Unitary up to a factor Representations of the inhomogeneous Galilei group and the non-relativistic Schrödinger equation.

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Özet.— Dalga mekaniğinde bir Φ hali muhtelif g, g' koordinat sistemlerinde Φ_g, $\Phi_{g'}$ gibi birbirinden farklı dalga fonksiyonlarıyla gösterilir. İki Φ, ψ hali arasındaki atlama ihtimalinin (transition probability) değişmezliği sebebiyle, $\Phi_{g'}$ nün zaten itibarî olan fazı o şekilde seçilebilir ki, bütün Φ_g, $\Phi_{g'}$ ler için $\Phi_{g'} = D(N)\Phi_g$ carî olur ; burada $D(N)$ lineer ve üniter bir operatör ve N, g sistemini g' sistemine götüren transformasyondur $(g' = Ng)$. Rölativistik olmıyan dalga mekaniğinde genel Galile transformasyonlarına karşı değişmezlik düşünüldüğü için, $D(N)$ operatörleri inhomogen Galile grubunun bir çarpan farkıyla bir röprezantasyonunu meydana getirirler ; yani

$$D(N_1)\ D(N_2) = w(N_1, N_2)\ D(N_1N_2)$$

olur ki burada $w(N_1, N_2)$ nün mutlak değeri 1 dir. Wigner, Lorentz grubu için bu operatörleri $w = \mp 1$ olacak şekilde normlamak mümkün olduğunu göstermişti. Bu yazıda Wigner'in metodu bizi ilgilendiren Galile grubuna tatbik edilmektedir. Normlama sonucunda bir sabit farkıyla belirli özel bir röprezantasyon elde edildiği ve bu röprezantasyonun da esas itibariyle rölativistik olmıyan Schrödinger denkleminin düzlem-dalga (plane-wave) çözümlerinin meydana getirdiği röprezantasyondan başka bir şey olmadığı görülecektir.

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Introduction. The inhomogeneous proper Galilei group contains, in addition to proper Galilei transformations (given by $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$), displacements of the origin both in space (given by $\mathbf{x}' = \mathbf{x} + \mathbf{a}$) and in time $(t' = t + b)$. Consider all the frames of reference that can be obtained from each other by transformations of this group (ie By an inhomogeneous Galilei transformation). In non-relativistic quantum mechanics, any two such frames should be physically equivalent. Since the transition probability between two states Φ and Ψ, defined as the square of the modulus of the unitary scalar product (Φ, Ψ) of the two normalized wave functions Φ, Ψ has an invariant physical meaning, it must have the same value in both frames. Thus if the states are described by Φ_g, Ψ_g in the g frame and by $\Phi_{g'}, \Psi_{g'}$ in the g' frame, one must have

$$|(\Phi_g, \Psi_g)|^2 = |(\Phi_{g'}, \Psi_{g'})|^2 \tag{1}$$

By an appropriate choice of the physically meaningless constants in $\Phi_{g'}$ one can derive from (1), [1], the existence of a linear unitary operator $D(N)$ such that

$$\Phi_{g'} = D(N)\,\Phi_g$$

for all functions $\Phi_g, \Phi_{g'}$ where N is the transformation that carries g into $g' = Ng$. The operator $D(N)$ is determined by the physical content of the theory only up to a constant of modulus unity which can depend on g and g'. Consequently the $D(N)$ form a representation up to a factor of the inhomogeneous Galilei group:

$$D(N_1)\,D(N_2) = w(N_1, N_2)\,D(N_1N_2) \tag{2}$$

where w is a number whose phase can depend on N_1, N_2 but whose modulus is equal to unity. This whole argument is taken bodily from the discussion given by Wigner for relativistic quantum mechanics in his paper on «Unitary representations of the inhomogeneous Lorentz group» [2], where he shows that the operators which transform relativistic wave functions in different Lorentz frames into each other form a representation up to a factor of the inhomogeneous Lorentz group.

In the case of the Lorentz group, by making use of the mathematical properties of the group, Wigner could further show that it is possible to give a definite phase to each operator $D(L)$ which leaves only the sign undetermined and thus obtain for these normalized operators $U(L)$,

$$U(L_1)\,U(L_2) = \mp\, U(L_1L_2). \tag{3}$$

In this note we want to applv his mathematical arguments to the case of the Galilei group and see to what extent the general representations up to a factor can be simplified by a proper normalization. It will be seen that such a normalization is still fruitful and leads essentially to the special representation up to a factor formed by the plane-wave solutions of the non-relativistic Schrödinger equation. Prof. Bargmann has independently obtained the same result by a more general method [3].

After discussing briefly the Galilei group we shall carry out the normalization step by step, following closely Wigner's method. In fact some of the arguments that he developed for the Lorentz group are equally valid for the Galilei group; in those cases we shall simply refer to his paper (Ref. 2).

