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On The Constant $\mathfrak{C}$.

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Özet. $\mathfrak{C}$ *sabiti hakkında.* 1929 da Landau $\mathfrak{L} < \mathfrak{A}$ eşitsizliğini ispat etmiştir. Evvelki bir yazıda bir $\mathfrak{C}$ sabiti tarif ederek bu eşitsizliği, muhtemel bir eşitliği ihtiva eden bir eşitsizlikle takviye ettik. Böylelikle $\mathfrak{A}$ nın islâh edilmiş yeni bir alt sınırı tarif edilmiş oldu. Şimdiki yazıda Bloch fonksiyonların benzeri olan bir ekstremal fonksiyonun varlığı gösterildikten sonra $\mathfrak{C}$ nın bir üst sınırı verilmektedir. Bu sınırı temin eden analitik fonksiyon, simetri bakımından Ahlfors-Grunsky ve Rademacher in tetkik etmiş oldukları misâllerin benzeridir. Bu sebepten dolayı bu müelliflerin $\mathfrak{B}$ ve $\mathfrak{L}$ hakkındaki tahminleri aynen $\mathfrak{C}$ ye teşmil edilebilir.

1. In 1929 Landau proved the inequality $\mathfrak{L} < \mathfrak{A}$ [1]. We wish however to obtain an inequality with a possible equality. In this direction we have defined in [2] a constant $\mathfrak{C}$ based on the general results obtained in [2] and [3] about Bloch functions introduced by R. M. Robinson [4]. Thus a better lower bound is defined for $\mathfrak{A}$. In fact it was natural to introduce the constant $\mathfrak{C}$ since the associated Riemann surface stands always «somewhere» between a Bloch function of the second and third kind. We recall first the definition of $\mathfrak{C}$. Afterward we shall obtain an upper estimate for $\mathfrak{C}$. In the selection of the example yielding this bound we have been guided mainly by the Theorem on the existence of a Typical extremal function similar to a Bloch function.

Definition. We consider the subclass N of normalized analytic functions $w = f(z) = z + \cdots$ defined in the unit circle K such that the corresponding maps R are the universal covering surfaces of planes slit along analytic arcs with a diameter bounded away from zero by $b > 0$, say. It is clear that for all these functions $f'(z) \neq 0$ in K. Now the totality of these analytic functions falls into two classes. If the fundamental group of the slit plane Ω is not the unit element then to every interior point of Ω correspond in K more than one point, exactly a denumerable infinity. In this case we say that $w = f(z)$ belongs to the class C. Then $\mathfrak{C}$ is defined as the minimum of the numbers $\mathfrak{C}' = \mathfrak{C}'(f)$ where $\mathfrak{C}'$ is the upper bound of the radii of all circles contained in the slit plane Ω. $\mathfrak{C}$ depends on the lower bound b.

Functions associated to the number $\mathfrak{C}$ are called extremals. If on the contrary the Fundamental Group of Ω is the unit element then to every interior point of Ω corresponds in K a single point. i. e., $w = f(z)$ is schlicht in K. In this case we say that $w = f(z)$ belongs to the class S. Then $\mathfrak{S}$ is defined as the minimum of the numbers $\mathfrak{S}' = \mathfrak{S}'(f)$ where $\mathfrak{S}'$ is the upper bound of the radii of all circles contained in the slit plane Ω

We have shown that for any b

$$\mathfrak{L} < \mathfrak{C} \leq \mathfrak{S} = \mathfrak{A}$$

2. For a fixed b we wish to show the existence of a Typical extremal function $f = z + \cdots$ analytic in K such that the corresponding map is a slit plane which does not contain any circle of radius greater than $\mathfrak{C}$.

