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İki Boyutlu Düzlemde Hareketin Kübit Temsili

Qubit Representation of Motion on the 2D Plane

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Özet. Bu çalışmada bir spin-1/2 parçacığın spin serbesti derecesi olarak bir kübiti (kuantum bit) ele alıyoruz. İki boyutlu düzlem ve kuzey kutbu çıkartılmış 2-küre topolojik olarak eşdeğer olduklarından, iki boyutlu düzlemdeki hareket 2-küre üzerinde temsil edilebilir. Ters stereografik fonksiyonu kullanarak iki boyutlu düzlemi 2-küre üzerine aktarıyoruz. Öte yandan 2-küre, üzerindeki her noktanın bir kübiti temsil ettiği Bloch küresi olarak düşünülebildiğinden, bir parçacığın iki boyutlu düzlem üzerindeki zaman içindeki evrimi, buna karşılık gelen kübitin Bloch küresi üzerinde zaman içindeki evrimine karşılık gelir. Manyetik alanın spini döndürebilme özelliğini kullanarak iki boyutlu bir düzlemdeki parçacığın hareketinin tek bir kübit kullanılarak simüle edilmesini sağlayacak zamana bağımlı manyetik alanın tam formunu veriyoruz.

Anahtar Kelimeler: Kübit, manyetik alan, düzlemsel hareket, stereografik fonksiyon.

Abstract. In this study we consider a qubit (quantum bit) as the spin degree of freedom of a spin-1/2 particle. Since the 2D plane and the 2-sphere minus the north pole are topologically equivalent, one can represent motion on the 2D plane on the 2-sphere. We use inverse stereographic projection to map the 2D plane to the 2-sphere minus the north pole. Because the 2-sphere can be thought of as the Bloch sphere where any point represents a qubit, the time evolution of a particle on the 2D plane corresponds to the time evolution of the corresponding qubit on the Bloch sphere. We use the property of magnetic field to rotate spins to provide exact time dependent magnetic field to simulate, using a single qubit, the path followed by a particle on the 2D plane.

Keywords: Qubit, magnetic field, planar motion, stereographic projection.

1. Introduction

Quantum computers are based on operations on qubits (quantum bits), whereas digital computers are based on operations on bits. A bit can take the value 0 or 1, where as a qubit can take values $\alpha|0\rangle + \beta|1\rangle$ where α, β are complex numbers such that the sum $|\alpha|^2 + |\beta|^2$ is unity and $|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1}$ are orthogonal basis vectors. For an introduction to quantum information theory, readers may find Refs. [1, 2, 6] useful.

Bloch sphere is the space where any qubit is represented by a point on its surface. Technically speaking it is the 2-sphere (S^2), that is, the two dimensional sphere sitting inside $\mathbb{R}^3$. A point $|\theta, \phi\rangle$ corresponds to the eigenvector of the $\hat{n} \cdot \vec{S}$ spin operator with the positive eigenvalue, where $\hat{n}$ is a unit vector in $\mathbb{R}^3$ given as

$$\hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)). \tag{1}$$

In this paper we considered the qubit as the spin degree of freedom of a spin-1/2 particle, *i.e.* an electron. Then we considered motion of a particle on the 2D plane. By using the inverse stereographic projection we mapped the motion of a particle on the plane to S^2 minus the north pole. By considering

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the corresponding point on the S^2 with the qubit it represents, we found out the exact magnetic field, as a function of particle (polar) coordinates on the plane, that should be applied to the qubit in order to simulate the motion on the 2D plane.

The organization of the paper is as follows: in Section 2 we find the Hamiltonian to make the qubit trace any curve on the Bloch sphere, in Section 3 using the inverse stereographic projection we map 2D plane to S^2 minus the north pole, in Section 4 we find the magnetic field as a function of particle trajectory on 2D plane, in Section 5 we give details on how to prepare the qubit in the intial position, in Section 6 we provide two simple applications of our model and finally in Section 7 we conclude the paper.

2. The motion of a qubit on any curve on the Bloch sphere

In this Section we find the Hamiltonian that causes the spin to trace any curve on the Bloch sphere. We then find the magnetic field as a function of angles and angular velocities that reproduces the aforesaid motion.

