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Tome II

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Osman Yalçın Matbaası, İstanbul

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# TABLE DES MATIÈRES

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Tome II 1949


# On a Class of Recurrence Relations


by J. A. Strang

(Department of Mathematics of Ankara University)


Let $$F(z) = z^{\rho} \sum_{n=0}^{\infty} c_n z^n,$$

where the coefficients are determined by a recurrence relation

$$c_n = g(c_0, c_1, \ldots, c_{n-1}). \tag{1}$$

in which $g$ is an algebraic polynomial of degree $m$ in the variables $c_r$, that is, it is the sum of homogeneous polynomials $g_0, g_1, \ldots, g_m$. Each $g_r$ is of the form

$$g_r = \sum_{p_1, p_2, \ldots, p_r} f c_{p_1} c_{p_2} \ldots c_{p_r}$$

in which $f$ is a coeficient depending on $p_1, \ldots, p_r$, and these are positive integers whose values may range from 1 to $n-1$, subject as a rule to the condition that their sum differs from $n$ by a fixed integer $N$, so that

$$p_1 + p_2 + \ldots + p_r = n - N.$$

The integer $N$ may but need not be the same for all $g_r$, and $g_r$ may be the sum of two or more groups in which $N$ has different values. Thus for instance $g_1$ may be of the form

$$g_1 = \sum_{r=p}^{q} a_r\, c_{n-r},$$

where $p$ and $q$ are fixed, and the $a_r$ depend on $n$ and $r$.

It is assumed that either all $c_r$ and all coefficients in $g$ are zera or positive, or if not they are replaced by their moduli, in which case the recurrence relation is replaced by the inequality

$$c_n \leq g\,(c_0\,, c_1\,, \ldots, c_{n-1}) \tag{1}$$

and the same letters now represent moduli for the sake of brevity.

It is stated above that $g$ does not contain $c_n$, but this is a convenience, not a necessity. There would be little change in the argument if $g$ contained coefficients of order higher than $n$, except that $N$ might be negative.

The object of this paper is to investigate the conditions for the existence of certain simple types of dominant function for $F(z)$. The case $m=0$ is trivial. The factor $z^\rho$ is omitted in what follows. It does not affect the argument and can be inserted when required.

## 2. The linear recurrence relation

$$c_n = a_p\, c_{n-p} + \ldots + a_q\, c_{n-q} \quad (p > q > 0)$$

determines $c_n$ for $n \geq p$, but leaves $c_0, c_1, \ldots, c_{p-1}$ arbitrary.

(i) Let $c = \max\,(c_0\,, c_1\,, \ldots, c_{\nu-1})$

and let the numbers $K$, $\lambda$ independent of $n$ be such that either

$$\begin{array}{l} c_0 \ \leq c = K \\ c_1 \ \leq c \leq K\lambda \\ \cdots\cdots\cdots \\ c_{\nu-1} \leq c \leq K\lambda^{\nu-1} \end{array} \tag{I}$$

so that $K = c$, $\lambda \geq 1$; or

$$\begin{array}{l} c_0 \ \leq c \leq K \\ c_1 \ \leq c \leq K\lambda \\ \cdots\cdots\cdots \\ c_{\nu-1} \leq c = K\lambda^{\nu-1} \end{array} \tag{II}$$

so that $K = c\,\lambda^{1-\nu}$, $\lambda \leq 1$.

Then we can prove that

$$c_n \leq K\lambda^n \quad \text{for all } n \tag{2}$$

provided that

$$K(a_p \lambda^{n-p} + \ldots + a_q \lambda^{n-q}) \leq K\lambda^n \quad \text{for all } n \geq \nu.$$

i. e. provided that

$$\sum_{r=p}^{q} a_r \lambda^{-r} \leq 1 \quad \text{for all } n \geq \nu. \tag{3}$$

This condition can be satisfied by a $\lambda$ independent of $n$ if for each coefficient $a_r$ there exists a $k_r$ independent of $n$ such that

$$a_r \leq k_r \quad \text{for all} \quad n \geq \nu.$$

or if there is a constant s such that

$$\sum_{r=p}^{q} a_r \leq s \quad \text{for all } n \geq \nu.$$

These conditions are of course equivalent.

