Self-superposable Fluid Motions

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Özet: Lüzucî sıkıştırılamaz homogen bir aşışkanın hareket diferensiyel denklemleri lineer olmadığından onların genel çözümlerini bulmak güçtür. Bu yüzden çözümü kolaylaştırmak için ekseriya bazı kabuller yapılır. Bizim burada kullanacağımız kabul hareketin kendi kendisi üzerine bindirilebilme (self-superposability) veya kısaca ss, özeliğidir. Prof. J. A. Strang (9). bindirilebilmenin gerek ve yeter şartını ifade etmiştir : Eğer hareket denklemlerinin iki farklı çözümü U_1 ve U_2 ise bunların toplamının da bir çözüm olması için gerek ve yeter şart

$$U_1 \times (\nabla \times U_2) + U_2 \times (\nabla \times U_1) = \nabla\chi$$

olup burada χ, x, y, z ve t nin keyfî skaler bir fonksiyonu ve $\nabla = i\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$ dir. İşte bu bindirilebilme şartıdır. Eğer $U_1 = U_2 = U$ alınırsa

$$U \times (\nabla \times U) = \nabla\chi_1$$

kendi kendisi üzerine bindirilebilme (ss.) şartı elde edilir, χ_1 evvelki gibi herhangi bir skalerdir.

Bu sonuncu şart kullanıldığı takdirde hareketin vektörlü denklemi birisi yalnız lineer olmayan terimleri ve öteki yalnız lineer olanları ihtiva eden iki denkleme ayrılır. Bu iki denklemin çözümü ise esas denkleminkinden çok daha kolaydır.

İşte bu yazıda kullanılan çözüm metodu budur. Eser dört bölüme ayrılmıştır. İlk iki bölümde iki boyutlular ele alınmıştır.

1. The equations of motion.

The equations of motion of a viscous incompressible homogeneous fluid in vector form are

$$\frac{DU}{Dt} - \nu\nabla^2 U = F - \frac{1}{\delta}\nabla p, \tag{1·1}$$

$$\nabla \cdot U = 0, \tag{1·2}$$

where $U = (u, v, w)$ is the velocity vector, $u = u(x, y, z, t)$,

$v = v(x, y, z, t)$ and $w = w(x, y, z, t)$ the velocity components at a point $M(x, y, z)$ at time t. $F(M, t)$ the force on unit mass, $p(M, t)$ the pressure, $\rho(M, t)$ the dencity, $\mu(M, t)$ the coefficient of viscosity and

$$\nu = \frac{\mu}{\rho}$$

is the kinematic coefficient of viscosity. We assume ν, μ and ν to be constants.

$$\frac{DU}{Dt} = \frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + w\frac{\partial U}{\partial z}$$

is the acceleration following the motion.

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

are operators.

W shall suppose that F is derived from a potential function Ω, so that

$$F = -\nabla\Omega.$$

Equation (1·1) can be written in the form

$$\frac{\partial U}{\partial t} - U \times (\nabla \times U) + \nu\nabla x(\nabla \times U = -\nabla H, \qquad (1\cdot 3)$$

where $\nabla \times U$ is the vorticity vector, and

$$H = \frac{p}{\rho} + \Omega + \frac{1}{2}U^2. \qquad (1\cdot 4)$$

If we apply the operator to both sides of (1·3) we obtain

$$\nabla \times \frac{\partial U}{\partial t} - \nabla \times [U \times (\nabla \times U)] + \nu\nabla \times [\nabla \times (\nabla \times U)] = 0. \quad (1.5)$$

This equation contains only the kinematic elements of the motion, and is called "The kinematic consistency equation". It is the consistency condition of the three scalar equations in (1·3); that is, If (1·5) is satisfied U is a solution of (1·3).

Hence the determination of a fluid motion will consist of two successive processes; The first is to determine the velocity field by (1·5), and the second is to determine the pressure, i.e. the function H by (1·4).

The general solution of the equation (1·5) is difficult because of the non-linear terms. For this reason always some assumption are made to simplify the equation. The principal assumption which we shall make in this memoir is to use the self-superposability property of the motion.

Since the equations of motion are not linear their solutions are not, in general, superposable

If $U_1 = (u_1, v_1, w_1)$ and $U_1 = (u_2, v_2, u_2)$ are any two solutions of the equations of motion of a viscous incompressible fluid corresponding to given external forces, initial and boundary conditions, not necessarily the same in both cases, they are superposable on each other if and only if

$$U_1 \times (\nabla \times U_2) + U_2 \times (\nabla \times U_1) = \nabla \chi, \qquad (1\cdot 6)$$

where χ is an arbitrary scalar function of x, y z and t. This is the superposability condition [9] If $U_1 = U_2 = U$ we obtain the self-superposability condition.

$$U \times (\nabla \times U) = \nabla \chi, \qquad (1\cdot 7)$$

where as before χ means any scalar function From this onwards we shall denote the compound word "self-superposable" simply by ss. in order to save writing.

If U is ss. the middle term in the equation (1·5) disappear

$$\nabla \times [U \times (\nabla \times U)] = 0, \qquad (1\cdot 8)$$

and the consistency equation reduces to

$$\nabla \times \frac{\partial U}{\partial t} + \nu \nabla \times [\nabla \times (\nabla \times U)] = 0. \qquad (1\cdot 9)$$

Hence the use of the self-superposability condition is to remove the non-linear terms from the equations of motion.

Prof. Kampé de Fériet [6] has used this fact in a rather different way in the case of pure plane motion. Assuming that the vorticity is constant along a stream line, i.e. by taking a relation

$$\psi = f(\zeta) \qquad (1\cdot 10)$$

between the stream function and vorticity, he has got rid of the non-linear terms in the equations of a plane motion, where $\zeta = - \nabla_1^2 \psi$ is the vorticity.

But the relation (1·10) is just the self-superposability condition in plane We must note the self superposability condition is more general than (1·10). They mean the same thing only in the case of plane motion.

The object of this thesis is to find exact solutions of the equations of motion, in various cases, when they ~~are~~ simplified by the self-superposability condition. The work is divided into four chapters. Each chapter begins by the definition of the particular motion to which it refers The coordinate systems which will be used are indicated. The forms of the velocity and the vorticity components, the self-superposability and the consistency equations are shown, the pressure equation is given in each system.