By definition of $\mathfrak{C}$ to every $\varepsilon > 0$ there exists a function $f \in C$ whose values do not completely fill the interior of any circle of radius $\mathfrak{C} + \varepsilon$. Hence for every integer $n > 0$ we can choose a function $f_n \in C$ whose values do no completely fill the interior of any circle of radius $\mathfrak{C} + 1/n$ and which satisfies the inequality (E. Landau, loc. cit. p. 618, or R. M. Robinson, loc. cit. p. 454)

$$|f_n'(z)| \leq \frac{1}{1 - |z|^2} \qquad \text{for} \qquad |z| < 1.$$

Hence the sequence $\{f_n(z)\}$ is normal in K. Therefore there exists in $\{f_n(z)\}$ a subsequence $\{f_{n_p}(z)\}$ which converges uniformly in K to the analytic function $f(z) = z + \cdots$. Exactly as in the case of Bloch functions (R. M. Robinson, loc. cit.) it can be shown that the map of $f(z)$ cannot contain any circle of radius greater than $\mathfrak{C}$. Clearly the limiting function is in the Class N.

One can arrive at the same result by defining a larger Class. e. g. we may consider all normalized analytic functions defined in K such that the corresponding Riemann surfaces are without inner branch points and with boundary consisting of continua with a diameter bounded away from zero. Using same definitions and notations, the proof of the existence of a Typical extremal function is the same as before except that the map of the limiting function f is the universal covering surface of a slit plane is no longer evident. In this case one may apply the proof for a Bloch function of the first or second kind [3] except that for sufficiently smooth boundary arc it must be replaced by the following one. Suppose R possesses boundary arc γ which is sufficiently smooth to allow a circle of radius $\mathfrak{C}$ to roll along it inside R. There will be no loss of generality by assuming γ non analytic. In this case there exists a neighbourhood N_i inside R such that (i) part of the boundary of N_i consists of a subarc of γ (ii) a branch of the inverse function is analytic in N_i. If n_i is the image of N_i under this branch then the Theorem of Carathéodory on the one to one correspondence between points of Jordan boundary arcs can be applied to these regions. But then N_i is continuable univalently beyond R just in the same way as for the schlicht case, thus yielding the desired contradiction [2].

Note that the proof reduces to the proof for the schlicht case when R is reduced to a single sheet. i. e., it makes no difference between schlicht and non schlicht case. On the other hand because of its simplicity and generality this proof can replace advantageously the one given for Bloch functions of the first and second kind [3] but not conversely.

If $f \in C$ then f is called a Bloch function of the first type (not to be confused with Bloch functions of the first or second kind introduced by R. M. Robinson). If $f \in S$ then f is called a Bloch function of the second type.

Obviously f is schlicht in case for every n $f_n(z)$ is schlicht in K. On the other hand for functions $f_n(z)$ not schlicht in K it is not known whether the limiting function f is of the first or second type or if there exist limiting functions f which are Bloch functions of both types. But if f is of the second type then again it coincides with a Bloch function of the third kind.

In what follows we let b vary. In other words we are interested in the g. l. b. $\mathfrak{C}$ for $b > 0$.

3. Now we wish to obtain an upper estimate for $\mathfrak{C}$ (see end of § 2). We consider in the z-plane the circular polygon P : GAG′EBE′FCF′, where G′E, E′F, F′G and GAG′, EBE′, FCF′ are arcs of circle orthogonal and belonging to the circumference of the unit circle $|z| < 1$ respectively. The points A, B, C have for affixes, say, $z = -1$, $e^{-\pi i/3}$, $e^{\pi i/3}$. Moreover P is symmetric with respect to the axes OA, OB, OC and is thus circumscribed to the concentric circle of radius s