If all $k_r$ are chosen as small as possible, and $\lambda$ is the least positive solution of

$$\sum_{r=p}^{q} k_r \lambda^{-r} \leq 1, \tag{4}$$

(if $p > q > 0$ there is one and only one such solution) it follows that $K(1 - \lambda z)^{-1}$ is a dominant function for $F(z)$ within the circle $z = \lambda^{-1}$; and no greater circle of convergence can be obtained for a dominant function of this form.

The radius of convergence does not depend on c; it depends only on the coefficients $a_r$. $K$ depends on $c$ through the initial conditions I or II. There is no restriction on $c$.

From (3) it follows that we take the initial conditions I or II according as

$$\sum_{r=p}^{q} a_r > \text{or} < 1.$$

If $\lim_{n=\infty} a_r = 0$ for all $a_r$ we may take $\lambda$ as small as we please, so that $F(z)$ is an integral function.

If $\lim_{n=\infty} a_r = \infty$ for any $a_r$ no dominant function of the above form exists.

If one or more of the numbers $p, \dots q$ is zero or negative the same may be true even if constants $k_r$ exist. If for instance $q = 0$ the inequality (4) cannot be satisfied by a positive value of $\lambda$ unless $k_q < 1$, and if $q$ is negative a corresponding condition is necessary. For instance if (4) is

$$k_1 \lambda^{-p} + k_2 \lambda^q \leq 1,$$

where $p$ and $q$ are now positive, the necessary condition is

$$k_1^q \, k_2^p \leq \frac{p^p \, q^q}{(p+q)^{p+q}} .$$

(ii) The initial conditions

$$\begin{aligned} c_0 &\leq c = K \\ c_1 &\leq c \leq K\lambda \\ &\dots\dots\dots \\ c_{\nu-1} &\leq c \leq K\lambda^{\nu-1} / (\nu-1) \end{aligned} \qquad \text{III.}$$

are satisfied if $K = c$ and $\lambda^r / r \geq 1$ for $r = 1, 2, \dots, \nu - 1$, i.e. if

$$\lambda \geq \max r^{1/r} = 3^{1/3} = 1.442 \text{ approx. if } \nu \geq 4.$$

The initial conditions

$$\begin{aligned} c_0 &\leq c \leq K \\ c_1 &\leq c \leq K\lambda \\ &\dots\dots\dots \\ c_{\nu-1} &\leq c = K\lambda^{\nu-1} / (\nu-1) \end{aligned} \qquad \text{IV.}$$

are satisfied if $K = (\nu - 1) c \lambda^{1-\nu}$ and $\lambda \leq 1$.

Using either III or IV we can establish

$$c_n \leq K\lambda^n / n \quad \text{for all } n$$

provided that $\sum_{r=p}^{q} a_r \dfrac{\lambda^{n-r}}{n-r} \leq \lambda^n / n$ for all $n \geq \nu$,

i. e. provided that

$$\sum_{r=p}^{q} a_r \frac{\lambda^{-r}}{n-r} \leq \frac{1}{n} \quad \text{for all } n \geq \nu. \tag{5}$$

If

$$\sum_{r=p}^{q} \frac{a_r}{n-r} < \frac{1}{n} \qquad \text{for } n \geqq \nu$$

we can find $\lambda \leqq 1$ and independent of $n$ to satisfy (5), so that we can use IV, and it follows that

$$K[1-\log(1-\lambda z)]$$

is a dominant function for $F(z)$ within the circle $z=\lambda^{-1}$. The maximum radius of convergence is determined by the least constant value of $\lambda$ for which (5) is true.

If constant numbers $k_r$ exist such that

$$a_r \leqq k_r \qquad \text{for } n \geqq \nu,$$

it follows that provided $\nu$ be suitably chosen we can make

$$na_r / (n-r) \leqq k_r \qquad \text{for } n \geqq \nu,$$

and we can replace (5) by

$$\sum_{r=p}^{q} k_r \lambda^{-r} \leqq 1,$$

which is (4) again, and can always be satisfied by sufficiently large values of $\lambda$ provided that $p > q > 0$. Hence when the numbers $k_r$ exist, and $p > q > 0$, a dominant function

$$K[1-\log(1-\lambda z)]$$

always exists in virtue of III. But as in the previous case further restrictions are required if one or more of the numbers $p, \ldots, q$ are zero or negative.

Similar results are obtained from the assumptions

$$c_n \leqq K\, p_m(n) \lambda^n$$

and

$$c_n \leqq K \lambda^n / p_m(n)$$

where $p_m(n)$ is a polynomial of degree $m$ in $n$, and $m$ is independent of $n$.