Description of the inhomogeneous Galilei group. We shall denote the general element of the proper inhomogeneous Galilei group by $N = (\mathbf{a}, b, \mathbf{v}, R)$ where $\mathbf{a}$ represents the space translation $\mathbf{x}' = \mathbf{x} + \mathbf{a}$, b represents the time displacement $t' = t + b$, $\mathbf{v}$ represents the uniform acceleration $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$ and R represents the rotation $\mathbf{x}' = R\mathbf{x}$. The order of the transformations is from right to left. By direct substitutions one easily obtains the following relations between the elementary transformations:

(4) $(\mathbf{a}_1)(\mathbf{a}_2) = (\mathbf{a}_2)(\mathbf{a}_1) = (\mathbf{a}_1 + \mathbf{a}_2)$

(5) $(b_1)(b_2) = (b_2)(b_1) = (b_1 + b_2)$

(6) $(\mathbf{v}_1)(\mathbf{v}_2) = (\mathbf{v}_2)(\mathbf{v}_1) = (\mathbf{v}_1 + \mathbf{v}_2)$

(7) $(R_1)(R_2) = (R_1 R_2)$

(8) $(\mathbf{a})(b) = (b)(\mathbf{a})$

(9) $(\mathbf{a})(\mathbf{v}) = (\mathbf{v})(\mathbf{a})$

(10) $(R)(\mathbf{a}) = (\mathbf{a}')(R)$ where $\mathbf{a}' = R\mathbf{a}$

(11) $(R)(b) = (b)(R)$

(12) $(R)(\mathbf{v}) = (\mathbf{v}')(R)$ where $\mathbf{v}' = R\mathbf{v}$

(13) $(\mathbf{v})(b) = (b)(\mathbf{a})(\mathbf{v})$ where $\mathbf{a} = b\mathbf{v}$.

By means of the relations (4-13), the product of two inhomogeneous Galilei transformations is easily calculated to be another such transformation:

$$(\mathbf{a}_1, b_1, \mathbf{v}_1, R_1)(\mathbf{a}_2, b_2, \mathbf{v}_2, R_2) = (\mathbf{a}_{12}, b_{12}, \mathbf{v}_{12}, R_{12})$$

where

(14)
$$\begin{aligned} \mathbf{a}_{12} &= \mathbf{a} + b_2\mathbf{v}_1 + R_1\mathbf{a}_2 \\ b_{12} &= b_1 + b_2 \\ \mathbf{v}_{12} &= \mathbf{v}_1 + R_1\mathbf{v}_2 \\ R_{12} &= R_1 R_2. \end{aligned}$$

Operators. The operator $D(N)$ for the general transformation can be decomposed into four elementary operators corresponding to the four elementary transformations:

(15) $$D(N) = T(\mathbf{a})\,\Theta(\mathrm{b})\,G(\mathbf{v})\,O(R).$$

Then using the commutation relations (4-13), we obtain from (2) the equivalent relations:

(16) $$T(\mathbf{a}_1)\,T(\mathbf{a}_2) = w(\mathbf{a}_1, \mathbf{a}_2)\,T(\mathbf{a}_1 + \mathbf{a}_2)$$
(17) $$\Theta(\mathrm{b}_1)\,\Theta(\mathrm{b}_2) = w(\mathrm{b}_1, \mathrm{b}_2)\,\Theta(\mathrm{b}_1 + \mathrm{b}_2)$$
(18) $$G(\mathbf{v}_1)\,G(\mathbf{v}_2) = w(\mathbf{v}_1, \mathbf{v}_2)\,G(\mathbf{v}_1 + \mathbf{v}_2)$$
(19) $$O(R_1)\,O(R_2) = w(R_1, R_2)\,O(R_1 R_2)$$
(20) $$O(R)\,T(\mathbf{a}) = w(R, \mathbf{a})\,T(R\mathbf{a})\,O(R)$$
(21) $$O(R)\,G(\mathbf{v}) = w(R, \mathbf{v})\,G(R\mathbf{v})\,O(R)$$
(22) $$O(R)\,\Theta(\mathrm{b}) = w(R, \mathrm{b})\,\Theta(\mathrm{b})\,O(R)$$
(23) $$T(\mathbf{a})\,\Theta(\mathrm{b}) = w(\mathbf{a}, \mathrm{b})\,\Theta(\mathrm{b})\,T(\mathbf{a})$$
(24) $$T(\mathbf{a})\,G(\mathbf{v}) = w(\mathbf{a}, \mathbf{v})\,G(\mathbf{v})\,T(\mathbf{a})$$
(25) $$G(\mathbf{v})\,\Theta(\mathrm{b}) = w(\mathbf{v}, \mathrm{b})\,\Theta(\mathrm{b})\,T(\mathrm{b}\mathbf{v})\,G(\mathbf{v}).$$

Normalization. The purpose of the normalization is to eliminate the arbitrariness in the w's of (16-25) as much as possible. To this end we point out the following theorems.

I. All $T(\mathbf{a})$ commute.

Proof: From (16) we have

$$T(\mathbf{a}_2) = T(\mathbf{a}_2)\,T(\mathbf{a}_1)\,T(\mathbf{a}_1)^{-1} = w(\mathbf{a}_2, \mathbf{a}_1)\,w(\mathbf{a}_2 + \mathbf{a}_1, -\mathbf{a}_1)\,T(\mathbf{a}_2)$$

or

(26) $$w(\mathbf{a}_1 + \mathbf{a}_2, -\mathbf{a}_1) = w(\mathbf{a}_2, \mathbf{a}_1)^{-1}.$$

