

then,  $\tilde{M}$  is called a nearly Cosymplectic manifold, where  $\tilde{\nabla}$  is the Levi-Civita connection of  $g$ . We have also on nearly Cosymplectic manifold  $\tilde{M}$

$$(2.5) \quad \tilde{\nabla}_X \xi = 0,$$

for any  $X \in \Gamma(T\tilde{M})$ .

Now, let  $M$  be a submanifold of a contact metric manifold  $\tilde{M}$  with the induced metric  $g$  and  $\xi$  be tangent to  $M$ . Also, let  $\nabla$  and  $\nabla^\perp$  be the induced connections on the tangent bundle  $TM$  and the normal bundle  $T^\perp M$  of  $M$ , respectively. Then the Gauss and Weingarten formulas are respectively, given by

$$(2.6) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$(2.7) \quad \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

where  $h$  and  $A_V$  are the second fundamental form and the shape operator (corresponding to the normal vector field  $V$ ), respectively, for the immersion of  $M$  into  $\tilde{M}$ . The second fundamental form  $h$  and shape operator  $A_V$  are related by

$$(2.8) \quad g(A_V X, Y) = g(h(X, Y), V),$$

for all  $X, Y \in \Gamma(TM)$  and  $V \in \Gamma(T^\perp M)$ .

In equation (2.6), for  $Y = \xi$ , we have

$$\tilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

Using (2.5), the tangential and normal parts of the last equation give, respectively, us

$$(2.9) \quad \nabla_X \xi = 0$$

and

$$(2.10) \quad h(X, \xi) = 0.$$

The mean curvature vector  $H$  of  $M$  is given by

$$(2.11) \quad H = \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i)$$

where  $n$  is the dimension of  $M$  and  $\{e_1, e_2, \dots, e_n\}$  is a local orthonormal frame of  $M$ .

A submanifold  $M$  of an contact metric manifold  $\widetilde{M}$  is said to be totally umbilical if

$$(2.12) \quad h(X, Y) = g(X, Y)H,$$

where  $H$  is the mean curvature vector. A submanifold  $M$  is said to be totally geodesic if  $h(X, Y) = 0$ , for each  $X, Y \in \Gamma(TM)$  and  $M$  is said to be minimal if  $H = 0$ .

### 3. PSEUDO-SLANT SUBMANIFOLDS OF A NEARLY COSYMPLECTIC MANIFOLD

In this section we will obtain the integrability conditions of the distributions of pseudo-slant submanifold of a nearly Cosymplectic manifold. Also, we obtain some results on a totally umbilical pseudo-slant in a nearly Cosymplectic manifold.

Let  $M$  be a submanifold of an almost contact metric manifold  $\widetilde{M}$ . Then for any  $X \in \Gamma(TM)$ , we can write

$$(3.1) \quad \phi X = TX + NX,$$

where  $TX$  is the tangential component and  $NX$  is the normal component of  $\phi X$ .

Similarly for  $V \in \Gamma(T^\perp M)$ , we can write

$$(3.2) \quad \phi V = tV + nV,$$

where  $tV$  is the tangential component and  $nV$  is the normal component of  $\phi V$ .

Thus by using (2.1), (3.1) and (3.2), we obtain

$$(3.3) \quad T^2 = -I + \eta \otimes \xi - tN, \quad NT + nN = 0$$

and

$$(3.4) \quad Tt + tn = 0, \quad Nt + n^2 = -I.$$

Furthermore, for any  $X, Y \in \Gamma(TM)$  and  $V, U \in \Gamma(T^\perp M)$ , we have  $g(TX, Y) = -g(X, TY)$  and  $g(U, nV) = -g(nU, V)$ . These show that  $T$  and  $n$  are also skew-symmetric tensor fields. Moreover, for any  $X \in \Gamma(TM)$  and  $V \in \Gamma(T^\perp M)$ , we can easily see

$$(3.5) \quad g(NX, V) = -g(X, tV),$$

which gives the relation between  $N$  and  $t$ .

Furthermore, the covariant derivatives of the tensor field  $T$ ,  $N$ ,  $t$  and  $n$  are, respectively, defined by

$$(3.6) \quad (\nabla_X T)Y = \nabla_X T Y - T \nabla_X Y,$$

$$(3.7) \quad (\nabla_X N)Y = \nabla_X^\perp N Y - N \nabla_X Y,$$

$$(3.8) \quad (\nabla_X t)V = \nabla_X t V - t \nabla_X^\perp V$$

and

$$(3.9) \quad (\nabla_X n)V = \nabla_X^\perp n V - n \nabla_X^\perp V,$$

for any  $X, Y \in \Gamma(TM)$ .