After these, first the steady solutions and then the non-steady solutions of the equations are examined.

CHAPTER I.

Self-superposable Plane Motions

2. Plane motion.

A plane motion is defined as motion in which the stream lines are plane curves in planes parallel to a fixed plane, say XOY, and the motion is the same in all such planes. Therefore the velocity components do not depend on z Hence in rectangular coordinates these are

$$u = u(x, y, t), \qquad v = v(x, y, t), \qquad w = 0. \tag{2·1}$$

The continuity equation (1·2) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2·1}$$

This shows that there is a stream function $\psi = \psi(x, y, t)$ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{2·2}$$

The vorticity components are

$$\xi = 0, \quad \eta = 0. \quad \zeta = -\nabla_1^2 \psi \tag{2·3}$$

where

$$\nabla_1^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}.$$

Hence vorticity is always normal to the plane of motion.

The ss. condition (1·8) and the consistency equation (1·9) furnish respectively(*)

$$\frac{D(\psi, \nabla_1^2 \psi)}{D(x, y)} = 0 \tag{2·4}$$

and

$$\nu \nabla_1^2 (\nabla_1^2 \psi) - (\nabla_1^2 \psi)_t = 0 \tag{2·5}$$

The equation (1·3) shows that H does not depend on z

$$H = H(x, y, t),$$

and is obtained from the equations

$$\left.\begin{aligned} H_x &= -\psi_{yt} + \psi_x \cdot \nabla_1^2 \psi + \nu (\nabla_1^2 \psi)_y, \\ H_y &= \quad \psi_{xt} + \psi_y \cdot \nabla_1^2 \psi - \nu (\nabla_1^2 \psi)_x, \end{aligned}\right\} \tag{2·6}$$

after ψ is determined from (2·4) and (2·5).

In plane polar coordinates r, θ the velocity components are

$$v_1 = v_1(r, \theta, t), \quad v_2 = v_2(r, \theta, t), \quad v_3 = 0. \tag{2·6}$$

The continuity equation is

$$\frac{\partial (r v_1)}{\partial r} + \frac{\partial v_2}{\partial \theta} = 0.$$

Hence there is a stream function $\psi = \psi(r, \theta, t)$, such that

$$v_1 = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \qquad v_2 = -\frac{\partial \psi}{\partial r}. \tag{2·7}$$

The vorticity components are

$$\zeta_1 = 0, \quad \zeta_2 = 0, \quad \zeta_3 = \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_2) - \frac{\partial v_1}{\partial \theta} \right].$$

The vorticity vector is always normal to the plane of motion.

The ss. and the consistency equations become respectively

(*) Letters as suffixes denote differentiation.

$$\frac{D(\psi, \nabla_1^2\psi)}{D(r, \theta)} = 0, \tag{2·8}$$

$$\nu\nabla_1^2(\nabla_1^2\psi) - (\nabla_1^2\psi)_t = 0, \tag{2·9}$$

where $\nabla_1^2\psi$ is the Laplacian of ψ

$$\nabla_1^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}.$$

Finally, the equation (1·3) gives for $H = H(r, \theta, t)$ the following two equations:

$$\left.\begin{aligned} H_r &= -\frac{1}{r}\psi_{\theta t} + \psi_r\cdot\nabla_1^2\psi + \frac{\nu}{r}(\nabla_1^2\psi)_\theta, \\ \frac{1}{r}H_\theta &= \psi_{rt} + \frac{1}{r}\psi_\theta\cdot\nabla_1^2\psi - \nu(\nabla_1^2\psi)_r. \end{aligned}\right\} \tag{2·10}$$

In a real fluid motion both the velocity components and the pressure must be uniform. Hence if the fluid extends to a region where θ may vary from 0 to 2π this uniformity must be secured.

3. Solution of the equations in the steady case.

In Cartesian coordinates the stream function ψ must be a solution of the system of equations

$$\frac{D(\psi, \nabla_1^2\psi)}{D(x, y)} = 0, \quad \nu\nabla_1^2(\nabla_1^2\psi) = 0. \tag{3·1}$$

In plane polar coordinates the first of these equations must be replaced by

$$\frac{D(\psi, \nabla_1^2\psi)}{D(r, \theta)} = 0, \text{ where } \nabla_1^2\psi = \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta}.$$

In perfect fluids $\nu = 0$, and the only condition which will be imposed on ψ is

$$\nabla_1^2\psi = f(\psi).$$

H. Lamb (8.) referring to one of the papers of Stokes is giving this as the existence of steady plane motion in a homogeneous incompressible non-viscous fluid. It is in fact the ss condition, and as it is clear from the equations (2·4) and (2·5), it is one of the conditions of the existence of any ss. plane motion, steady or otherwise, in a viscous or non-viscous homogeneous incompressible fluid.

The full solution of the system (3·1) is know [2]. The possible solutions are the following:

1°) Let
$$\nabla_1^2\psi = K, \tag{3·2}$$

where K is an arbitrary constant. The vorticity vector which is always normal to the plane of motion is constant in magnitude. The velocity components are harmonics. This class of motions includes all irrotational plane motions, which correspond to $K=0$.

2°) Motion in parallel straight lines. If we take the x-axis parallel to the direction of motion the stream function ψ is of the form

$$\psi = \psi\,(y).$$

The first of the equations (3·1) is satisfied identically, and the second requires

$$\psi'''' = 0.$$
$$\psi(y) = Ay^3 + By^2 + Cy, \tag{3·3}$$

where A, B, C are arbitrary constants of integration. The velocity components and the pressure equation are

$$u = 3Ay^2 + 2By + C, \quad v = 0,$$

$$\frac{p}{\rho} + \Omega = 6A\nu x - \frac{1}{2}\,C^2.$$

3°) Motion along concentric circles. If we take the origin at the centre of the circles the stream function is of the form

$$\psi = \psi(r).$$

The first of the equations (3·1) is satisfied identically, and the second requires

$$\psi'''' + \frac{2}{r}\,\psi''' - \frac{1}{r}\,\psi'' + \frac{1}{r^3}\,\psi' = 0,$$

of which the solution is

$$\psi(r) = Ar^2 \log r + B \log r + Cr^2, \tag{3·4}$$

where A, B, C are constants of integration. The uniformity of the pressure requires $A = 0$, and therefore the motion reduces toone with constant vorticity. The velocity components in plane polar coordinates are

$$v_1 = 0, \quad v_2 = -2Cr - \frac{B}{r}.$$

If the fluid contains the origin we must take $B = 0$

4°) Radial motion. The stream function is of the form

$$\psi = \psi(\theta)$$

$$\therefore \nabla_1^2 \psi = \frac{1}{r^2} \psi_{\theta\theta}.$$

The ss. condition requires $\psi_\theta = a$ constant, say C. Hence

$$\psi = C\theta. \tag{3·5}$$

It follows that $\nabla_1^2 \psi = 0$, and the solution becomes a special case of (3·2). The velocity components are

$$v_1 = \frac{C}{r}, \quad v_2 = 0.$$

The motion is irrotational.