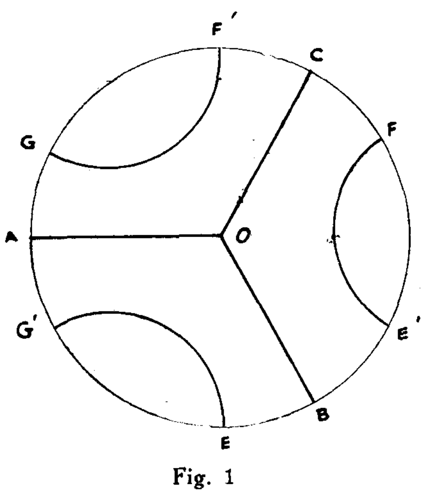

Fig. 1

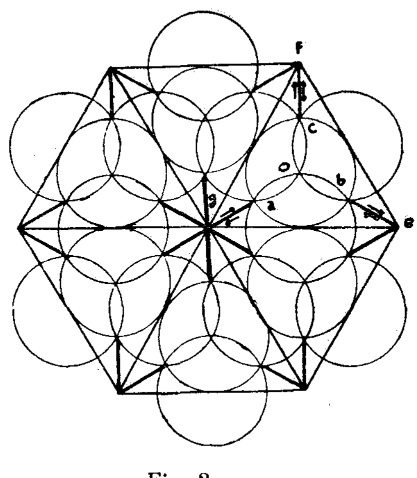

Fig. 2

$$s_0 < s < 1$$

where s_0 is the radius of the circle inscribed in the zero-angled circular triangle ABC, fig. 1.

Next we consider in the w-plane a straight equilateral triangle T. Let e, f, g denote the vertices and o the centre of T. Consider the cuts ga, eb, fc half way along the radii og, oe, of. We shall see that s can be fixed uniquely in a conformal mapping of P onto the given slit triangle. In this mapping the arc, say, GAG′ goes over into the slit ga as indicated by the arrows in fig. 2. The function thus obtained can be continued analytically across the sides of P by means of Schwarz' principle of reflection. In fact the repeated reflections of P in the sides interior to the unit circle just fill the unit circle, whereas the corresponding images in the plane obtained by reflections in the sides of T will build up a Riemann surface with infinitely many sheets. Clearly the slit plane Ω covered by the Riemann surface cannot contain circles of radius greater than the radius of the circle inscribed in T. Thus it is seen that our example belongs to the class C and has the properties of a Bloch function of the first type. Let $\mathfrak{L}^*$ be the radius of these circles. It remains to evaluate $\mathfrak{L}^*$ which we do in the following sections.

4. Let $x = x(z)$ be the function which maps P conformally onto the unit circle with centre at the origin such that A, B, C go over into $x = -1$, $e^{-\pi i/3}$, $e^{\pi i/3}$ respectively.

Let

$$y = \{k^{-1}[pk(x^3)]\}^{1/3}, \qquad k(x) = \frac{x}{(1-x)^2}, \qquad p = \frac{4r}{(1+r)^2}$$

be the function which maps $|x| < 1$ conformally onto $|y| < 1$ with three equal cuts along the three equidistant rays issued from the origin such that one of the cuts extends from $y = -1$ to a point whose distance from the origin is $r^{1/3}$ [5].

Let

$$y = \frac{\zeta - a}{\zeta - \bar{a}}$$

be the linear substitution which transforms the slit circle $|y| < 1$ into the slit upper half ζ-plane. Finally

$$w = K\int_a^\zeta \zeta^{-2/3}(\zeta - 1)^{-2/3}\, d\zeta$$

will transform the slit half ζ-plane into the slit equilateral triangle. Once the parameters p and s are fixed suitably it will suffice to combine these special mappings in order to obtain the required function.

Lemma 1. $p = 8/9$.

Proof. Let us consider the integral

(1) $$v = \int \zeta^{-2/3} (\zeta - 1)^{-2/3} \, d\zeta$$

Differentiating (1) with respect to ζ we obtain

(2) $$(d\zeta/dv)^3 = \zeta^2 (1 - \zeta)^2$$

We put as usually

(3) $$d\zeta/dv = x^2$$

(here x has not the same meaning as in the beginning of § 4).

Then (2) becomes

$$\zeta^2 - \zeta + x^3 = 0$$

We consider the root

(4) $$\zeta = \frac{1}{2} + \frac{1}{2} (4x^3 - 1)^{\frac{1}{2}} i$$

determined by the conditions $x = 0, \quad \zeta = 0$

Differentiating (4) and then taking account of (3) we have

$$d\,(v/3i) = dx/(4x^3 - 1)^{\frac{1}{2}}$$

Putting $u = v/3i$ we have

$$u = \int_{\infty}^{x} \frac{dx}{(4x^3 - 1)^{\frac{1}{2}}}$$

whose inverse function is

$$x = \mathbf{P}u$$

with

$$\mathbf{P}'^2 (u) = 4 \mathbf{P}^3 (u) - 1$$

and which corresponds to the well known equiharmonic case, finally we can write (4) as