By direct calculations, we obtain the following formulas

$$(3.10) \quad (\nabla_X T)Y + (\nabla_Y T)X = A_{NX}Y + A_{NY}X + 2th(X, Y)$$

and

$$(3.11) \quad (\nabla_X N)Y + (\nabla_Y N)X = 2nh(X, Y) - h(X, TY) - h(Y, TX).$$

Similarly, for any  $V \in \Gamma(T^\perp M)$ , we obtain

$$(3.12) \quad (\nabla_X t)V = A_{nV}X - T A_V X$$

and

$$(3.13) \quad (\nabla_X n)V = -h(tV, X) - N A_V X.$$

In contact geometry, A. Lotta introduced slant immersions as follows [13].

**Definition 3.1.** Let  $M$  be a submanifold of a nearly Cosymplectic manifold  $\widetilde{M}$ . For each non-zero vector  $X$  tangent to  $M$  at  $x$ , the angle  $\theta(x) \in [0, \frac{\pi}{2}]$ , between  $\phi X$  and  $TX$  is called the slant angle or the Wirtinger angle of  $M$ . If the slant angle is constant for each  $X \in \Gamma(TM)$  and  $x \in M$ , then the submanifold is also called the slant submanifold. If  $\theta = 0$  the submanifold is *invariant submanifold*. If  $\theta = \frac{\pi}{2}$  then it is called *anti-invariant submanifold*. If  $\theta(x) \in (0, \frac{\pi}{2})$ , then it is called *proper-slant submanifold* [13].

If  $M$  is a slant submanifold of an almost contact metric manifold, then the tangent bundle  $TM$  of  $M$  can be decomposed as

$$(3.14) \quad TM = D_\theta \oplus \xi,$$

where  $\xi$  denotes the distribution spanned by the structure vector field  $\xi$  and  $D_\theta$  is complementary of distribution of  $\xi$  in  $TM$ , known as the slant distribution on  $M$ . Recently, Cabrerizo et al. [6, 7] extended the above result in to a characterization for a slant submanifold in a contact metric manifold. In fact, they obtained the following crucial theorem.

**Theorem 3.2.** Let  $M$  be a slant submanifold of an almost contact metric manifold  $\widetilde{M}$  such that  $\xi \in \Gamma(TM)$ . Then  $M$  is slant submanifold if and only if there exists a constant  $\lambda \in [0, 1]$  such that

$$(3.15) \quad T^2 = -\lambda(I - \eta \otimes \xi)$$

furthermore, in such case, if  $\theta$  is the slant angle of  $M$ , then  $\lambda = \cos^2 \theta$  [6].

**Corollary 3.3.** [6]. *Let  $M$  be a slant submanifold of an almost contact metric manifold  $\widetilde{M}$  with slant angle  $\theta$ . Then for any  $X, Y \in \Gamma(TM)$ , we have*

$$(3.16) \quad g(TX, TY) = \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}$$

and

$$(3.17) \quad g(NX, NY) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}.$$

**Definition 3.4.** We say that  $M$  is a pseudo-slant submanifold of an almost contact metric manifold  $\widetilde{M}$  if there exist two orthogonal distributions  $D_\theta$  and  $D^\perp$  on  $M$  such that

- 1)  $TM$  admits the orthogonal direct decomposition  $TM = D^\perp \oplus D_\theta$ ,  $\xi \in \Gamma(D_\theta)$
- 2) The distribution  $D^\perp$  is anti-invariant i.e.,  $\phi D^\perp \subset (T^\perp M)$ ,
- 3) The distribution  $D_\theta$  is a slant with slant angle  $\theta \neq \frac{\pi}{2}$ , that is, the angle between  $D_\theta$  and  $\phi(D_\theta)$  is a constant [1, 10].

From the definition, it is clear that if  $\theta = 0$ , then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand, if  $\theta = \frac{\pi}{2}$ , submanifold becomes an anti-invariant.