Although the more general solution of (2·4) or (2·8) is

$$\nabla_1^2 \psi = f(\psi), \tag{3·6}$$

where $f(\psi)$ ie an arbitrary function of ψ, and ψ depends both on x and y, or in polar coordinates on r and θ, the equation $\nabla_1^2(\nabla_1^2\psi) = 0$ requires

$$\nabla_1^2 f = 0,$$

and this set of equations gives nothing new other than (3·2), (3·3) and (3·4).

M. Kampé de Fériet (6·1) has shown that the solutions (3·2), (3·3) and (3·4) are the only solutions which satisfy the required conditions, i. e. the system of equations (3·1).

The system (3·1) shows the solutions given above are valid both for viscous and non-viscous fluids.

These are the only plane steady motions of viscous incompressible fluids in which the vorticity is constant along a stream line.

4. Solution in the non-steady case.

In Cartesian coordinates the equations to be satisfied are

$$\frac{D(\psi, \nabla_1^2\psi)}{D(x, y)} = 0, \tag{2·4}$$

$$\nu\nabla_1^2(\nabla_1^2\psi) - (\nabla_1^2\psi)_t = 0, \text{ where } \nabla_1^2\psi = \psi_{xx} + \psi_{yy}. \quad (2\cdot5)$$

The vorticity vector being always normal to the plane of motion, its magnitude changes with time according to the diffusion equation.

In plane polar coordinates x, y will be replaced by r, θ and $\nabla_1^2\psi$ will be

$$\nabla_1^2\psi = \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi_{\theta\theta}.$$

The consistency equation $(2\cdot5)$ shows that $\nabla_1^2\psi$ can not be a function of t only.

Prof. Kampé de Fériet [7] has studied these equations also. Although he has not given the exact forms of the solutions, he has indicated the possible motions. obtainable. These are (i) motions with constant vorticity, (ii) motions on parallel straight lines, (iii) motions on concentric circles, (iv) motions of the type

$$\psi(x, y, t) = e^{-\nu kt}\,\psi_0(x, y),$$

where $\psi_0(x, y)$ is an arbitrary solution of the equation

$$\nabla_1^2\psi_0 + k\psi_0 = 0.$$

We shall set in order the possible solutions of the equations:

1°) Motions with constant vorticity. There may be motions of the fluid where although the velocity changes with time, the vorticity is constant both in magnitude and direction. Hence we can write

$$\nabla_1^2\psi = K,$$

where K is an absolute constant. The form of ψ is

$$\psi = V(x, y, t) + Ax^2 + Bxy + Cy^2, \quad (4\cdot1)$$

where $2(A + C) = K$, B may be an arbitrary function of t, and $V(x, y, t)$ is an arbitrary harmonic function. The velocity components are harmonics.

2°) Motion in parallel straight lines. If we take x-axis parallel to the direction of motion the stream function ψ is of the form

$$\psi = \psi(y, t).$$

The ss. condition $(2\cdot4)$ is satisfied identically.

The consistency equation (2·5) becomes

$$\frac{\partial^2}{\partial y^2}(\nu\psi_{yy} - \psi_t) = 0,$$

$$\therefore \quad \nu\psi_{yy} - \psi_t = a(t)y + b(t),$$

where $a(t)$, $b(t)$ are arbitrary functions of t.

A particular solution of this equation is

$$\alpha y^3 + \beta y^2 + \gamma y,$$

where α, β are arbitrary constants, and γ is a function of t, such that

$$6\alpha\nu - \gamma' = a(t),$$

$$2\beta\nu = b(t), \quad \therefore \quad b(t) \text{ is constant.}$$

Hence the form of ψ is

$$\psi(y, t) = V(y, t) + \alpha y^3 + \beta y^2 + \gamma y, \tag{4·2}$$

where $V(y, t)$ is an arbitrary solution of

$$\nu\psi_{yy} - \psi_t = 0;$$

i. e. the equation of heat flow in one dimension.

A particular value of $V(y, t)$ is

$$A \cos(ky + \varepsilon)e^{-\nu k^2 t},$$

where, A, k, ε are real arbitrary constants.

Since the equation is linear and the solutions

$$A_1 \cos(k_1 y + \varepsilon_1)\, e^{-\nu k_1^2 t} \quad \text{and} \quad A_2 \cos(k_2 y + \varepsilon_2) e^{-\nu k_2^2 t},$$

corresponding to different values of the constants A, k and ε are both self-superposable on each other, they may be superposed. Hence a more general value of V is

$$V(y, t) = \sum_k A_k \cos(ky + \varepsilon_k)\, e^{-\nu k^2 t}, \tag{4·3}$$

and this is also ss. (9, p. 7).