Hence

(27) $$T(\mathbf{a}_1)\,T(\mathbf{a})\,T(\mathbf{a}_1)^{-1} = \frac{w(\mathbf{a}_1, \mathbf{a}_2)}{w(\mathbf{a}_2, \mathbf{a}_1)}\,T(\mathbf{a}_2) = \mathrm{c}(\mathbf{a}_1, \mathbf{a}_2)\,T(\mathbf{a}_2).$$

with

(28) $$\mathrm{c}(\mathbf{a}_1, \mathbf{a}_2) = \mathrm{c}(\mathbf{a}_2, \mathbf{a}_1)^{-1}.$$

Transforming (27) with $T(\mathbf{a}_3)$ we obtain

$$T(\mathbf{a}_3)\,T(\mathbf{a}_1)\,T(\mathbf{a}_2)\,T(\mathbf{a}_1)^{-1}\,T(\mathbf{a}_3)^{-1} = \mathrm{c}(\mathbf{a}_1, \mathbf{a}_2)\,T(\mathbf{a}_3)\,T(\mathbf{a}_2)\,T(\mathbf{a}_3)^{-1}$$

or

$$w(\mathbf{a}_3, \mathbf{a}_1)\,T(\mathbf{a}_3 + \mathbf{a}_1)\,T(\mathbf{a}_2)\,w(\mathbf{a}_3, \mathbf{a}_1)^{-1}\,T(\mathbf{a}_3 + \mathbf{a}_1)^{-1} = \mathrm{c}(\mathbf{a}_1, \mathbf{a}_2)\,\mathrm{c}(\mathbf{a}_3, \mathbf{a}_2)\,T(\mathbf{a}_2)$$

or

$$c(\mathbf{a}_3 + \mathbf{a}_1, \mathbf{a}_2) = c(\mathbf{a}_3, \mathbf{a}_2)\, c(\mathbf{a}_1, \mathbf{a}_2). \tag{29}$$

It follows from (29) that (Ref. 2. page 171)

$$c(\mathbf{a}_1, \mathbf{a}_2) = \exp\left\{2\pi i \sum_{\varkappa=1}^{3} a_{1\varkappa} f_\varkappa(\mathbf{a}_2)\right\} \tag{30}$$

and using (28) we obtain

$$\sum_{\varkappa=1}^{3} \left[a_{1\varkappa} f_\varkappa(\mathbf{a}_2) + a_{2\varkappa} f_\varkappa(\mathbf{a}_1)\right] = n(\mathbf{a}_1, \mathbf{a}_2) \tag{31}$$

where $n(\mathbf{a}_1, \mathbf{a}_2)$ is an integer. Setting for $\mathbf{a}_2$ in (31) the three unit vectors $\mathbf{e}_\lambda$ (λ component of which is 1, the others being zero) in turn and letting $f_\varkappa(\mathbf{e}_\lambda) = -f_{\varkappa\lambda}$ yields

$$f_\lambda(\mathbf{a}_1) = n(\mathbf{a}_1, \mathbf{e}_\lambda) + \sum_{\varkappa=1}^{3} a_{1\varkappa} f_{\varkappa\lambda}$$

and putting this back into (31) we find

$$\sum_{\varkappa,\lambda=1}^{3} (a_{1\varkappa} a_{2\lambda} + a_{1\lambda} a_{2\varkappa}) f_{\lambda\varkappa} + \sum_{\varkappa=1}^{3} \left[a_{1\varkappa} n(\mathbf{a}_2, \mathbf{e}_\varkappa) + a_{2\varkappa} n(\mathbf{a}_1, \mathbf{e}_\varkappa)\right] = n(\mathbf{a}_1, \mathbf{a}_2). \tag{32}$$

Now, by assuming for the components of $\mathbf{a}_1$ and $\mathbf{a}_2$ such values that are transcendental both with respect to each other and the $f_{\varkappa\lambda}$ (which are fixed numbers) we see that (32) can not hold except if the coefficient of every term vanishes; i.e.

$$f_{\varkappa\lambda} + f_{\lambda\varkappa} = 0 \quad \text{and} \quad n(\mathbf{a}, \mathbf{e}_\lambda) = 0. \tag{33}$$

Thus (30) becomes

$$c(\mathbf{a}_1, \mathbf{a}_2) = \exp\left\{2\pi i \sum_{\varkappa,\lambda=1}^{3} a_{1\varkappa} a_{2\lambda} f_{\varkappa\lambda}\right\} \tag{34}$$

We now transform the equation (27) by the operator $O(R)$ and using also (20) we obtain,

$$O(R)\, T(\mathbf{a}_1)\, T(\mathbf{a}_2)\, T(\mathbf{a}_1)^{-1}\, O(R)^{-1} = c(\mathbf{a}_1, \mathbf{a}_2)\, w(R, \mathbf{a}_2)\, T(R\mathbf{a}_2),$$

on the other hand,

$$\begin{aligned} O(R)\, T(\mathbf{a}_1)\, O(R)^{-1}\, O(R)\, T(\mathbf{a}_2)\, O(R)^{-1}\, O(R)\, T(\mathbf{a}_1)^{-1}\, O(R)^{-1} \\ = w(R, \mathbf{a}_1)\, T(R\mathbf{a}_1)\, w(R, \mathbf{a}_2)\, T(R, \mathbf{a}_2)\, w(R, \mathbf{a}_1)^{-1}\, T(R\mathbf{a}_1)^{-1} \\ = w(R, \mathbf{a}_2)\, c(R\mathbf{a}_1, R\mathbf{a}_2)\, T(R\mathbf{a}_2) \end{aligned}$$