We suppose that  $M$  is a pseudo-slant submanifold of an almost contact metric manifold  $\widetilde{M}$  and we denote the dimensions of distributions  $D^\perp$  and  $D_\theta$  by  $d_1$  and  $d_2$ , respectively, then we have the following cases:

- 1) If  $d_2 = 0$  then  $M$  is an anti-invariant submanifold,
- 2) If  $d_1 = 0$  and  $\theta = 0$ , then  $M$  is an invariant submanifold,
- 3) If  $d_1 = 0$  and  $\theta \neq 0$ , then  $M$  is a proper slant submanifold with slant angle  $\theta$ ,
- 4) If  $d_1, d_2 \neq 0$  and  $\theta \in (0, \frac{\pi}{2})$  then  $M$  is a proper pseudo-slant submanifold.

Let  $M$  a proper pseudo-slant submanifold of a contact metric manifold  $\widetilde{M}$  and we denote the projections on  $D^\perp$  and  $D_\theta$  by  $P_1$  and  $P_2$ , respectively, then for any vector field  $X \in \Gamma(TM)$ , we can write.

$$(3.18) \quad X = P_1X + P_2X + \eta(X)\xi.$$

Now applying  $\phi$  on both sides of equation (3.18), we obtain

$$\phi X = \phi P_1X + \phi P_2X,$$

that is,

$$(3.19) \quad TX + NX = NP_1X + TP_2X + NP_2X.$$

We can easily to see

$$TX = TP_2X, \quad NX = NP_1X + NP_2X$$

and

$$(3.20) \quad \phi P_1X = NP_1X, \quad TP_1X = 0, \quad \phi P_2X = TP_2X + NP_2X$$

$$TP_2X \in \Gamma(D_\theta).$$

If we denote the orthogonal complementary of  $\phi TM$  in  $T^\perp M$  by  $\mu$ , then the normal bundle  $T^\perp M$  can be decomposed as follows

$$(3.21) \quad T^\perp M = N(D^\perp) \oplus N(D_\theta) \oplus \mu,$$

where  $\mu$  is an invariant sub bundle of  $T^\perp M$  as  $N(D^\perp)$  and  $N(D_\theta)$  are orthogonal distribution on  $M$ . Indeed,  $g(Z, X) = 0$  for each  $Z \in \Gamma(D^\perp)$  and  $X \in \Gamma(D_\theta)$ . Thus, by equation (2.3) and (3.1), we can write

$$g(NZ, NX) = g(\phi Z, \phi X) = g(Z, X) = 0$$

that is, the distributions  $N(D^\perp)$  and  $N(D_\theta)$  are mutually perpendicular. In fact, the decomposition (3.21) is an orthogonal direct decomposition.

**Lemma 3.5.**  $D_\theta$  is slant distribution if only and if there is a constant  $\lambda \in [0, 1]$  such that

$$(TP_2)^2 X = -\lambda X,$$

for all  $X \in \Gamma(D_\theta)$ . In such case, if  $\theta$  is the slant angle of  $M$ , then  $\lambda = \cos^2 \theta$ [6].

**Proposition 3.6.** Let  $M$  be a pseudo-slant of a nearly Cosymplectic manifold  $\widetilde{M}$ . Then

$$(3.22) \quad h(X, \xi) = 0$$

$$(3.23) \quad h(TX, \xi) = 0$$

and

$$(3.24) \quad \nabla_\xi \xi = 0,$$

$\forall X, Y \in \Gamma(TM)$ .

*Proof.* Since  $\xi$  is tangent to  $M$ , we have

$$\widetilde{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).$$

for any  $X, Y \in \Gamma(TM)$ . This yields to  $h(X, \xi) = 0$  and  $\nabla_\xi \xi = 0$ . □

**Theorem 3.7.** Let  $M$  be a pseudo-slant of a nearly Cosymplectic manifold  $\widetilde{M}$ . Then the anti-invariant distribution  $D^\perp$  is integrable if and only if

$$(3.25) \quad A_{NW}Z + A_{NZ}W + 2T\nabla_Z W + 2th(W, Z) = 0$$

for any  $Z, W \in \Gamma(D^\perp)$ .