On the other hand, the equation

$$\nu\psi_{yy} - \psi_t = 0$$

has a particular solution of the form

$$\psi = t^{-1/2}\, e^{-y^2/4\nu t},$$

which vanishes when $y = \infty$ or $t = \infty$. The same is true for

$$\psi = At^{-1/2}\, e^{-(y-k)^2/4\nu t},$$

and for different values of the constant k these are both ss. and superposable on each other. Hence since the equation is linear, in the solution (4·2) we can take

$$V(y, t) = \sum_k A_k t^{-1/2} e^{-(y-k)^2/4\nu t}. \qquad (4\cdot4)$$

The stream lines are straight lines parallel to x-axis
The pressure equation for both cases is

$$\frac{p}{\rho} + \Omega = C + (\gamma' - 6\alpha\nu)\, x,$$

where C is an arbitrary function of time

3°) Motion along concentric circles. If we take the origin at the centre of the circles the stream function is of the form

$$\psi = \psi\,(r, t).$$

The ss. condition (2·4) is satisfied identically. The consistency condition (2·5) becomes

$$\nabla_1^2 (\nu \nabla_1^2 \psi - \psi_t) = 0.$$

$$\therefore \quad \nu\,(\psi_{rr} + \frac{1}{r}\,\psi_r) - \psi_t = a(t) \log r + b(t),$$

where $a(t)$, $b(t)$ are arbitrary functions of t.
A particular solution of this equation is

$$(\alpha r^2 + \beta) \log r + \gamma r^2,$$

where α, γ are arbitrary constants, and β is a function of t, such that

$$4\alpha\nu - \beta' = a(t),$$
$$4\nu(\alpha + \gamma) = b(t), \quad \therefore \quad b(t) \text{ is constant.}$$

Hence the form of ψ is

$$\psi(r, t) = V(r, t) + (\alpha r^2 + \beta) \log r + \gamma r^2, \qquad (4\cdot5)$$

where $V(r, t)$ is an arbitrary solution of

$$\nu\left(\psi_{rr} + \frac{1}{r}\,\psi_r\right) - \psi_t = 0.$$

A particular value of $V(r, t)$ is

$$[A J_0\,(kr) + B\, Y_0\,(kr)]\, e^{-\nu k^2 t},$$

where A, B, k are real arbitrary constants; J_0 and Y_0 are Bessel's functions of order zero of the first and second kinds respectively. Since the two solutions

$[A_1J_0(k_1r)+B_1Y_0(k_1r)]e^{-\nu k_1^2t}$ and $[A_2J_0(k_2r)+B_2Y_0(k_0r)]e^{-\nu k_2^2t}$,

corresponding to different values of the constants A, B and k are both ss. and superposable on each other they may be superposed at any rate. Hence a more general value of V is

$$V(r,t)=\sum_k\,[A_kJ_0(kr)+B_kY_0(kr)]\,e^{-\nu k^2t}, \tag{4·6}$$

and this is also ss.

Again an other form of V is [10]

$$V(r,t)=t^{-k}\{F_{11}(k,1;g)(A+B\log\mid g\mid)+\Sigma C_mg^m\}, \tag{4·7}$$

provided k is not zero or a negative integer, where A, B, m are arbitrary constants, $m\geqq 1$, $g=-r^2/4\nu t$,

$$F_{11}(k,1;g)=1+\frac{k}{1!}\frac{g}{1!}+\frac{k(k+1)}{2!}\frac{g^2}{2!}+\cdots,$$

and

$$C_m=\left\{\frac{1}{k}+\frac{1}{k+1}+\ldots+\frac{1}{k+m-1}-\frac{2}{1}-\frac{2}{2}-\ldots-\frac{2}{m}\right\}\cdot\frac{k(k+1)\ldots(k+m-1)}{(m!)^2}.$$

If k is zero or a negative integer the series $F_{11}(k,1;g)$ terminates, otherwise it is an infinite series which is convergent for all values of g.

The stream lines are concentric circles with their centres on the z-axis. The pressure equation is

$$\frac{p}{\rho}+\Omega=\int\psi_r^2\cdot\frac{dr}{r}+(\beta'-4\alpha\nu)\,\theta+C$$

where C is an arbitrary function of time. The uniformity of the pressure, in a region where θ may vary from 0 to 2π, requires $\beta'-4\alpha\nu=0$, i.e. $\beta(t)=4\alpha\nu t+a$ constant.

4°) Radial motion. The stream function is of the form

$$\psi=\psi(\theta,t)\,;$$

$$\therefore\quad \nabla_1^2\psi=\frac{1}{r^2}\,\psi_{\theta\theta}.$$

Now the ss. condition requires $\psi_{\theta\theta} = 0$, and since $\psi_\theta \neq 0$ it follows that

$$\psi(\theta, t) = a(t)\,\theta + b(t), \tag{4·8}$$

and

$$\nabla_1^2\psi = 0.$$

Hence this reduces to the case 1°) with $K = 0$. The motion is irrotational. This shows that the vorticity can not depend on θ and t only, i. e. the constant vorticity lines can not be concurrent straight lines.

5°) Motions of the type

$$\psi(x, y, t) = e^{\nu k t}\,\psi_0(x, y),$$

where k is *a* parameter. The equations (2·4) and (2·5) are satisfied if

$$\nabla_1^2\psi_0 - k\psi_0 = 0.$$

If $k < 0$, put $k = -\lambda^2$. Then a solution of the equation $\nabla_1^2\psi_0 + \lambda^2\psi_0 = 0$ is

$$\psi_0 = A\cos(\lambda x + \varepsilon) + B\cos(\lambda y + \delta),$$

$$\therefore\ \psi(x, y, t) = e^{-\nu\lambda^2 t}\,[A\cos(\lambda x + \varepsilon) + B\cos(\lambda y + \delta)] \tag{4·9}$$

and

$$\nabla_1^2\psi = -\lambda^2\psi.$$

Have we the right to superpose the motions of the (4·9) in order to obtain more general solutions? Will the resulting motion be ss. ?

If we indicate the two different values of ψ by ψ_1 and ψ_2 corresponding to different values of the constants A, B, λ, ε and δ, we shall have

$$\nabla_1^2\psi_1 = -\lambda_1^2\psi_1$$

$$\nabla_1^2\psi_2 = -\lambda_2^2\psi_2.$$

These show that both ψ_1 and ψ_2 are ss.. In order that $\psi_1+\psi_2$ may be ss. ψ_1 must be superposable on ψ_2 (9, p, 15). But this is only true if $\lambda_1 = \lambda_2$. Hence a more general form of ψ is

$$\psi(x,y,t) = e^{-\nu\lambda^2 t}\cdot\sum_m [A_m\cos(\lambda x + \varepsilon_m) + B_m\cos(\lambda y + \delta_m)]. \tag{4·10}$$

If $k > 0$, put $k = \lambda^2$ in the equation $\nabla_1^2\psi_0 - k\psi_0 = 0$. Then a solution is

$$\psi_0 = A\,ch(\lambda x + \varepsilon) + B\,ch(\lambda y + \delta),$$

$$\therefore\ \psi = e^{\nu\lambda^2 t}\,[A\,ch(\lambda x + \varepsilon) + B\,ch(\lambda y + \delta)] \tag{4·11}$$

and $$\nabla_1^2\psi = \lambda^2\psi.$$

Similarly a more general solution is obtained by summation

$$\psi(x,y,t) = e^{\nu\lambda^2 t}\cdot\sum_n [A_n ch(\lambda x + \varepsilon_n) + B_n ch(\lambda y + \delta_n)], \quad (4\cdot 12)$$

if the series is convergent.