(5) $$\zeta = \frac{1}{2} + \frac{1}{2} \mathbf{P}'(u)\, i \quad \text{or} \quad \zeta = \frac{1}{2} + \frac{1}{2} (4 \mathbf{P}^3 u - 1)^{\frac{1}{2}} i$$

It maps conformally an equilateral triangle with the vertex at the origin $u = 0$ corresponding to $\zeta = \infty$ onto the upper half ζ-plane, the other vertices correspond of course to $\zeta = 0$ and 1 respectively. The real half period ω_2 of $\mathbf{P}u$ being the height oh of the equilateral triangle, the image M on $R\zeta = \frac{1}{2}$ of the point m where $om = oh/3$, has for affixe

$$\zeta_M = \frac{1}{2} + \frac{1}{2}\, \mathrm{P}'(\omega_2/3)\, i$$

On the other hand

$$y = \frac{\zeta - a}{\zeta - \bar{a}},$$

where $a = \frac{1}{2} + \frac{1}{2}\, 3^{\frac{1}{2}}\, i$, transforms the upper half ζ-plane into the unit circle such that $\zeta = \infty,\ 0,\ 1$ go over into

$$y = 1, \qquad e^{2\pi i/3}, \qquad e^{-2\pi i/3}$$

respectively. The image μ of M is then on the real positive axis with the affixe

$$y_\mu = \frac{\zeta_M - a}{\zeta_M - \bar{a}}$$

We may now put $0\mu = r^{1/3}$, where 0 is the centre of the unit circle. We have

$$r^{1/3} = \frac{\mathrm{P}'(\omega_2/3) - 3^{\frac{1}{2}}}{\mathrm{P}'(\omega_2/3) + 3^{\frac{1}{2}}} \tag{6}$$

We wish to show that $r = \frac{1}{2}$. We notice, by (5), that

$$\mathrm{P}(2\omega_2/3) = 1$$

Taking account of

$$\mathrm{P}(u + \omega_2) = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{\mathrm{P}(u) - e_2}$$

and replacing the unknowns by their values

$$u = -\omega_2/3$$

and

$$e_1 = \frac{e^{2\pi i/3}}{2^{2/3}}, \qquad e_2 = \frac{1}{2^{2/3}}, \qquad e_3 = \frac{e^{-2\pi i/3}}{2^{2/3}}$$

we obtain

$$\mathrm{P}(-\omega_2/3) = \mathrm{P}(\omega_2/3) = \frac{1}{2^{1/3} - 1} \tag{7}$$

It suffices to write (6) as

$$\mathrm{P}'(\omega_2/3) = 3\left(\frac{1 + r^{1/3}}{1 - r^{1/3}}\right)^2 \tag{6'}$$

to see, when compared with (7), that $r = \frac{1}{2}$ is the required root. Consequently $p = 8/9$ which had to be shown.

Lemma 2. $\mathfrak{L}^* = \frac{\Gamma^2(1/3)}{4 \cdot 3^{1/3}\,\Gamma(2/3)} \cdot C$ where $C = \left|\frac{dz}{dx}\right|_0$ is a positive constant less than 1.

Proof. Denoting the vertices of T corresponding to $\zeta = 0, 1, \infty$ by w_1, w_2, w_3 respectively the mapping function can be written as

$$w = K \int_{\zeta}^{\infty} \zeta^{-2/3} (\zeta - 1)^{-2/3} d\zeta + w_3$$

Hence

$$K \int_0^1 \zeta^{-2/3} (\zeta - 1)^{-2/3} d\zeta = w_2 - w_1$$

Consequently

$$| K | = \frac{| w_2 - w_1 |}{B\left(\frac{1}{3}, \frac{1}{3}\right)}$$

Hence

$$| dw/d\zeta |_{\zeta = a} = | K | = 6.3^{-\frac{1}{2}}\, \Gamma^{-2}(1/3)\, \Gamma(2/3)\, \mathfrak{L}^*$$