*Proof.* For any  $Z, W \in \Gamma(D^\perp)$ , by using equation (2.4), we have

$$(\widetilde{\nabla}_Z \phi)W + (\widetilde{\nabla}_W \phi)Z = 0$$

which is equivalent to

$$\widetilde{\nabla}_Z \phi W - \phi \widetilde{\nabla}_Z W + \widetilde{\nabla}_W \phi Z - \phi \widetilde{\nabla}_W Z = 0.$$

By using (2.6), (2.7), (3.1) and (3.2), we have

$$\begin{aligned} 0 &= \widetilde{\nabla}_Z NW - T\nabla_Z W - N\nabla_Z W - th(W, Z) - nh(W, Z) \\ &\quad + \widetilde{\nabla}_W NZ - T\nabla_W Z - N\nabla_W Z - th(W, Z) - nh(W, Z). \end{aligned}$$

So we have

$$\begin{aligned} 0 &= -A_{NW}Z + \nabla_Z^\perp NW - T\nabla_Z W - N\nabla_Z W - 2th(W, Z) \\ &\quad - A_{NZ}W + \nabla_W^\perp NZ - T\nabla_W Z - N\nabla_W Z - 2nh(W, Z). \end{aligned}$$

Corresponding the tangent components of the last equation, we conclude

$$A_{NW}Z + A_{NZ}W + T\nabla_Z W + T\nabla_W Z + 2th(W, Z) = 0.$$

From the above equation, we can infer

$$A_{NW}Z + A_{NZ}W + 2T\nabla_Z W - T(\nabla_Z W - \nabla_W Z) + 2th(W, Z) = 0$$

$$T[Z, W] = A_{NW}Z + A_{NZ}W + 2T\nabla_Z W + 2th(W, Z).$$

Thus  $[W, Z] \in \Gamma(D^\perp)$  if and only if (3.25) is satisfied.  $\square$

**Theorem 3.8.** *Let  $M$  be a pseudo-slant submanifold of a nearly Cosymplectic manifold  $\tilde{M}$ . Then the slant distribution  $D_\theta$  is integrable if and only if*

$$(3.26) \quad P_1\{\nabla_X T Y - T\nabla_Y X + (\nabla_Y T)X - A_{NX}Y - A_{NY}X - 2th(X, Y)\} = 0.$$

for any  $X, Y \in \Gamma(D_\theta)$ .

*Proof.* For any  $X, Y \in \Gamma(D_\theta)$  and we denote the projections on  $D^\perp$  and  $D_\theta$  by  $P_1$  and  $P_2$ , respectively, then for any vector fields  $X, Y \in \Gamma(D_\theta)$ , by using equation (2.4), we obtain

$$(\tilde{\nabla}_X \phi)Y + (\tilde{\nabla}_Y \phi)X = 0,$$

or

$$\tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y + \tilde{\nabla}_Y \phi X - \phi \tilde{\nabla}_Y X = 0.$$

By using equations (2.6), (3.6) (3.1), and (3.2), we can write

$$0 = \tilde{\nabla}_X T Y + \tilde{\nabla}_X N Y - \phi(\nabla_X Y + h(X, Y)) + \tilde{\nabla}_Y T X + \tilde{\nabla}_Y N X - \phi(\nabla_Y X + h(X, Y)),$$

that is,

$$(3.27) \quad \nabla_X T Y + h(X, T Y) - A_{NY}X + \nabla_X^\perp N Y - T\nabla_X Y - N\nabla_X Y - th(X, Y) - nh(X, Y) + \nabla_Y T X + h(Y, T X) - A_{NX}Y + \nabla_Y^\perp N X - T\nabla_Y X - N\nabla_Y X - th(X, Y) - nh(X, Y) = 0.$$

From tangential components of (3.27) reach

$$(3.28) \quad \nabla_X T Y - T\nabla_X Y + (\nabla_Y T)X - A_{NX}Y - A_{NY}X - 2th(X, Y) = 0,$$

which implies that

$$(3.29) \quad T[X, Y] = \nabla_X T Y - T\nabla_Y X + (\nabla_Y T)X - A_{NX}Y - A_{NY}X - 2th(X, Y).$$