Both for the solutions (4·10) and (4·12) the function

$H = \frac{p}{\rho} + \Omega + \frac{1}{2}(\psi_x^2 + \psi_y^2)$ is given by

$$H = \frac{1}{2} k\psi^2,$$

where $k = -\lambda^2$ in the first and $k = \lambda^2$ in the second.

All motions obtained in in this chapter have a common property. The vorticity is constant on each stream line. This results from the equation (2·4), i.e. it is a consequence of the ss. condition. It states that either $\nabla_1^2\psi = f(\psi)$, or both ψ and $\nabla_1^2\psi$ depend on the same space variable. Therefore on each stream line, where ψ is constant, $\nabla_1^2\psi$, that is the vorticity is constant. This is true only in the case of ss. plane motions.

CHAPTER II.

Axially Simmetric Ss. Motions in Planes Passing Through OZ.

5. The stream lines are contained in planes passing through OZ, and it is supposed that the motion in each of these planes is the same; therefore the velocity components are independent of θ. In cylindrical coordinates these components are

$$v_1 = v_1(r, z, t), \quad v_2 = 0, \quad v_3 = v_3(r, z, t). \qquad (5\cdot 1)$$

The continuity equation is

$$\frac{\partial(rv_1)}{\partial r} + \frac{\partial(rv_3)}{\partial z} = 0.$$

Hence there is a stream function $\psi = \psi(r, z, t)$ such that

$$v_1 = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad v_3 = -\frac{1}{r}\frac{\partial\psi}{\partial r}. \tag{5·2}$$

The vorticity components are

$$\xi_1 = 0, \quad \xi_2 = \frac{1}{r} D_2\psi, \quad \xi_3 = 0, \tag{5·3}$$

where

$$D_2\psi = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}.$$

Hence vorticity is always normal to the plane of motion.

The ss. condition (1·8) and the consistency equation (1·9) become respectively

$$\frac{D\left(\psi, \frac{1}{r^2} D_2\psi\right)}{D(r, z)} = 0, \tag{5·4}$$

$$\nu D_2(D_2\psi) - (D_2\psi)_t = 0. \tag{5·5}$$

Equation (1·3) shows that H does not depend on θ,

$$H = H(r, z, t),$$

and is determined by the equations

$$\left.\begin{aligned} H_r &= -\frac{1}{r}\psi_{zt} + \frac{1}{r^2}\psi_r \cdot D_2\psi + \frac{\nu}{r}(D_2\psi)_z \\ H_z &= \frac{1}{r}\psi_{rt} + \frac{1}{r^2}\psi_z \cdot D_2\psi - \frac{\nu}{r}(D_2\psi)_r \end{aligned}\right\}. \tag{5·6}$$

After ψ is determined by the system (5·4) and (5·5), H will be determined by (5·6).

If the motion is steady the equation (5·5) becomes

$$D_2(D_2\psi) = 0 \tag{5·7}$$

6. Solution of the equations in the steady case.

In cylindrical coordinates the stream function ψ must be a solution of the system of equations (5·4), (5·7)

$$\frac{D\left(\psi, \frac{1}{r^2} D_2\psi\right)}{D(r, z)} = 0, \tag{5·4}$$

$$D_2(D_2\psi) = 0, \tag{5·7}$$

where

$$D_2\psi = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}.$$

The first requires

$$\frac{1}{r^2} D_2\psi = f(\psi), \tag{6·1}$$

where $f(\psi)$ is an arbitrary function of ψ. Now (5·7) furnishes

$$f''(\psi_r^2 + \psi_z^2) + f'(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz}) = 0,$$

or, taking account of the equation (6·1) this becomes

$$\frac{1}{r^2} f''(\psi_r^2 + \psi_z^2) + ff' + \frac{4}{r^3}\psi_r \cdot f' = 0. \tag{6·2}$$

In particular, if $f(\psi)$ is constant the equation is satisfied, and (6·1) becomes

$$D_2\psi = Cr^2, \tag{6·3}$$

where C is an arbitrary constant.

U. Crudeli has studied the solutions of this equation [3,4,5]. The following formal solutions can be obtained easily:

$$\left.\begin{aligned} \psi &= \psi_1 + \sum_k (A_k r^2 + B_k)(a_k z + b_k), \\ \psi &= \psi_1 + \sum_k r[A_k J_1(kr) + B_k Y_1(kr)](C_k e^{kz} + D_k e^{-kz}) \end{aligned}\right\} \tag{6·4}$$

where J_1 and Y_1 are Bessel's functions of order unity, and ψ_1 is a particular solution of (6·3). U. Crudeli gives for ψ_1

$$\psi_1 = \frac{1}{8} Cr^4 + Er^2,$$

where E is an arbitrary constant.

If $f(\psi)$ is not constant, the equation (6·2) is integrable only if

$$\psi_r^2 + \psi_z^2 = r^2 g(\psi) \tag{6·5}$$

and

$$\frac{4}{r^3}\psi_r = h(\psi), \tag{6·6}$$

where $g(\psi)$ and $h(\psi)$ are functions of ψ. Now the equation (6·2) is written

$$f''g + ff' + hf' = 0.$$

By eliminating ψ_r and ψ_z between the equations (6·1), (6·5) and (6·6) we obtain the following relations:

$$h + g' = f, \tag{6·7}$$
$$r^4(3h^2 + 4gh'' - 2hg') = 16\,g. \tag{6·8}$$

The second shows that, either $g = h = 0$, or ψ is a function of r only. The first possibility is excluded, because then the equation (6·7) requires $f = 0$, and we had assumed that f is not a constant. Hence

$$\psi = \psi(r).$$

This gives the well-known solution on straight lines parallel to OZ, i.e. Poiseuille motion. The stream function is of the form

$$\psi(r) = Ar^2 \log r + Br^4 + Cr^2, \tag{6·9}$$

where A, B, C are arbitrary constants.