2.1. The Hamiltonian to trace any curve on the Bloch sphere

Let us denote by the ket $|\theta,\phi\rangle$, the spin state with the positive eigenvalue of $\hbar/2$ of the operator $\hat{n}\cdot\vec{S}$ where $\hat{n}=(\sin(\theta)\cos(\phi),\sin(\theta)\sin(\phi),\cos(\theta))$. Here both θ and ϕ are functions of time t. It is known in the literature that we can write $|\theta,\phi\rangle$ as a superposition of $|+\rangle$ and $|-\rangle$ states (in the qubit notation $|+\rangle$ and $|-\rangle$ correspond to $|0\rangle$ and $|1\rangle$ respectively) where $|\pm\rangle$ are eigenvectors of S_z with eigenvalues $\pm\hbar/2$:

$$|\theta,\phi\rangle=\begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2)\end{pmatrix}. \tag{2}$$

The Schrödinger equation is as follows:

$$H|\theta,\phi\rangle=i\hbar\partial_t|\theta,\phi\rangle. \tag{3}$$

By writing the Hamiltonian as a 2×2 matrix as

$$H=\begin{pmatrix} a & b \\ c & d\end{pmatrix}. \tag{4}$$

The Schrödinger equation reduces to the following two complex quations:

$$ae^{-i\phi/2}\cos(\frac{\theta}{2})+be^{i\phi/2}\sin(\frac{\theta}{2})=\frac{\hbar}{2}\dot{\phi}e^{-i\phi/2}\cos(\frac{\theta}{2})-\frac{i\hbar}{2}\dot{\theta}e^{-i\phi/2}\sin(\frac{\theta}{2}), \tag{5}$$

$$ce^{-i\phi/2}\cos(\frac{\theta}{2})+de^{i\phi/2}\sin(\frac{\theta}{2})=-\frac{\hbar}{2}\dot{\phi}e^{i\phi/2}\sin(\frac{\theta}{2})+\frac{i\hbar}{2}\dot{\theta}e^{i\phi/2}\cos(\frac{\theta}{2}). \tag{6}$$

By equating the coefficients of $\sin(\theta/2)$ and $\cos(\theta/2)$ on both side of equations we can solve for a,b,c,d. The solution is as follows:

$$a=\frac{\hbar}{2}\dot{\phi}, \tag{7}$$

$$b=-\frac{i\hbar}{2}e^{-i\phi}\dot{\theta}, \tag{8}$$

$$c=\frac{i\hbar}{2}e^{i\phi}\dot{\theta}, \tag{9}$$

$$d=-\frac{\hbar}{2}\dot{\phi}. \tag{10}$$

In the end, we can write down the Hamiltonian in the following form:

$$H=\frac{\hbar}{2}\begin{pmatrix} \dot{\phi} & -ie^{-i\phi}\dot{\theta} \\ ie^{i\phi}\dot{\theta} & -\dot{\phi}\end{pmatrix}. \tag{11}$$

If the initial state of the system is $|\theta(0),\phi(0)\rangle$ this Hamiltonian evolves the system along a path $|\theta(t),\phi(t)\rangle$ on the unit sphere. We can then give the following theorem.

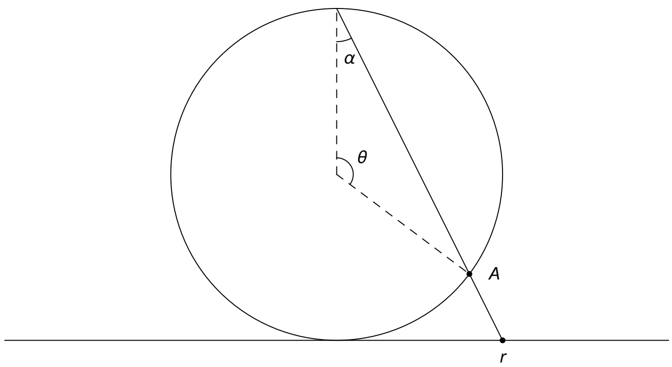

Figure 1. The inverse stereographic projection maps each point on the x, y plane to a unique point on S^2 minus the north pole. Here we illustrate a 1D example for fixed ϕ.

Theorem 1. *Let the state of the qubit is given by $|\theta, \phi\rangle$ where θ, ϕ are functions of time, then a curve represented on the Bloch sphere by $\theta(t), \phi(t)$ parameters is traced if the Hamiltonian given in Eq. (11) is applied to the qubit.*

2.2. Magnetic field to make qubit trace any curve on S^2

First, let us remember the spin operators:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}$$

We can write the Hamiltonian operator, H, we found in Eq. (11) as a sum over spin operators as follows:

$$H = -\sin(\phi)\dot{\theta} S_x + \cos(\phi)\dot{\theta} S_y + \dot{\phi} S_z. \tag{13}$$

We give the following theorem about the magnetic field vector applied to the spin-1/2 particle that will make its spin follow any curve on the Bloch sphere.