Applying  $P_1$  to (3.29), we get (3.26).  $\square$

**Theorem 3.9.** *Let  $M$  be a pseudo-slant submanifold of a nearly Cosymplectic manifold  $\tilde{M}$ . Then the distribution  $D^\perp \oplus \xi$  is integrable if and only if*

$$A_{\phi Z}W = A_{\phi W}Z$$

for any  $Z, W \in \Gamma(D^\perp \oplus \xi)$

*Proof.* For any  $Z, W \in \Gamma(D^\perp \oplus \xi)$  and  $U \in \Gamma(TM)$ , by using (2.8), we can write

$$2g(A_{\phi Z}W, U) = g(h(U, W), \phi Z) + g(h(U, W), \phi Z).$$

By using (2.6), we have

$$\begin{aligned} 2g(A_{\phi Z}W, U) &= g(\widetilde{\nabla}_W U, \phi Z) + g(\widetilde{\nabla}_U W, \phi Z) \\ &= -g(\phi \widetilde{\nabla}_W U, Z) - g(\phi \widetilde{\nabla}_U W, Z). \end{aligned}$$

So we have

$$2g(A_{\phi Z}W, U) = g((\widetilde{\nabla}_W \phi)U + (\widetilde{\nabla}_U \phi)W, Z) - g(\widetilde{\nabla}_W \phi U, Z) - g(\widetilde{\nabla}_U \phi W, Z).$$

By using equation (2.4), we obtain

$$\begin{aligned} 2g(A_{\phi Z}W, U) &= -g(\widetilde{\nabla}_W \phi U, Z) - g(\widetilde{\nabla}_U \phi W, Z) \\ &= g(\widetilde{\nabla}_W Z, \phi U) - g(-A_{\phi W}U, Z) \\ &= -g(\phi \widetilde{\nabla}_W Z, U) + g(A_{\phi W}U, Z) \\ &= -g(\phi \widetilde{\nabla}_W Z, U) + g(A_{\phi W}Z, U) \\ &= -g(T\nabla_W Z + th(Z, W), U) + g(A_{\phi W}Z, U) \end{aligned}$$

which is equivalent to

$$(3.30) \quad 2A_{\phi Z}W = A_{\phi W}Z - T\nabla_W Z - th(Z, W).$$

Interchanging  $W$  by  $Z$  in (3.30), we derive

$$(3.31) \quad 2A_{\phi W}Z = A_{\phi Z}W - T\nabla_Z W - th(W, Z).$$

By using equation (3.30) and (3.31), we obtain

$$(3.32) \quad 3(A_{\phi Z}W - A_{\phi W}Z) = T[Z, W]$$

thus the distribution  $D^\perp \oplus \xi$  is integrable if and only if  $T[Z, W] = 0$  which proves our assertion.  $\square$

**Theorem 3.10.** *Let  $M$  be a totally umbilical pseudo-slant submanifold of a nearly Cosymplectic manifold  $\widetilde{M}$ . Then at least one of the following statements is true;*

- 1)  $\dim(D^\perp) = 1$ ,
- 2)  $H \in \Gamma(\mu)$ ,
- 3)  $M$  is proper pseudo-slant submanifold

*Proof.* For any  $Z \in \Gamma(D^\perp)$ , by using (2.4), we have

$$\begin{aligned} (\widetilde{\nabla}_Z \phi)Z &= 0 \\ \widetilde{\nabla}_Z NZ - \phi(\nabla_Z Z + h(Z, Z)) &= 0. \end{aligned}$$

From the last equation, we have

$$(3.33) \quad -A_{NZ}Z + \nabla_Z^\perp NZ - N\nabla_Z Z - th(Z, Z) - nh(Z, Z) = 0.$$

From (3.7) and from the tangential components of (3.33), we obtain

$$(3.34) \quad A_{NZ}Z + th(Z, Z) = 0,$$



Taking the product by  $W \in \Gamma(D^\perp)$ , we obtain

$$g(A_{NZ}Z + th(Z, Z), W) = 0.$$

It implies that

$$(3.35) \quad g(h(Z, W), NZ) + g(th(Z, Z), W) = 0.$$

Since  $M$  is totally umbilical submanifold, we obtain

$$(3.36) \quad g(Z, W)g(H, NZ) + g(Z, Z)g(tH, W) = 0,$$

that is,

$$(3.37) \quad g(tH, W)Z - g(tH, Z)W = 0.$$

Here  $tH$  is either zero or  $Z$  and  $W$  are linearly dependent. vector fields If  $tH \neq 0$ , then  $\dim \Gamma(D^\perp) = 1$ .