The velocity parallel to OZ is

$$v_3 = -2A \log r - 4Br^2 - (A + 2C).$$

The pressure equation is

$$\frac{p}{\rho} + \Omega = -16\,B\nu z - \frac{1}{2}(A + 2C)^2.$$

If the region in which the motion is taking place includes the z-axis, we must take $A = 0$ in order to prevent the velocity from being infinite on the axis.

Thus, the only ss. steady motions in planes passing through OZ are (6·9) and the motions represented by the solutions of the equation (6·3).

8. Solution in the non-steady case.

We shall use cylindrical coordinates. The equations to be satisfied are (5·4) and (5·5)

$$\frac{D\left(\psi, \frac{1}{r^2} D_2\psi\right)}{D(r,z)} = 0, \tag{5·4}$$

$$\nu D_2(D_2\psi) - (D_2\psi)_t = 0, \tag{5·5}$$

where $D_2\psi = \psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz}$.

Let $$\frac{1}{r^2} D_2\psi = s\,(r, z, t).$$

The equation (5·4) suggests that it can be satisfied in particular if

1°) s is a function of r and t only,

2°) s is a function of z and t only, and in general if

3°) s is a function of ψ, i.e. $s = f(\psi)$.

We shall discuss each in turn.

1°) $s = \frac{1}{r^2} D_2\psi$ is a function of r and t only. The equation (5·4) requires

$$\psi_z \cdot s_r = 0 ;$$

hence either $s_r = 0$, or $\psi_z = 0$.

If $s_r = 0$, s may depend only on t. Let

$$s = \frac{1}{r^2} D_2\psi = a(t),$$

$$D_2\psi = r^2 \cdot a(t).$$

Now (5·5) requires $a' = 0$. Hence we obtain the equation

$$D_2\psi = ar^2 \tag{7·1}$$

where a is an arbitrary absolute constant. The equations (5·4) and (5·5) are satisfied. We obtain the solutions, similar to (6·4)

$$\left.\begin{aligned} \psi(r, z, t) &= \psi_1 + \sum_k (A_k r^2 + B_k)(C_k z + D_k), \\ \psi(r,z,t) &= \psi_1 + \sum_k r[A_k J_1(kr) + B_k Y_1(kr)](C_k e^{kz} + D_k e^{-kz}) \end{aligned}\right\} \tag{7·2}$$

where the capital letters A, B, C, D denote arbitrary functions of t, and ψ_1 is a particular solution of (7·1). For example we can take for ψ_1 the expression

$$\psi_1 = \frac{1}{8} ar^4 + br^2,$$

where b may be an arbitrary function of t.

If $\psi_z = 0$, then $\psi = \psi(r, t)$, and inserting

$$D_2\psi = r^2 s$$

into the equation (5·5) we obtain

$$\nu(rs_{rr} + 3s_r) - rs_t = 0.$$

By the method of separation of the variables we find a finite solution when $r = 0$, i.e.

$$\psi(r,t) = A e^{-\nu k^2 t} \cdot \int r\,dr \int \frac{dr}{r} \int r J_0(kr)dr + B(t)r^2, \qquad (7\cdot3)$$

where A, k are real arbitrary constants, and $B(t)$ is an arbitrary function of time.

For different values of the constants A, k and the function $B(t)$ the solutions obtained are both ss. and superposable on each other. This can easily be tested by the formula (1·6). Hence a more general solution is

$$\psi(r,t) = \sum_k [A_k e^{-\nu k^2 t} \int r\,dr \int \frac{dr}{r} \int r J_0(kr)dr + B_k(t)r^2]. \quad (7\cdot3')$$

2°) $s = \frac{1}{r^2} D_2\psi$ is a function of z and t only. The equation (5·4) requires

$$\psi_r \cdot s_z = 0;$$

hence either $s_z = 0$, or $\psi_r = 0$.

If $s_z = 0$, s depends on t only. We have discussed this case in 1°).

If $\psi_r = 0$, then $\psi = \psi(z, t)$, and the relation

$$D_2\psi = \psi_{zz} = r^2 s$$

is possible only if $\psi_{xz} = 0$, $s = 0$. Hence this is a particular case of (7·1) with $a = 0$. There is no new solution of the equations in this case.

3°) In general, s must be a function of ψ, say

$$s = \frac{1}{r^2} D_2\psi = f(\psi). \qquad (7.4)$$

Since we have discussed the cases where $s = s(r,t)$ and $s = s(z,t)$ in the previous paragraphs, it is reasonable to assume here that both ψ_r and ψ_z are different from zero. (5.4) is satisfied identically, and (5.5) furnishes

$$f''(\psi_r^2 + \psi_z^2) + f'(\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz} - \frac{1}{\nu}\psi_t) = 0. \qquad (7.5)$$

We can satisfy this in three different ways:

(i) $f' = 0$. $\therefore$ $f(\psi) = a$ constant, say a. Hence

$$D_2\psi = ar^2,$$

and we are led to the equation (7.1) of No: 1°).

(ii) $f' \neq 0$, but $f'' = 0$ and $\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz} - \frac{1}{\nu}\psi_t = 0$.
This reduces to an impossibility. For, we have

$$f'' = 0,$$
$$\therefore\ f' = C,$$
$$\therefore\ f = C\psi + D,$$

where C, D are arbitrary constants, and this last equation gives

$$\left.\begin{aligned} &\psi_{rr} - \frac{1}{r}\psi_r + \psi_{zz} - r^2(C\psi + D) = 0 \text{ by (7.4)}, \\ &\text{in addition to} \\ &\psi_{rr} + \frac{3}{r}\psi_r + \psi_{zz} - \frac{1}{\nu}\psi_t = 0. \end{aligned}\right\} \quad (7.6)$$

By subtracting the first from the second we find

$$4\nu\psi_r - r\psi_t + \nu r^3(C\psi + D) = 0,$$

a linear partial differential equation of the first order to determine ψ as a function of r, z and t. The general solution is

$$C\psi + D = \varphi(z, 4\nu t + \frac{1}{2}r^2)\, e^{-Cr^4/16},$$

or

$$C\psi + D = \varphi(z, \alpha)\, e^{-Cr^4/16};$$

where $\alpha = 4\nu t + \frac{1}{2}r^2$, and φ is an arbitrary function. If we insert this into the equations (7.6), we find that both are satisfieds if $C = 1$ and

$$r^2\varphi_{\alpha\alpha} - \frac{1}{2}r^4\varphi_\alpha + \left(\frac{1}{16}r^4 - \frac{3}{2}\right)r^2\varphi + \varphi_{zz} = 0.$$

But this is not integrable, since the coefficients are not functions of α or z.