hence

$$c(\mathbf{a}_1, \mathbf{a}_2) = c(R\mathbf{a}_1, R\mathbf{a}_2) \tag{35}$$

for any rotation R. Combined with (34) this gives,

$$\sum_{\varkappa,\lambda=1}^{3}\left(f_{\varkappa\lambda}a_{1\varkappa}a_{2\lambda}-\sum_{\mu,\nu=1}^{3}f_{\mu\nu}R_{\mu\varkappa}R_{\nu\varkappa}a_{1\varkappa}a_{2\lambda}\right)=\mathrm{n}'(\mathbf{a}_1,\mathbf{a}_2)$$

where $\mathrm{n}(\mathbf{a}_1,\mathbf{a}_2)$ is an integer. Since this is true for every $\mathbf{a}_1,\mathbf{a}_2$ we again obtain

$$f_{\varkappa\lambda}=\sum_{\mu,\nu=1}^{3}f_{\mu\nu}R_{\mu\varkappa}R_{\nu\lambda}$$

or

$$f=R'fR=R^{-1}fR$$

for any rotation. As the only matrix which is invariant under all rotations is the identity matrix, it follows from (33) that f vanishes identically.

Hence

$$\mathrm{c}(\mathbf{a}_1,\mathbf{a}_2)=1$$

and

(36) $$T(\mathbf{a}_1)\,T(\mathbf{a}_2)=T(\mathbf{a}_2)\,T(\mathbf{a}_1).$$

II. All $G(\mathbf{v})$ commute. Proof is identical with that for I.

III. All $\Theta(\mathrm{b})$ commute. Proof is very similar. Instead of (30), (31) we obtain in the same way,

(37) $$\mathrm{c}(\mathrm{b}_1,\mathrm{b}_2)=\exp\{2\pi i\,\mathrm{b}_1 f(\mathrm{b}_2)\}$$

where f is a scalar and

(38) $$\mathrm{b}_1 f(\mathrm{b}_2)+\mathrm{b}_2 f(\mathrm{b}_1)=n(\mathrm{b}_1,\mathrm{b}_2)$$

where $\mathrm{n}(\mathrm{b}_1,\mathrm{b}_2)$ is an integer; this gives for $\mathrm{b}_1=\mathrm{b}_2=\mathrm{b}$

$$f(\mathrm{b})=\frac{n(\mathrm{b},\mathrm{b})}{2\mathrm{b}}$$

and putting it back into (38) we have

$$\frac{\mathrm{b}_1}{\mathrm{b}_2}\mathrm{n}(\mathrm{b}_2,\mathrm{b}_2)+\frac{\mathrm{b}_2}{\mathrm{b}_1}\mathrm{n}(\mathrm{b}_1,\mathrm{b}_1)=2\mathrm{n}(\mathrm{b}_1,\mathrm{b}_2)$$

By choosing a transcendental value for $\frac{\mathrm{b}_1}{\mathrm{b}_2}$, we see that this equation can only be satisfied if

$$\mathrm{n}(\mathrm{b},\mathrm{b})=0,\quad \mathrm{n}(\mathrm{b}_1,\mathrm{b}_2)=0.$$

Consequently

$$f(\mathrm{b}) = 0, \quad c(\mathrm{b}_1, \mathrm{b}_2) = 1$$

and

$$\Theta(\mathrm{b}_1)\,\Theta(\mathrm{b}_2) = \Theta(\mathrm{b}_2)\,\Theta(\mathrm{b}_1).$$

IV. Normalization of $\Theta(\mathrm{b})$. Prof. Bargmann has shown very simply how every $\Theta(\mathrm{b})$ can be multiplied with a definite phase factor $e^{-i\eta(b)}$ so as to get $w(\mathrm{b}_1, \mathrm{b}_2) = 1$ for the new operators

$$V(\mathrm{b}) = e^{-i\eta(b)}(\mathrm{b}).$$

We shall reproduce here his proof for the sake of completeness.

By multiplying with the phase factors we obtain from (17)

(40) $$V(\mathrm{b}_1)\,V(\mathrm{b}_2) = w'(\mathrm{b}_1, \mathrm{b}_2)\,V(\mathrm{b}_1, + \mathrm{b}_2)$$

where

(41) $$w'(\mathrm{b}_1,\mathrm{b}_2) = w(\mathrm{b}_1, \mathrm{b}_2) \exp\{- i\,[\eta(\mathrm{b}_1) + \eta(\mathrm{b}_2) - \eta(\mathrm{b}_1+\mathrm{b}_2)]\}$$

with

(42) $$w(\mathrm{b}_1, \mathrm{b}_2) = w(\mathrm{b}_2, \mathrm{b}_1)\,;$$

or letting

$$w(\mathrm{b}_1,\mathrm{b}_2) = \exp\{i\xi(\mathrm{b}_1, \mathrm{b}_2)\}$$

we have

$$\xi'(\mathrm{b}_1, \mathrm{b}_2) = \xi(\mathrm{b}_1, \mathrm{b}_2) - \eta(\mathrm{b}_1) - \eta(\mathrm{b}_2) + \eta(\mathrm{b}_1 + \mathrm{b}_2).$$

The function $\xi(\mathrm{b}_1, \mathrm{b}_2)$ satisfies two relations :

(43) $$\xi(\mathrm{b}_1, \mathrm{b}_2) = \xi(\mathrm{b}_2, \mathrm{b}_1)$$

which follows from (42) ; and

(44) $$\xi(\mathrm{b}_1, \mathrm{b}_2) + \xi(\mathrm{b}_1 + \mathrm{b}_2, \mathrm{b}_3) = \xi(\mathrm{b}_1, \mathrm{b}_2 + \mathrm{b}_3) + \xi(\mathrm{b}_2, \mathrm{b}_3)$$

which follows from the associativity of $\Theta(\mathrm{b})$.