Theorem 2. *The time dependent magnetic field that make the spin trace any curve on S^2 is given as follows:*

$$\vec{B} = \frac{1}{\gamma}(\sin(\phi)\dot{\theta}, -\cos(\phi)\dot{\theta}, -\dot{\phi}). \tag{14}$$

Proof. It is known that the Hamiltonian for a spin-1/2 particle under a magnetic field $\vec{B}$ is given by $H_m = -\gamma \vec{B} \cdot \vec{S}$ where γ is the gyromagnetic ratio [4]. So, by comparing this form with Eq. (13) we see that the time dependent magnetic field that make the spin trace any curve on S^2 is given by the form in the theorem. □

3. Mapping 2D plane to S^2

As is well known in topology, one can map 2D plane to S^2 with the north pole removed, in a continuous manner. In this paper we will use the inverse stereographic projection (stereographic projection is the one that maps unit sphere onto a plane, so we use its inverse). The main reason why we chose this mapping is that it has axial symmetry that is ϕ coordinate on the plane is mapped to ϕ coordinate on the sphere and θ coordinate on the sphere is a monotonic function of r coordinate on the plane. Moreoever the (inverse) stereographic mapping has an intuitive geometric meaning. For a review of the stereographic projection, readers may find Ref. [5] useful.

In order to define the inverse stereographic projection, we consider a unit sphere with center at the coordinate $(0, 0, 1)$ in $\mathbb{R}^3$. On the other hand the 2D plane passes though the origin, $(0, 0, 0)$, and extends in the x, y plane. In order to define the inverse stereographic projection, we will adopt the polar coordinates r, ϕ on the plane and θ, ϕ on the unit sphere. There is no need to include r coordinate in the unit sphere since it is always equal to 1.

The inverse stereographic projection maps a point (r, ϕ) on the x, y plane to a point (θ, ϕ) where $\theta = \pi - 2\alpha = \pi - 2\arctan(r/2)$ since $\alpha = \arctan(r/2)$. Fig. 1 illustrates an example mapping.

4. The magnetic field as a function of particle trajectory

The trajectory of a particle is given as a function of time as $(r(t), \phi(t))$ in polar coordinates in the x, y plane. We have seen in Theorem 2 (see Eq. (14)) the magnetic field that is necessary to represent a trajectory $|\theta, \phi\rangle$ on the Bloch sphere is given by:

$$\vec{B} = \frac{1}{\gamma}(\sin(\phi)\dot{\theta}, -\cos(\phi)\dot{\theta}, -\dot{\phi}). \tag{15}$$

The inverse stereographic mapping leaves the ϕ coordinate intact, so all we need to calculate is the time derivative of θ in terms of r. We begin with $\theta = \pi - 2\arctan(r/2)$. It is known that $\partial_r \arctan(r) = 1/(1 + r^2)$. So we obtain:

$$\dot{\theta} = -\frac{4\dot{r}}{4 + r^2}. \tag{16}$$

We have now all the information to present the following theorem.

Theorem 3. *The magnetic field to make the spin trace a curve on the Bloch sphere that corresponds to a motion on the 2D plane is given as follows:*

$$\vec{B} = \frac{1}{\gamma}(-\sin(\phi)\frac{4\dot{r}}{4 + r^2}, \cos(\phi)\frac{4\dot{r}}{4 + r^2}, -\dot{\phi}). \tag{17}$$

Here $r(t)$ and $\phi(t)$ are the polar coordinates of a particle on the 2D plane that depend on time.

Proof. The proof is obtained by substituting the time derivative of θ calculated in Eq. (16) in Theorem 2 (see Eq. (14)). Since ϕ is mapped to ϕ by the inverse stereographic mapping, it is not a function of r, and $\dot{\phi}$ is left as it is. □

5. Setting up the initial conditions

In this section we deal with preparing the qubit in the initial condition. Suppose the qubit is given in the $|0\rangle$ state, which means it is given in the $|z, +\rangle$ or $|\theta, \phi\rangle$ for $\theta = 0, \phi = 0$. This state can be prepared by passing the spin-1/2 particle through a Stern-Gerlach apparatus (for a review see Ref. [7]) and selecting the ones with $|z, +\rangle$. Then applying a magnetic field one can rotate the qubit and prepare it in the $|\theta_0, \phi_0\rangle$ state. In order to achieve the initial state $|\theta_0, \phi_0\rangle$ the magnetic field should be in the direction $\hat{\phi}$. Another way to obtain the initial state $|\theta_0, \phi_0\rangle$ is to arrange the magnetic field of the Stern-Gerlach apparatus in the direction $|\hat{n}\rangle$ and select the ones with positive eigenvalues.

6. Some applications

In this Section, we give a few applications of the representation of a particle trajectory on a plane by a spin on the Bloch sphere.

6.1. Free particle

Let us suppose that the particle moves on the line (r, ϕ) with ϕ constant. If the speed of the particle is v, then its position on the plane is $(vt - r_0, \phi)$. By a coordinate transformation on the plane we can represent the point on the plane as $(vt, 0)$. The trajectory of the particle as represented on the Bloch sphere is a great circle passing through the north pole and the south pole. The magnetic field to represent the motion of a particle on the Bloch sphere is as follows:

$$\vec{B} = \frac{4v}{\gamma(4 + v^2t^2)}(0, 1, 0), \tag{18}$$

Figure 2 illustrates the path of a qubit on the Bloch sphere to represent the motion of a free particle on the plane.