(iii) Neither f' nor f'' is zero.

The equation (7.5) indicates that ψ_t contains the factor ν. Let

$$\psi_t = \nu k\psi,$$

where k is an arbitrary constant. Hence

$$\psi = e^{\nu kt}\,\varphi(r,z) \tag{7.7}$$

and
$$D_2\,\psi = e^{\nu kt}\,D_2\,\varphi\,,$$

where $\varphi(r,z)$ is an arbitrary function of r and z, and neither φ_r nor φ_z is zero.

Inserting these into the equations (5.4) and (5.5) we obtain

$$\frac{D(\varphi,\ \frac{1}{r^2}\,D_2\,\varphi)}{D(r,z)} = 0,$$

$$D_2(D_2\,\varphi) - k.D_2\,\varphi = 0.$$

The first requires

$$\frac{1}{r^2}\,D_2\,\varphi = F(\varphi), \tag{7.8}$$

where F is an arbitrary function, and according to the equation (7.7) we have

$$s = f(\psi) = e^{\nu kt}\,F.$$

The second requires

$$F''(\varphi_r^2 + \varphi_z^2) + F'(\varphi_{rr} + \frac{3}{r}\,\varphi_r + \varphi_{zz}) - kF = 0.$$

Since $f' \neq 0$, $f'' \neq 0$, it follows that $F' \neq 0$ and $F'' \neq 0$. Hence this is integrable only if the coefficients are functions of φ, say

$$\varphi_r^2 + \varphi_z^2 = g^2,$$

$$\varphi_{rr} + \frac{3}{r}\,\varphi_r + \varphi_{zz} = h,$$

where g and h are functions of φ. And we have the equation (7.8) at our disposal. Subtracting the equations

$$\varphi_{rr} + \frac{3}{r}\,\varphi_r + \varphi_{zz} = h,$$

$$\varphi_{rr} - \frac{1}{r}\,\varphi_r + \varphi_{zz} = r^2F$$

we obtain
$$\frac{4}{r}\,\varphi_r = h - r^2F,\ \text{i.e.}\ \varphi_r = \frac{r}{4}\,(h - r^2\,F)$$

$$\varphi_{rr} = \frac{1}{16}\,(4h - 12r^3\,h + r^2\,hh' - r^4\,Fh' - r^4\,hF' + r^6\,FF').$$

And from

$$\varphi_z^2 = g^2 - \varphi_r^2 \quad \text{we find}$$

$$\varphi_z^2 = g^2 - \frac{1}{16}\,r^2\,(h - r^2\,F)^2.$$

Differentiating both sides with respect to z and dividing by $2\varphi_z$, we find

$$\varphi_{zz} = gg' - \frac{1}{16}\,r^2\,(h - r^3\,F)\,(h' - r^2\,F\,).$$

Hence

$$D_2\,\varphi = \varphi_{rr} - \frac{1}{r}\,\varphi_r + \varphi_{zz} = gg' - \frac{1}{2}\,r^2\,F = r^2\,F \quad \text{by} \qquad (7.8)$$

$$2gg' - 3r^2\,F = 0.$$

Under the assumptions we have made this is impossible. For this equation is valid only if $F = g' = 0$, or φ is a function of r only. But we have excluded both cases by assuming $F' \neq 0$, $\varphi_r \neq 0$ and $\varphi_z \neq 0$.

It seems that, there is no *ss*. non-steady solution in planes passing through OZ (symmetrical about OZ) other than of the forms $\psi = \psi\,(r,t)$, $\psi = \psi\,(z,t)$ and the solution of $D_2\,\psi = a\,r^2$, though $\psi = (z,t)$ leads to a particular solution of the last one. These possible solutions are given by (7.2), (7.3), and (7.3').

If $\psi = \psi\,(r,t)$ the stream lines are straight lines parallel to OZ, and since $r =$ constant on a stream line, $\frac{1}{r}\,D_2\,\psi$, i.e. the vorticity is constant also. It is always normal to the plane of motion. Hence in this case only constant vorticity lines are coincident with the stream lines.

CHAPTER III

Three-dimensional Ss. Motions

8. Ss. pseudo-plane motions of the first kind.

A plane motion is defined by the following two conditions:

(i) The stream lines are contained in planes parallel to a fixed plane, say XOY; this means

$$w = 0.$$

(ii) The motion is the same in all such planes, that is, the velocity components are independent of z.

But these two conditions may not be satisfied simultaneously. We may consider the motions which satisfy only one of the conditions, but not the other. Thus we obtain two classes of motions, each containing the plane motion as a particular case.

The first class of motions satisfy only the first condition, but not the second. Hence $w = 0$, but u and v may depend on z. The second class of motions satisfy only the second condition, but not the first. Hence the velocity components are independent of z, but $w \neq 0$.

R. Berker [2] calls these two classes of motions "pseudo-plane motions" of the first and of the second kind respectively.

In this chapter we shall discuss the *ss.* pseudo-plane motions of the first kind, leaving the other to the next chapter.

For these motions the velocity components are of the form

$$u = u\,(x, y, z, t), \qquad v = v\,(x, y, z, t), \qquad w = 0.$$

The continuity condition is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

hence there is a stream function $\psi = \psi\,(x, y, z, t)$ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{8.1}$$

and ψ is determined except for an arbitrary additive function of z and t.