We now claim that for any given continuous $\xi(\mathrm{b}_1, \mathrm{b}_2)$ that satisfies (43-44), one can find a function $\eta(\mathrm{b})$ such that

(45) $$\xi'(\mathrm{b}_1, \mathrm{b}_2) = 0 \quad \text{or} \quad \xi(\mathrm{b}_1, \mathrm{b}_2) = \eta(\mathrm{b}_1) + \eta(\mathrm{b}_2) - \eta(\mathrm{b}_1 + \mathrm{b}_2).$$

To prove this assertion, we remark first that if $\eta_0(\mathrm{b})$ satisfies (45), so does $\eta(\mathrm{b}) = \eta_0(\mathrm{b}) + C\mathrm{b}$ where C is any constant. In particular for $C = -\eta_0(1)$ we have

(46) $$\eta(1) = 0.$$

We shall suppose that (46) is satisfied. Integrating (45) with respect to b_2 and interchanging b and b_1 we obtain

$$\int_0^b \xi(b_1, b_2)\, db_2 = b\eta(b_1) + \int_0^b \eta(b_2) db_2 - \int_0^b \eta(b_1 + b_2) db_2$$

$$\int_0^{b_1} \xi(b, b_2) db_2 = b_1\eta(b) + \int_0^{b_1} \eta(b_2)\, db_2 - \int_0^{b_1} (b + b_2) db_2$$

and after subtracting the first equation from the second and setting $b_1 = 1$,

(47) $$\eta(b) = \int_0^1 \xi(b, b_2) db_2 - \int_0^b \xi(1, b_2)\, db_2.$$

It remains to show that (47) indeed satisfies (45). We have

$$\eta(b_1) + \eta(b_2) - \eta(b_1 + b_2) = \int_0^1 \left[\xi(b_1, b) + \xi(b_2, b) - \xi(b_1 + b_2, b)\right] db$$
$$- \int_0^{b_1} \xi(1, b) db - \int_0^{b_2} \xi(1, b) db + \int_0^{b_1+b_2} \xi(1, b)\, db$$

or by using (44),

$$= \xi(b_1, b_2) + F(b_1, b_2)$$

where

$$F(b_1, b_2) = \int_0^1 \xi(b_1, b) db - \int_{b_2}^{b_2+1} \xi(b_1, b) db +$$
$$\int_{b_1}^{b_1+b_2} \xi(1, b) db - \int_0^{b_2} \xi(1, b) db;$$

but since

$$F(b_1, 0) = 0$$

and again by (44)

$$\frac{\partial F(b_1, b_2)}{\partial b_2} = -\xi(b_1, b_2 + 1) + \xi(b_1, b_2) + \xi(1, b_1 + b_2) - \xi(1, b_2) = 0$$

we must have

$$F(b_1, b_2) \equiv 0$$

which proves the theorem. Consequently we shall put in what follows

(48) $$w(b_1, b_2) = 1.$$

V. Normalization of $T(\mathbf{a})$. Consider the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in the x, y, z directions. The arguments of the previous section show that one can normalize $T(a_1\mathbf{e}_1)$, $T(a_2\mathbf{e}_2)$, $T(a_3\mathbf{e}_3)$ in such a way that

$$T(a_i\mathbf{e}_i)\, T(b_i\mathbf{e}_i) = T\big[(a_i + b_i)\mathbf{e}_i\big] \quad , \; i = 1, 2, 3. \tag{49}$$

One can also fix the equality

$$T(\mathbf{a}) = T(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = T(a_1\mathbf{e}_1)\,T(a_2\mathbf{e}_2)\,T(a_3\mathbf{e}_3) \tag{50}$$

as this essentially defines the operation $T(\mathbf{a})$. Then we shall have using (49), (50) and (36),

$$\begin{aligned} T(\mathbf{a})T(\mathbf{b}) &= T(a_1\mathbf{e}_1)T(a_2\mathbf{e}_2)T(a_3\mathbf{e}_3)T(b_1\mathbf{e}_1)T(b_2\mathbf{e}_2)T(b_3\mathbf{e}_3) \\ &= T\big[(a_1+b_1)\mathbf{e}_1\big]T\big[(a_2+b_2)\mathbf{e}_2\big]T\big[(a_3+b_3)\mathbf{e}_3\big] = T(\mathbf{a}+\mathbf{b}); \end{aligned}$$

hence

$$w(\mathbf{a}, \mathbf{b}) = 1. \tag{51}$$

The equation $T(\mathbf{a})T(\mathbf{b}) = T(\mathbf{a}+\mathbf{b})$ remains valid if one replaces $T(\mathbf{a})$ by $e^{2\pi i \mathbf{a}.\mathbf{c}}T(\mathbf{a})$, where $\mathbf{c}$ is an arbitrary constant vector. Following Wigner we shall make use of this remaining freedom to eliminate $w(R, \mathbf{a})$. From (20) we have