On the other hand

$$| dy/dx |_0 = p^{1/3}, \qquad | d\zeta/dy |_0 = 3^{\frac{1}{2}}$$

Taking account of the identity

$$\frac{dw}{dz} = \frac{dw}{d\zeta} \frac{d\zeta}{dy} \frac{dy}{dx} \frac{dx}{dz}$$

we get after putting $| dw/dz |_{z=0} = 1$

$$\mathfrak{L}^* = \frac{C \cdot \Gamma^2(1/3)}{4 \cdot 3^{1/3}\, \Gamma(2/3)}$$

where $C = | dz/dx |_0$.

The theorem now follows since $z = z(x)$ fulfils all the conditions of Schwarz' Lemma as soon as we dispose of the positive scale constant C so as to make the circle unity, i. e., $| z | < 1$.

5. Consider on the circumference $| x | = 1$ the points G_1, A_1, G_1', E_1, B_1, E_1', F_1, C_1, F_1', where F_1, F_1', say, are symmetrical with respect to C_1 whose affixe is $e^{\pi i/3}$, the affixes of A_1

and B_1 being -1 and $e^{-\pi i/3}$ respectively. Let $a = e^{i\psi}$, $0 < \psi < \pi/3$, designate the affixe of F_1. Those of G_1 and E_1 are εa and $\varepsilon^2 a$ where $1, \varepsilon, \varepsilon^2$ are the roots of unity. The affixes of E_1', F_1', G_1' will be the conjugates of the affixes of F_1, E_1, G_1 respectively, i. e., $\bar{a}$, $\varepsilon\bar{a}$, $\varepsilon^2\bar{a}$.

Theorem. $z = z(x)$ satisfies the differential equation

$$\{z, x\} = 3(1-\alpha^2)\, x \cdot \frac{2(\cos 3\psi - \cos\psi)\, x^6 + (\cos 6\psi + 4\cos 3\psi\cos\psi - 5)\, x^3 + 2(\cos 3\psi - \cos\psi)}{(x^6 - 2x^3\cos 3\psi + 1)^2}$$

where $\{z, x\}$ is the well known Schwarzian derivative, $\alpha = \frac{1}{2}$.

Proof. In the conformal mapping of P, inscribed in the circle $|z| < 1$, onto $|x| < 1$ (the arcs of circle G′E, E′F, F′G are not necessarily orthogonal to the unit circle) the points $G_1, A_1, G_1', \ldots, F_1'$ correspond to $G, A, G', \ldots, F'$ respectively. Consequently the corresponding differential equation is

$$\begin{aligned}(8) \qquad \{z, x\} = \frac{1}{2}(1-\alpha^2)\,[&(x-a)^{-2} + (x-\varepsilon a)^{-2} + (x-\varepsilon^2 a)^{-2} \\ &+ (x-\bar{a})^{-2} + (x-\varepsilon^2\bar{a})^{-2} + (x-\varepsilon\bar{a})^{-2}] \\ &+ h/(x-a) + h_2/(x-\varepsilon a) + h_3/(x-\varepsilon^2 a) \\ &+ h_4/(x-\bar{a}) + h_5/(x-\varepsilon^2\bar{a}) + h_6/(x-\varepsilon\bar{a})\end{aligned}$$

with the conditions

$$h + h_2 + \cdots + h_6 = 0$$

$$a(h + \varepsilon h_2 + \varepsilon^2 h_3) + \bar{a}(h_4 + \varepsilon^2 h_5 + \varepsilon h_6) = -3(1-\alpha^2)$$

$$a^2(h + \varepsilon^2 h_2 + \varepsilon h_3) + \bar{a}^2(h_4 + \varepsilon h_5 + \varepsilon^2 h_6) = 0$$