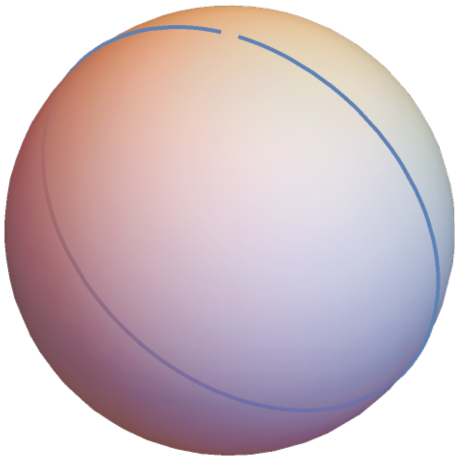

Figure 2. The trajectory of a qubit on the Bloch sphere to represent the motion of a free particle on a plane that passes through the origin.

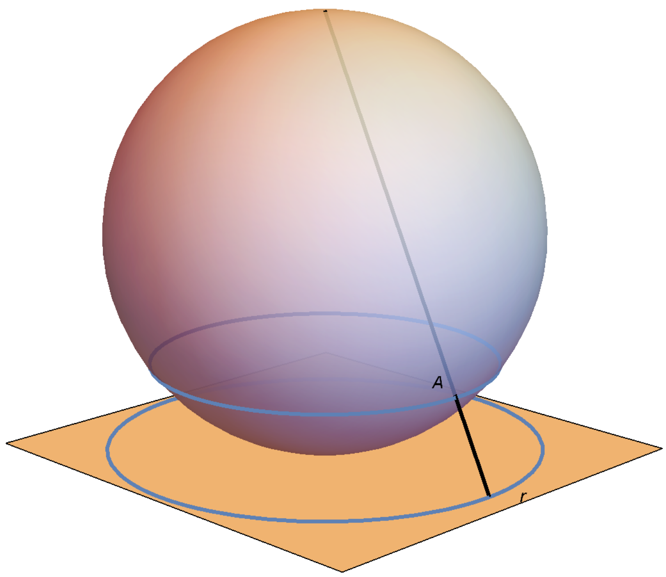

Figure 3. Representation of a circular path for the Kepler problem on the Bloch sphere.

6.2. A simple example of the Kepler problem

The classic textbook [3] describes the two-body problem under Newtonian gravity in a detailed manner. This is the Kepler problem. A simple example of the Kepler problem is that when the orbit is circular: so r is constant and $\phi = \omega t$ for some angular frequency ω. One can also deal with the most general case when the path on the plane is 1) circle, 2) ellipse, 3) parabola and 4) hyperbola. In this example we deal with the circular case. Because r is constant for a circle, $\dot{\theta} = 0$. Hence the magnetic field is given as follows:

$$\vec{B} = -\frac{\omega}{\gamma}(0, 0, 1). \tag{19}$$

Figure 3 illustrates the path traced on the Bloch sphere by the qubit.

7. Conclusion

In this paper we considered a qubit implemented as the spin of a spin-1/2 particle such as an electron or proton under a specific magnetic field to represent the motion of a particle on the 2D plane, on the Bloch sphere. For that purpose we found out the Hamiltonian as a function of angles and angular velocities on S^2 (which is the Bloch sphere) so that the state $|\theta, \phi\rangle$ traces any desired curve. Then by writing the Hamiltonian as $H = -\gamma \vec{B} \cdot \vec{S}$ as a sum of spin operators, we calculated the magnetic field necessary to produce the desired curve on the Bloch sphere, where γ is the gyromagnetic ratio. By using the inverse stereographic projection of the plane onto S^2 with the north pole removed, we calculated the magnetic field as a function of polar coordinates on the plane that represent the position of particle at time t, *i.e.* $r(t), \phi(t)$. In order to illustrate the procedure, we have given two examples in Section 6.

All in all, in this study, we have represented the motion of a particle on the 2D plane as a path on the Bloch sphere and have provided the magnetic field (to make the spin of a spin-1/2 particle trace this trajectory) as a function of time to represent this motion. What we found out may be useful on its own as an interesting mathematical connection or to prepare input qubits for a quantum computer that may do some calculations with the motion on the 2D plane. Another possibility is to model interactions between particles in a classical many body problem on the 2D plane as quantum interactions between the qubits which the positions of particles correspond to. However, in the latter case, it remains an open problem what an entanglement between qubits corresponds to in terms of particle positions.

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