$$O(R)T(\mathbf{a})O(R)^{-1} = w(R, \mathbf{a})T(R\mathbf{a}) \tag{52}$$

and transforming it with $O(S)$,

$$O(S)O(R)T(\mathbf{a})O(R)^{-1}O(S)^{-1} = w(R,\mathbf{a})O(S)T(R\mathbf{a})O(S)^{-1}$$

which gives, using (19),

$$Rw(S,R)O(SR)T(\mathbf{a})w(S,R)^{-1}O(SR)^{-1} = w(R,\mathbf{a})w(S,R\mathbf{a})T(SR\mathbf{a})$$

or

$$w(SR,\mathbf{a}) = w(R,\mathbf{a})\; w(S,R\mathbf{a}). \tag{53}$$

On the other hand, applying $O(R)$ to $T(\mathbf{a}_1)T(\mathbf{a}_2) = T(\mathbf{a}_1+\mathbf{a}_2)$ we obtain

$$\begin{aligned} O(R)T(\mathbf{a}_1)T(\mathbf{a}_2) &= w(R,\mathbf{a}_1)T(R\mathbf{a}_1)O(R)T(\mathbf{a}_2) \\ &= w(R,\mathbf{a}_1)w(R,\mathbf{a}_2)T(R\mathbf{a}_1)T(R\mathbf{a}_2)O(R) \end{aligned}$$

or

$$w(R,\mathbf{a}_1+\mathbf{a}_2) = w(R,\mathbf{a}_1)w(R,\mathbf{a}_2), \tag{54}$$

therefore

$$w(R,\mathbf{a}) = \exp\{2\pi i \mathbf{a}.\mathbf{f}(R)\} \quad \text{where } \mathbf{f} \text{ is a vector.} \tag{55}$$

Inserting this back into (53) we obtain

$$\mathbf{a}\cdot\{\mathbf{f}(SR)-\mathbf{f}(R)-R^{-1}\mathbf{f}(S)\}=\mathrm{n}$$

where n is integer; which shows, since n depends linearly on the arbitrary vector **a**, that n = 0 and

(56) $$\mathbf{f}(SR)=\mathbf{f}(R)+R^{-1}\mathbf{f}(S).$$

It is shown in Ref. 2 (page 175) that (56) is equivalent to

$$\mathbf{f}(R)=(1-R^{-1})\,\mathbf{r}_0$$

where $\mathbf{r}_0$ is an arbitrary constant vector. We thus obtain

$$w(R,\mathbf{a})=\exp\{2\pi i\,\mathbf{a}\,(1-R^{-1})\,\mathbf{r}_0\}$$

and

$$O(R)T(\mathbf{a})=\exp\{2\pi i\mathbf{a}\cdot(1-R^{-1})\mathbf{r}_0\}\,T(R\mathbf{a})O(R)$$

or, after multiplying every $T(\mathbf{a})$ by $\exp\{-2\pi i\mathbf{a}\cdot\mathbf{r}_0\}$, for the new operators

(57) $$O(R)T(\mathbf{a})=T(R\mathbf{a})O(R).$$

This completes the normalization of $T(\mathbf{a})$.

VI. Normalization of $G(\mathbf{v})$ is done in the same way as in *V*. and results are

(58) $$w(\mathbf{v}_1,\mathbf{v}_2)=1\ ;\quad w(R,\mathbf{v})=1.$$

VII. $\Theta(\mathrm{b})$ commutes with $O(R)$.

Proof: By applying the same method as the one used in eliminating $w(R,\mathbf{a})$ we obtain the two relations:

(59) $$w(R,\mathrm{b}_1+\mathrm{b}_2)=w(R,\mathrm{b}_1)+w(R,\mathrm{b}_2)$$

(60) $$w(R_2R_1,\mathrm{b})=w(R_1,\mathrm{b})w(R_2,\mathrm{b}).$$

From (59) follows

$$w(R,\mathrm{b})=\exp\{2\pi i\mathrm{b}f(R)\}$$

where f is a scalar. Inserting this in (60) we obtain as usual,

(61) $$f(R_2R_1)=f(R_2)+f(R_1).$$

Consequently we have

(62) $$f(R^n)=\mathrm{n}f(R)\quad\text{and}\quad f(E)=0$$

where n is a positive integer and E is identity matrix. It follows from (62) that if R_q is a rotation around any axis by an angle $2\pi \frac{p}{q}$ (p, q being positive integers) we have

$$q f(R_q) = 0 \quad \text{or} \quad f(R_q) = 0.$$

Therefore $f(R)$ must vanish identically and we have

$$w(R, b) = 1. \tag{63}$$

VIII. $\Theta(b)$ commutes with $T(\mathbf{a})$.

Proof: From (23) we obtain as previously,

$$w(\mathbf{a}_1, b)\, w(\mathbf{a}_2, b) = w(\mathbf{a}_1 + \mathbf{a}_2, b) \tag{64}$$

$$w(\mathbf{a}, b_1)\, w(\mathbf{a}, b_2) = w(\mathbf{a}, b_1 + b_2). \tag{65}$$

and the resulting equations

$$w(\mathbf{a}, b) = \exp\{2\pi i \mathbf{a} \cdot \mathbf{f}(b)\}$$

where $\mathbf{f}$ is a vector; and

$$\mathbf{f}(b_1 + b_2) = \mathbf{f}(b_1) + \mathbf{f}(b_2). \tag{66}$$

The general solutions of (66) is

$$\mathbf{f}(b) = b\mathbf{B}$$

where $\mathbf{B}$ is an arbitrary constant vector. We have now

$$T(\mathbf{a})\Theta(b) = \exp\{2\pi i\, b\mathbf{a} \cdot \mathbf{B}\}\, \Theta(b) T(\mathbf{a}). \tag{67}$$