(iii) The initial conditions

$$\left.\begin{array}{l} c_0 \;\; \leq c = K \\ c_1 \;\; \leq c \leq K\lambda \\ \cdots\cdots\cdots \\ c_{\nu-1} \leq c \leq K\lambda^{\nu-1}/(\nu-1)! \end{array}\right\} \quad \text{V.}$$

are satisfied if $K = c$, $\lambda \geq \{(\nu-1)!\}^{1/(\nu-1)}$.

The conditions

$$\left.\begin{array}{l} c_0 \;\; \leq c \leq K \\ c_1 \;\; \leq c \leq K\lambda \\ \cdots\cdots\cdots \\ c_{\nu-1} \leq c = K\lambda^{\nu-1}/(\nu-1)! \end{array}\right\} \quad \text{VI.}$$

are satisfied if $K = (\nu-1)!\, c\lambda^{1-\nu}$, $\lambda \leq 1$.

Using either V or VI we can establish

$$c_n \leq K\lambda^n/n! \quad \text{for all } n$$

provided that $$\sum_{r=p}^{q} a_r \frac{\lambda^{-r}}{(n-r)!} \leq \frac{1}{n!} \qquad \text{for } n \geq \nu. \tag{6}$$

If $$\sum_{r=p}^{q} \frac{a_r}{(n-r)!} \leq \frac{1}{n!} \qquad \text{for } n \geq \nu$$

we can find $\lambda \leq 1$ and independent of $n$ to satisfy (6), and hence obtain for $F(z)$ the dominant function

$$K e^{\lambda z} \quad \text{by using VI.}$$

If there exist constants $k_r$ independent of $n$ such that for each $a_r$

$$n(n-1)(n-2)\ldots(n-r+1)a_r \leq k_r \quad \text{for all } n \geq \nu \tag{7}$$

we can replace (6) by (4), and use V to furnish a dominant function

$$c\, e^{\lambda z},$$

so that $F(z)$ is an integral function with a dominant exponential function. If the constants $k_r$ of (7) do not exist, the function $F(z)$ does not possess an exponential dominant function of this simple type; and as in the preceding sections if the $k_r$ exist there are additional conditions when one or more of the numbers $p, \dots, q$ are zero or negative: e.g. if $q = 0$, $k_q < 1$ is necessary.

### 3. The homogeneous polynomial.

Let
$$c_n \leqq g_m(c_0, c_1, \dots, c_{n-1}),$$

where $g_m$ is a homogeneous polynomial of degree $m$, the sum of $p$ groups of terms for which $N = N_1, N_2, \dots, N_p$ respectively; let $s_1, s_2, \dots, s_p$ be the sums of the coefficients in these groups, so that $s_r$ in general depends on $n$; and let

$$c = \max(c_0, c_1, \dots, c_{\nu-1}),$$

(i) With the initial conditions I or II we can establish

$$c_n \leqq K\lambda^n \quad \text{for all } n$$

provided that

$$K^m \sum_{r=1}^{p} s_r \lambda^{n-N_r} \leqq K\lambda^n \quad \text{for} \quad n \geqq \nu,$$

i. e. provided that

$$K^{m-1} \sum_{r=1}^{p} s_r \lambda^{-N_r} \leqq 1, \ (n \geqq \nu). \tag{8}$$

If we use I this is

$$c^{m-1} \sum_{r=1}^{p} s_r \lambda^{-N_r} \leqq 1, \ (n \geqq \nu) \tag{8$_1$}$$

and if II is used

$$c^{m-1} \lambda^{(m-1)(1-\nu)} \sum_{r=1}^{p} s_r \lambda^{-N_r} \leqq 1, \ (n \geqq \nu). \tag{8$_2$}$$

Neither of these inequalities depends on $n$ except through the $s_r$, but both depend on $c$ when $m > 1$. They differ only in one respect. Each index $N_r$ in (8$_1$) is replaced in (8$_2$) by $N_r + (m-1)(\nu - 1)$.

Assume that every $N_r > 0$.

If numbers $k_r$ independent of $n$ exist such that for each $s_r$

$$c^{m-1} s_r \leqq k_r \text{ for all } n \geqq \nu$$

the inequality $(8_1)$ reduces to

$$\sum_{r=1}^{p} k_r \lambda^{-N_r} \leqq 1,$$

which is in effect (4), and leads to the same conclusions. We obtain as before a dominant function $K(1-\lambda z)^{-1}$ within the circle $z=\lambda^{-1}$. But $\lambda$ now depends in general on $c$; the greater the value of $c$ the smaller the circle of convergence.