Otherwise  $H \in \Gamma(\mu)$ . Since  $D_\theta \neq 0$ ,  $M$  is pseudo-slant submanifold. Since  $\theta \neq 0$  and  $d_1.d_2 \neq 0$ ,  $M$  is proper pseudo-slant submanifold.  $\square$

**Theorem 3.11.** *Let  $M$  be totally umbilical proper pseudo-slant submanifold of a nearly Cosymplectic  $\tilde{M}$ . Then  $M$  is an either totally geodesic submanifold or it is an anti- invariant if  $H, \nabla_X^\perp H \in \Gamma(\mu)$ .*

*Proof.* Since the ambient space is a nearly Cosymplectic manifold, for any  $X \in \Gamma(TM)$ , by using (2.4), we have

$$(3.38) \quad \begin{aligned} (\tilde{\nabla}_X \phi)X &= 0, \\ \tilde{\nabla}_X \phi X &= \phi \tilde{\nabla}_X X. \end{aligned}$$

By using (2.6), (2.7), (2.12) and (3.1), (3.38) equation takes the from

$$(3.39) \quad \nabla_X TX + g(X, TX)H - A_{NX}X + \nabla_X^\perp NX = \phi \nabla_X X + g(X, X)\phi H$$

by taking the product with  $\phi H$ , we obtain

$$(3.40) \quad g(\nabla_X^\perp NX, \phi H) = g(N\nabla_X X, \phi H) + g(X, X) \|H\|^2$$

taking into account (2.7), we get

$$(3.41) \quad g(\tilde{\nabla}_X NX, \phi H) = g(X, X) \|H\|^2.$$

Now, for any  $X \in \Gamma(TM)$ , we have

$$(3.42) \quad \tilde{\nabla}_X \phi H = (\tilde{\nabla}_X \phi)H + \phi \tilde{\nabla}_X H$$

making use of then from (2.7), (2.12), (3.1), (3.2) and (3.42), takes the from

$$(3.43) \quad -A_{\phi H}X + \nabla_X^\perp \phi H = (\tilde{\nabla}_X \phi)H - TA_H X - NA_H X + t\nabla_X^\perp H + n\nabla_X^\perp H.$$

Taking the product with  $NX$  and view of fact  $n\nabla_X^\perp H \in \Gamma(\mu)$ , (3.43) becomes

$$g(\nabla_X^\perp \phi H, NX) = g((\tilde{\nabla}_X \phi)H, NX) - g(NA_H X, NX),$$

or

$$g(\tilde{\nabla}_X \phi H, NX) = g((\nabla_X n)H + h(tH, X) + NA_H X, NX) - g(NA_H X, NX).$$

By using (2.8), (2.12) and (3.17), we have

$$g(\widetilde{\nabla}_X \phi H, NX) = -\sin^2 \theta \{g(X, X) \|H\|^2 - g(h(X, \xi), H)\eta(X)\}.$$

From (2.10), we obtain

$$g(\widetilde{\nabla}_X \phi H, NX) = -\sin^2 \theta \{g(X, X) \|H\|^2\}$$

or

$$(3.44) \quad g(\widetilde{\nabla}_X NX, \phi H) = \sin^2 \theta \{g(X, X) \|H\|^2\}.$$

Thus, (3.41) and (3.44) imply

$$g(X, X) \|H\|^2 = \sin^2 \theta \{g(X, X) \|H\|^2\},$$

that is,

$$(3.45) \quad \cos^2 \theta g(X, X) \|H\|^2 = 0.$$

From (3.45), we conclude that  $g(X, X) \|H\|^2 = 0$ , for any  $X \in \Gamma(TM)$ . Since  $M$  proper pseudo slant submanifold we obtain  $H = 0$ . This tells us that  $M$  is totally geodesic in  $\widetilde{M}$ .  $\square$

**Theorem 3.12.** *Let  $M$  be totally umbilical proper pseudo-slant submanifold of nearly Cosymplectic  $\widetilde{M}$ . Then at least one of the following statements is true;*

- 1)  $H \in \mu$ ,
- 2)  $g(\nabla_{TX}\xi, X) = 0$ ,
- 3)  $\eta((\nabla_X T)X) = 0$ ,
- 4)  $M$  is a anti-invariant submanifold.
- 5) If  $M$  proper slant submanifold then,  $\dim(M) \geq 3$ ,  
for any  $X \in \Gamma(TM)$ .