However, transforming with $O(R)$ we can see that $\mathbf{B} = 0$; it gives in fact

$$O(R) T(\mathbf{a}) \Theta(b) O(R)^{-1} = \exp\{2\pi i b \mathbf{a} \cdot \mathbf{B}\}\, O(R) \Theta(b) T(\mathbf{a}) O(R)$$

or using V. and VII.,

$$T(R\mathbf{a})\Theta(b) = \exp\{2\pi i b \mathbf{a} \cdot \mathbf{B}\}\Theta(b) T(R\mathbf{a})\,; \tag{68}$$

but we also have from (67) directly

$$T(R\mathbf{a})\, \Theta(b) = \exp\{2\pi i b R\mathbf{a} \cdot \mathbf{B}\}\Theta(b) T(R\mathbf{a}). \tag{69}$$

Comparing (68) and (69) we obtain, since b is arbitrary,

$$\mathbf{a} \cdot \mathbf{B} = R\mathbf{a} \cdot \mathbf{B} \quad \text{or} \quad \mathbf{a} \cdot (\mathbf{B} - R^{-1}\mathbf{B}) = 0 \quad \text{for every } \mathbf{a}\,;$$

hence

$$R\mathbf{B} = \mathbf{B} \quad \text{for every } R.$$

Consequently

$$\mathbf{B} \equiv 0$$

and

(70) $$w(\mathbf{a}, \mathrm{b}) = 1.$$

IX. Normalization of $O(R)$ It is shown in Ref. 2 (pages 176-178) that $O(R)$ can be normalized so as to give

(71) $$w(R_1, R_2) = \mp 1.$$

Because of the form of the relations (19-22), this normalization does not interfere with other normalizations.

X. Determination of $w(\mathbf{a},\mathbf{v})$.

From (24) we obtain, in the usual way,

(72) $$w(\mathbf{a}_1,\mathbf{v})w(\mathbf{a}_2,\mathbf{v})=w(\mathbf{a}_1+\mathbf{a}_2,\mathbf{v})$$

(73) $$w(\mathbf{a},\mathbf{v}_1)w(\mathbf{a},\mathbf{v}_2)=w(\mathbf{a},\mathbf{v}_1+\mathbf{v}_2)$$

from which follows

$$w(\mathbf{a},\mathbf{v})=\exp\{2\pi i\mathbf{a}\cdot\mathbf{f}(\mathbf{v})\}$$

where $\mathbf{f}$ is a vector satisfying

(74) $$f(\mathbf{v}_1+\mathbf{v}_2)=f(\mathbf{v}_1)+f(\mathbf{v}_2).$$

The general solution of (74) is given by

$$\mathbf{f}(\mathbf{v}) = A\mathbf{v}$$

where A is a constant arbitrary matrix. Hence (24) becomes

(75) $$T(\mathbf{a})G(\mathbf{v})=\exp\{2\pi i\mathbf{a}\cdot A\mathbf{v}\}G(\mathbf{v})T(\mathbf{a}).$$

However, as in VIII., this expression is not symmetrical enough; in fact transforming with $O(R)$ we obtain

(76) $$T(R\mathbf{a})G(R\mathbf{v})=\exp\{2\pi i\mathbf{a}\cdot A\mathbf{v}\}G(R\mathbf{v})T(R\mathbf{a})$$

which combined with (75) gives

$$\mathbf{a}\cdot A\mathbf{v}=R\mathbf{a}\cdot AR\mathbf{v}=\mathbf{a}\cdot R^{-1}AR\mathbf{v} \quad \text{for every } \mathbf{a} \text{ and } \mathbf{v};$$

or

$$A=R^{-1}AR \qquad \text{for every } R.$$

It follows that A is a multiple of the unit matrix and can be taken as an ordinary number; we have thus finally,

(77) $$w(\mathbf{a},\mathbf{v})=\exp\{2\pi i A\mathbf{a}\cdot\mathbf{v}\}$$

where A is an arbitrary constant.

XI. Determination of $w(\mathbf{v},\mathrm{b})$.

From (25) we have,

(78) $$G(\mathbf{v}_1)\Theta(\mathrm{b})G(\mathbf{v}_1)^{-1}=w(\mathbf{v}_1,\mathrm{b})\Theta(\mathrm{b})T(\mathrm{b}\mathbf{v}_1);$$

transforming this with $G(\mathbf{v}_2)$ we obtain,

$$G(\mathbf{v}_2)G(\mathbf{v}_1)\Theta(\mathrm{b})G(\mathbf{v}_1)^{-1}G(\mathbf{v}_2)^{-1}=w(\mathbf{v}_1,\mathrm{b})G(\mathbf{v}_2)\Theta(\mathrm{b})T(\mathrm{b}\mathbf{v}_1)G(\mathbf{v}_2)^{-1}$$

or using the relations established so far,

(79) $$w(\mathbf{v}_1+\mathbf{v}_2,\mathrm{b})=w(\mathbf{v}_1\mathrm{b},)w(\mathbf{v}_2,\mathrm{b})\ \exp\{-2\pi \mathrm{i} A \mathrm{b}\mathbf{v}_1\cdot\mathbf{v}_2\}.$$