If $\lim_{n=\infty} s_r = 0$ for all $s_r$, $\lambda$ may be as small as we please, and $F(z)$ is an integral function.

If $\lim_{n=\infty} s_r = \infty$ for any $s_r$, no dominant function of the form $K(1-\lambda z)^{-1}$ exists.

When one or more of the numbers $N_r$ is zero or negative the result is as before.

The inequality $(8_2)$ differs in only one respect from $(8_1)$. In this case it is possible that $-N_r-(m-1)(\nu-1)$ may remain negative although $N_r$ changes sign.

(ii) Let us take the initial conditions III or IV together with

$$c_n \leqq K\lambda^n / n.$$

To establish this relation generally, and so obtain a dominant function $K[1-\log(1-\lambda z)]$, we require

$$K^{m-1} \sum_{r=1}^{p} \left[\lambda^{-N_r} \sum \frac{f}{p_1 p_2 \cdots p_m}\right] \leqq \frac{1}{n}, \quad (n \geqq \nu) \tag{9}$$

in place of (8), where in the inner sum the $p_r$ range from 0 to $n-1$ subject to the condition

$$p_1 + p_2 + \cdots + p_m = n - N_r .$$

If $a_r$ denotes max $f$ in the inner sum we can replace (9) by

$$K^{m-1} \sum_{r=1}^{p} \left[a_r \lambda^{-N_r} \sum \frac{1}{p_1 p_2 \cdots p_m}\right] \leqq \frac{1}{n}, \quad (n \geqq \nu).$$

Hence if there exist constants $k_r$ such that

$$na_r \sum \frac{1}{p_1 p_2 \cdots p_m} \leq k_r \quad \text{for } n \geq \nu, \tag{10}$$

the inequality (9) reduces to an analogue of (8), and can be treated similarly.

Since the sum in (10) is the coefficient of $z^{n-N_r}$ in the series expansion of $[-\log(1-z)]^m$ its order of magnitude is roughly

$$(\log n)^{m-1} / n,$$

so that to the same degree of approximation we may replace (10) by

$$a_r (\log n)^{m-1} \leq k_r \quad \text{for } n \geq \nu,$$

which brings out the essential similarity between this and (5); for when $m=1$ this is identical with the corresponding condition in the case of (5).

(iii) Finally let us take either V or VI together with

$$c_n \leq K\lambda^n / n!.$$

This can be established for all $n$ provided that

$$K^{m-1} \sum_r F_r \lambda^{-N_r} \leq \frac{1}{n!} \quad \text{for all } n \geq \nu \tag{11}$$

where

$$F_r = \sum \frac{f_r}{p_1! \, p_2! \cdots p_m!}$$

is the sum of the coefficients of terms containing $\lambda^{-N_r}$.

It is clear that this furnishes results similar to those already obtained. If $a_r = \max f_r$ in the sum $F_r$

$$F_r \leq a_r \sum \frac{1}{p_1! \, p_2! \cdots p_m!}$$

and this sum is the coefficient of $z^{n-N_r}$ in the series expansion of $(e^z)^m = e^{mz}$, i.e. it is

$$m^{n-N_r} / (n-N_r)!,$$

so that $$F_r \leqq a_r\, m^{n-N_r} \,/\, (n - N_r)\,!\,.$$

If now for each $a_r$ there exists a $k_r$ independent of $n$, such that

$$n\,!\, a_r\, m^{n-N_r} \,/\, (n - N_r)\,! \leqq k_r \quad \text{for all } n \geqq \nu \tag{12}$$

it follows that it is sufficient to choose $\lambda$ so that one or other set of initial conditions is satisfied, so that

$$K^{m-1} \sum_r k_r\, \lambda^{-N_r} \leqq 1\,; \tag{13}$$

and this inequality is independent of $n$.

**4.** The extension to the non-homogeneous polynomial is immediate and furnishes nothing essentially new, the principal change being that the condition (13), for example, is replaced by

$$\sum_m K^{m-1} \left(\sum k_r\, \lambda^{-N_r}\right) \leqq 1.$$

But the conclusions are similar, and it is evident that similar results are obtained for any assumption of the form

$$c_n \leqq \varphi\,(n)\,\lambda^n\,.$$

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