*Proof.* For any  $X \in \Gamma(TM)$ , from equation (2.4) and  $M$  is nearly Cosymplectic manifold, we have

$$\widetilde{\nabla}_X \phi X - \phi \widetilde{\nabla}_X X = 0.$$

By using (2.6), (2.7), (3.1) and (3.2), we have

$$(3.46) \quad \begin{aligned} \nabla_X TX + h(X, TX) - A_{NX}X + \nabla_X^\perp NX - T\nabla_X X - N\nabla_X X \\ -th(X, X) - nh(X, X) = 0 \end{aligned}$$

tangential components of (3.46), we obtain

$$(3.47) \quad \nabla_X TX - T\nabla_X X - th(X, X) - A_{NX}X = 0.$$

Since  $M$  is a totally umbilical pseudo-slant submanifold, by using (2.8) and (2.12), we can write

$$\begin{aligned} g(A_{NX}X, X) &= g(h(X, X), NX) \\ &= g(g(X, X)H, NX) \\ &= g(H, NX)g(X, X) \\ &= g(g(H, NX)X, X) = 0. \end{aligned}$$

If  $H \in \Gamma(\mu)$ , then from (3.47), we obtain

$$(3.48) \quad \nabla_X TX - T\nabla_X X = 0.$$

Taking the product of (3.48) by  $\xi$ , we obtain

$$g(\nabla_X TX, \xi) - \eta(T\nabla_X X) = 0,$$

that is,

$$(3.49) \quad g(\nabla_X TX, \xi) = 0.$$

Interchanging  $X$  by  $TX$  in (3.49), we derive

$$g(\nabla_{TX} T^2 X, \xi) = 0$$

or,

$$g(\nabla_{TX} \xi, T^2 X) = 0$$

by using (3.15), we have

$$g(\nabla_{TX} \xi, -\cos^2 \theta (X - \eta(X)\xi)) = 0$$

$$\cos^2 \theta g(\nabla_{TX} \xi, (X - \eta(X)\xi)) = 0.$$

Since,  $M$  is a proper pseudo-slant submanifold, we have

$$g(\nabla_{TX} \xi, (X - \eta(X)\xi)) = 0.$$

From which

$$(3.50) \quad g(\nabla_{TX} \xi, X) = \eta(X)g(\nabla_{TX} \xi, \xi).$$

Now, we have  $g(\xi, \xi) = 1$ . Taking the covariant derivative of above equation with respect to  $TX$  for any  $X \in \Gamma(TM)$ , we obtain  $g(\nabla_{TX} \xi, \xi) + g(\xi, \nabla_{TX} \xi) = 0$  which implies  $g(\nabla_{TX} \xi, \xi) = 0$  and then (3.50) gives

$$(3.51) \quad g(\nabla_{TX} \xi, X) = 0.$$

This proves (2) of theorem.

Now, Interchanging  $X$  by  $TX$  in the equation (3.51), we derive

$$g(\nabla_{T^2 X} \xi, TX) = g(\nabla_{\cos^2 \theta (-X + \eta(X)\xi)} \xi, TX) = 0,$$

that is,

$$-\cos^2 \theta g(\nabla_{(X - \eta(X)\xi)} \xi, TX) = 0,$$

or

$$-\cos^2 \theta g(\nabla_X \xi, TX) + \cos^2 \theta \eta(X)g(\nabla_X \xi, TX) = 0.$$

Since  $\nabla_X \xi = 0$ , we obtain

$$(3.52) \quad \cos^2 \theta g(\nabla_X \xi, TX) = 0.$$

From (3.52) if  $\cos \theta = 0$ ,  $\theta = \frac{\pi}{2}$  then  $M$  is an anti-invariant submanifold. On the other hand,  $g(\nabla_X \xi, TX) = 0$ , that is  $\nabla_X \xi = 0$ . This implies that  $\xi$  is a the Killing vector field on  $M$ . If the vector field  $\xi$  is not Killing, then we can take at least two linearly independent vectors  $X$  and  $TX$  to span  $D_\theta$ , that is, the  $\dim(M) \geq 3$ .  $\square$