On the other hand by means of

$$G(\mathbf{v})\Theta(\mathrm{b}_1)\Theta(\mathrm{b}_2)=w(\mathbf{v},\mathrm{b}_1)\Theta(\mathrm{b}_1)T(\mathrm{b}_1\mathbf{v})w(\mathbf{v},\mathrm{b}_2)\Theta(\mathrm{b}_2)T(\mathrm{b}_2\mathbf{v})G(\mathbf{v})$$

we find

(80) $$w(\mathbf{v},\mathrm{b}_1+\mathrm{b}_2)=w(\mathbf{v},\mathrm{b}_1)w(\mathbf{v},\mathrm{b}_2)$$

from which follows

$$w(\mathbf{v},\mathrm{b})=\exp\{2\pi \mathrm{i}\mathrm{b}f(\mathbf{v})\}$$

where f is a scalar. Inserting this in (79) we find

(81) $$f(\mathbf{v}_1+\mathbf{v}_2)=f(\mathbf{v}_1)+f(\mathbf{v}_2)-A\mathbf{v}_1\cdot\mathbf{v}_2$$

The general solution of (81) is given by

$$f(\mathbf{v})=-\frac{A}{2}\mathbf{v}^2+\mathbf{v}\cdot V_0:$$

but again by operating with $O(R)$ one can show that $V_0=0$. Thus we have

(82) $$w(\mathbf{v},\mathrm{b})=\exp\left\{-2\pi \mathrm{i} A\frac{\mathrm{b}}{2}\mathbf{v}^2\right\}.$$

Conclusion.

Using (48), (51), (57), (58), (63), (70), (71), (77) and (82 we have for the normalised operators

$$D(N_1)D(N)_2=T(\mathbf{a}_1)\Theta(\mathrm{b}_1)G(\mathbf{v}_1)O(R_1)T(\mathbf{a}_2)\Theta(\mathrm{b}_2)G(\mathbf{v}_2)O(R_2)$$

$$=\mp\exp\left\{-2\pi \mathrm{i} A(\mathbf{a}_2\cdot\mathbf{v}_1+\frac{\mathrm{b}_2}{2}\mathbf{v}_1^2)\right\}T(\mathbf{a}_1+\mathrm{b}_2\mathbf{v}_1+R_1\mathbf{a}_2)\Theta(\mathrm{b}_1+\mathrm{b}_2)$$
$$G(\mathbf{v}_1+R\mathbf{v}_2)O(R_1R_2)$$

or by (14)

(83) $$D(N_1)D(N_2)=\mp\exp\left\{-2\pi \mathrm{i} A(\mathbf{a}_2\cdot\mathbf{v}_1+\frac{1}{2}\mathrm{b}_2\mathbf{v}_1^2)\right\}D(N_1N_2)$$

where A is an arbitrary constant. This shows that by prope normalization all the represetations up to a factor of the Galilei group (of the form (2) where $w(N_1,N_2)$ is of modulus unity) can be brought to the same form. They will only differ from each other by the value of the arbitrary constant A. Furthermore, (83) is essentially the representation formed by the plane-wave solutions of the non-relativistic Schröndinger equation. Consider in fact the plane-wave solution for a particle with mass m, momentum **p** and spin zero,

$$\Psi(\mathbf{p})=\exp\left\{\frac{2\pi i}{h}\left(\mathbf{p}\cdot\mathbf{x}-\frac{\mathbf{p}^2}{2m}t\right)\right\} \tag{84}$$

One easily obtains for this solution

$$\begin{aligned} &T(\mathbf{a})\Psi(\mathbf{p})=\exp\left\{-\frac{2\pi i m}{h}\,\mathbf{a}\cdot\mathbf{v}\right\}\Psi(\mathbf{p}) \\ &\Theta(b)\Psi(\mathbf{p})=\exp\left\{\frac{2\pi i m}{h}\,\frac{b}{2}\,\mathbf{v}^2\right\}\Psi(\mathbf{p}) \\ &G(\mathbf{v})\Psi(\mathbf{p})=\Psi(\mathbf{p}-m\mathbf{v}) \\ &O(R)\Psi(\mathbf{p})=\Psi(R^{-1}\mathbf{p}) \end{aligned} \tag{85}$$

and consequently

$$D(N_1)D(N_2)\Psi(\mathbf{p})=\exp\left\{2\pi i\,\frac{m}{h}\,(\mathbf{a}_2\cdot\mathbf{v}_1)\,\frac{1}{2}\,b_2\mathbf{v}_1^2)\right\}D(N_1N_2)\,\Psi(\mathbf{p}). \tag{86}$$

This representation is identical with the one obtained from (83) by letting $A=-\frac{m}{h}$ and taking the positive sign for zero spin.

I am greatly indebted to Prof. Wigner for many illuminating discussions always resulting in extremely helpful suggestions and to Prof. Bargmann for very kindly communicating to me his unpublished results.

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(Manuscript received on August 14, 1952)