In addition to these conditions we must have the following symmetry conditions

$$\bar{h}_4 = h$$

$$\bar{h}_5 = h_2$$

$$\bar{h}_6 = h_3$$

$$h_2 = \varepsilon^2 h, \qquad h_3 = \varepsilon h$$

Finally in the limit case $\psi = 0\ (\alpha \to 1)$, $z = z(x)$ becomes $z = Cx$, where C is a constant. But

$$\{Cx, x\} \equiv 0$$

Hence the second member in (8) must be identically zero. This implies

$$h+\bar{h}=0$$

Hence for $0<\psi<\pi/3$ we obtain

$$h=\frac{\alpha^2-1}{a-\bar{a}}$$

Taking account of the symmetry conditions and putting this value in (8) we obtain the Theorem. Now if we put $y=-1$ in the equation of section 4 we find for $p=8/9$

$$x^3=-\frac{7}{9}+i\,\frac{4\cdot 2^{\frac{1}{2}}}{9}$$

We set

$$\cos\theta=-7/9, \qquad \sin\theta=4\cdot 2^{\frac{1}{2}}/9$$

with $\pi/2<\theta<\pi$. Consequently the argument of a is $\psi=\theta/3$. Setting $\alpha=\frac{1}{2}$ we obtain the required differential equation in this particular case. Numerically it can be written as

$$(9)\qquad \frac{1}{2}\{z,x\}=-\frac{9}{8}\,\frac{(81b+126)\,x+(126b+388)\,x^4+(81b+126)x^7}{81+252x^3+358x^6+252x^9+81x^{12}}$$

where

$$b=2\cos\psi=1\cdot 3634$$

For short, (9) is of the form

$$\frac{1}{2}\{z,x\}=\frac{B_1x+B_4x^4+B_1x^7}{A_0+A_3x^3+A_6x^6+A_3x^9+A_0x^{12}}$$

6. Consider the differential equation

$$\theta''+F(x)\,\theta=0$$

where $F(x)$ is the right member of the equation (9). In fact we have

$$(A_0+A_3x^3+A_6x^6+A_3x^9+A_0x^{12})\,\theta''+(B_1x+B_4x^4+B_1x^7)\,\theta=0$$

Setting in it a series of the form

$$\theta=c_0+c_1x+c_2x^2+\cdots$$

we find for the general integral an expression of the form

$$\theta = c_0\theta_0 + c_1\theta_1$$

where c_0 and c_1 are now two arbitrary constants.

The two particular solutions are of the form

$$\theta_0 = 1 + c_3x^3 + c_6x^6 + \cdots$$
$$\theta_1 = x + c_4x^4 + c_7x^7 + \cdots$$

which are valid in the neighbourhood of the origin and whose coefficients can be calculated step by step by means of the recurrence formula

$$(10) \qquad \begin{aligned}&(p-1)\,pc_pA_0 + (p-3)(p-4)\,c_{p-3}A_3 + (p-7)(p-6)\,c_{p-6}A_6\\ &+(p-10)(p-9)\,c_{p-9}A_3 + (p-13)(p-12)\,c_{p-12}A_0\\ &+c_{p-3}B_1 + c_{p-6}B_4 + c_{p-9}B_1 = 0,\end{aligned}$$

where $p=3n$ for θ_0 and $p=3n+1$ for θ_1, $n=1, 2, \ldots$

Now, it can be easily verified that θ_1/θ_0 is a solution of (9).

Hence the required mapping function up to an arbitrary scale constant C is given by

$$(11) \qquad z = \theta_1/\theta_0 = Cx\,\frac{1+c_4x^3+c_7x^6+\cdots}{1+c_3x^3+c_6x^6+\cdots}$$

The series (11) converges absolutely and uniformly in the interior of the unit circle $|x| < 1$. Moreover by a Theorem due to Féjer it converges at the point $x = -1$. (see e. g. G. Julia, Leçons sur la représentation conforme des aires multiplement connexes, 1934, p. 35.). Thus C can be determined by the condition $z(-1) = -1$. We find $C = 0.70$. and consequently

$$\text{g. l. b. } \mathfrak{C} \leq \mathfrak{L}^* = 0.64.$$

Example of the same type has been considered successively by Ahlfors and Grunsky [6] and Rademacher [7] for $\mathfrak{B}$ and $\mathfrak{L}$ respectively. For reason of symmetry it is very likely that $\mathfrak{L}^*$ is the exact value.

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