**Example 3.13.** Let  $\mathbb{R}^9$  be the semi- Euclidean space endowed with the usual semi- Euclidean metric tensor  $g = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + dx_3^2 + dy_3^2 + dx_4^2 + dy_4^2 + dz^2$  and with coordinates  $(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, z)$ . We define the almost contact metric structure on  $\mathbb{R}^9$  by

$$\begin{aligned}\phi\left(\frac{\partial}{\partial x_i}\right) &= \frac{\partial}{\partial y_i}, \quad 1 \leq i \leq 4 \\ \phi\left(\frac{\partial}{\partial y_j}\right) &= -\frac{\partial}{\partial x_j}, \quad 1 \leq j \leq 4 \\ \phi\left(\frac{\partial}{\partial z}\right) &= 0,\end{aligned}$$

and

$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz.$$

Then for any vector field  $W = \mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \lambda \frac{\partial}{\partial z} \in T(\mathbb{R}^9)$  we have

$$\begin{aligned}\phi W &= \mu_i \phi\left(\frac{\partial}{\partial x_i}\right) + \nu_j \phi\left(\frac{\partial}{\partial y_j}\right) + \lambda \phi\left(\frac{\partial}{\partial z}\right) \\ &= \mu_i \frac{\partial}{\partial y_j} - \nu_j \frac{\partial}{\partial x_i},\end{aligned}$$

$$g(\phi W, \phi W) = g\left(\mu_i \frac{\partial}{\partial y_j} - \nu_j \frac{\partial}{\partial x_i}, \mu_i \frac{\partial}{\partial y_j} - \nu_j \frac{\partial}{\partial x_i}\right) = \mu_i^2 + \nu_j^2,$$

$$g(W, W) = g\left(\mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \lambda \frac{\partial}{\partial z}, \mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \lambda \frac{\partial}{\partial z}\right) = \mu_i^2 + \nu_j^2 + \lambda^2,$$

$$\eta(W) = g(W, \xi) = g\left(\mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \lambda \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \lambda$$

and

$$\begin{aligned}\phi^2 W &= -\mu_i \frac{\partial}{\partial x_i} - \nu_j \frac{\partial}{\partial y_j} - \lambda \frac{\partial}{\partial z} + \lambda \frac{\partial}{\partial z} \\ &= -W + \eta(W)\xi\end{aligned}$$

which implies that  $g(\phi W, \phi W) = g(W, W) - \eta^2(W)$ .

Thus  $(\phi, \xi, \eta, g)$  is an almost contact metric structure on  $\mathbb{R}^9$ . We call the usual contact metric structure of  $\mathbb{R}^9$ .

Let  $M$  be a submanifold of  $\mathbb{R}^9$  defined by

$$(u, -\sqrt{2}v, v \sin \theta, v \cos \theta, s \cos t, s \sin t, -\cos t, \sin t, -\sin t, z).$$

We can easily to see that the tangent bundle of  $M$  is spanned by the tangent vectors

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1} \\ e_2 &= -\sqrt{2} \frac{\partial}{\partial y_1} + \sin \theta \frac{\partial}{\partial x_2} + \cos \theta \frac{\partial}{\partial y_2} \\ e_3 &= \cos t \frac{\partial}{\partial x_3} + \sin t \frac{\partial}{\partial x_4} \\ e_4 &= -s \sin t \frac{\partial}{\partial x_3} + \sin t \frac{\partial}{\partial y_3} + s \cos t \frac{\partial}{\partial x_4} - \cos t \frac{\partial}{\partial y_4} \\ e_5 &= \xi = \frac{\partial}{\partial z}. \end{aligned}$$

Then we have

$$\begin{aligned} \phi e_1 &= \frac{\partial}{\partial y_1} \\ \phi e_2 &= \sqrt{2} \frac{\partial}{\partial x_1} + \sin \theta \frac{\partial}{\partial y_2} - \cos \theta \frac{\partial}{\partial x_2} \\ \phi e_3 &= \cos t \frac{\partial}{\partial y_3} + \sin t \frac{\partial}{\partial y_4} \\ \phi e_4 &= -s \sin t \frac{\partial}{\partial y_3} - \sin t \frac{\partial}{\partial x_3} + s \cos t \frac{\partial}{\partial y_4} + \cos t \frac{\partial}{\partial x_4}. \end{aligned}$$

By direct calculations, we infer  $D_\theta = \text{span}\{e_1, e_2\}$  is a slant distribution with slant angle  $\alpha = \cos^{-1}(\frac{\sqrt{6}}{3})$ . Since  $\phi e_3$  and  $\phi e_4$  are orthogonal to  $M$ ,  $D^\perp = \text{span}\{e_3, e_4\}$  is an anti-invariant distribution. Thus  $M$  is a 5 -dimensional proper pseudo-slant submanifold of  $\mathbb{R}^9$  with its usual almost contact metric structure.

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