The vorticity components are

$$\xi = \frac{\partial^2 \psi}{\partial x \partial z}, \qquad \eta = \frac{\partial^2 \psi}{\partial y \partial z}, \qquad \xi = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right).$$

The *ss.* condition (1.8) gives three equations:

$$\left.\begin{aligned} &\frac{1}{2}(\psi_x^2 + \psi_y^2)_{yz} - (\psi_y \cdot \nabla_1^2 \psi)_z = 0, \\ &\frac{1}{2}(\psi_x^2 + \psi_y^2)_{xz} - (\psi_x \cdot \nabla_1^2 \psi)_z = 0, \\ &\frac{D(\psi, \nabla_1^2 \psi)}{D(x, y)} = 0. \end{aligned}\right\} \tag{8.2}$$

The consistency equation (1.9) also gives three equations:

$$\left.\begin{aligned} &\nu (\nabla_1^2 \psi)_{xz} - \psi_{xzt} = 0, \\ &\nu (\nabla_1^2 \psi)_{yz} - \psi_{yzt} = 0, \\ &\nu \nabla_1^2 (\nabla_1^2 \psi) - (\nabla_1^2 \psi)_t = 0 \end{aligned}\right\} \tag{8.3}$$

where ∇_1^2 and ∇^2 are the operators

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The first set of equations is equivalent to the system

$$\left.\begin{aligned} &\frac{1}{2}(\psi_x^2 + \psi_y^2)_y - \psi_y \cdot \nabla_1^2 \psi = Q_y \\ &\frac{1}{2}(\psi_x^2 + \psi_y^2)_x - \psi_x \cdot \nabla_1^2 \psi = Q_x, \end{aligned}\right\} \tag{8.4}$$

since the third is a consequence of these, where Q is an arbitrary function of x, y, z and t, such that $Q_{xz} = Q_{yz} = 0$; i.e. neither Q_x nor Q_y depends on z.

The second set of three equations is equivalent to

$$\left.\begin{aligned} \nu\nabla^2\psi - \psi_t &= F(x,y,t) + G(z,t) \\ \nabla_1^2 F &= 0, \end{aligned}\right\} \tag{8.5}$$

where the function $G(z,t)$ is entirely arbitrary.

Thus the systems (8.4) and (8.5) are equivalent respectively to the systems (8.2) and (8.3), and are therefore the kinematic conditions of the problem.

As for the dynamical condition, (1.3) shows that

$$H = P - Q + \frac{1}{2}(\psi_x^2 + \psi_y^2),$$

and Q does not depend on z, where Q is defined by the equations (8.4), and P is the harmonic conjugate of F given in (8.5). Hence the pressure equation is

$$\frac{p}{\rho} + \Omega = P(x,y,t) - Q(x,y,t), \tag{8.6}$$

i.e. the $\frac{p}{\rho} + \Omega$ does not depend on z. The function H, and therefore p, is determined only except for an arbitrary additive function of t.

9. Solution in the steady case.

The *ss.* condition is given by the equations

$$\frac{1}{2}(\psi_x^2 + \psi_y^2)_y - \psi_y \cdot \nabla_1^2\psi = Q_y, \tag{9.1}$$

$$\frac{1}{2}(\psi_x^2 + \psi_y^2)_x - \psi_x \cdot \nabla_1^2\psi = Q_x, \tag{9.2}$$

and the consistency equation is

$$\nabla^2\psi = F(x,y) + G(z), \tag{9.3}$$

where Q is an arbitrary function of x, y and $\nabla_1^2 F = 0$. The equation (8.6) becomes

$$\frac{p}{\rho} + \Omega = P(x,y) - Q(x,y). \tag{9.4}$$

If we choose $\nabla^2\psi$ as in (9.3), the equations to be satisfied are (9.1) and (9.2), and in all cases $Q_{xy} = Q_{yx}$ requires

$$\frac{D(\psi, \nabla_1^2\psi)}{D(x,y)} = 0. \tag{9.5}$$

We shall consider the following cases:

(1) $\nabla_1^2\psi =$ a function of z only, say $\nabla_1^2\psi = h(z)$;

(2) $\nabla_1^2\psi =$ a function of x and z, or y and z only, say $\nabla_1^2\psi = h(x, z)$;

(3) $\nabla_1^2\psi =$ a function of ψ, say $\nabla_1^2\psi = h(\psi)$.

To obtain a solution we shall start from the equation (9.5) and satisfy it in the ways indicated above. Then we shall determine the unknown function h by (9.3) and ψ by (9.1) and (9.2).

(1) Let $\nabla_1^2\psi$ be a function of z only, say $\nabla_1^2\psi = h(z)$.

$$\nabla^2\psi = h(z) + \psi_{zz} = F(x.y) + G(z).$$

$\nabla_1^2 F = 0$ requires

$$h_{zz} = 0,$$

$$h(z) = az + b, \tag{9.6}$$

where a, b are arbitrary constants. Then

$$\psi_{zz} = F(x,y) + G(z) - az - b.$$

Integrating twice with respect to, z, we find

$$\psi = \frac{1}{2} z^2 F(x,y) + z A(x,y) + B(x,y) + \iint (G - az - b)\, dz^2.$$

We can discard the last term by taking

$$G(z) = az + b,$$

since ψ is determined except for an arbitary additive function of. z. Hence ψ is

$$\psi = \frac{1}{2} z^2 F(x,y) + z A(x,y) + B(x,y). \tag{9,7}$$

$$\nabla_1^2\psi = \frac{1}{2} z^2 \nabla_1^2 F + z \cdot \nabla_1^2 A + \nabla_1^2 B = az + b.$$

This requires

$$\left.\begin{aligned}\nabla_1^2 A &= a \\ \nabla_1^2 B &= b,\end{aligned}\right\} \tag{9.8}$$

since $\nabla_1^2 F = 0$.

With these values of A, B and F, ψ satisfies the equations (9.3) and (9.5) There remains only to satisfy (9.1) and (9.2). They furnish

$$\left.\begin{aligned}F_x^2 + F_y^2 &= c_1, \\ F_x A_x + F_y A_y - aF &= c_2, \\ F_x B_x + F_y B_y + A_x^2 + A_y^2 - 2aA - bF &= c_3, \\ A_x B_x + A_y B_y - aB - bA &= c_4,\end{aligned}\right\} \tag{9.9}$$

where c's are arbitrary constants of integration.

The equations (9.8) and (9.9), together with $\nabla_1^2 F = 0$, determine A, B and F.

If $c_1 \neq 0$, $c_2 \neq 0$ they are

$$\left.\begin{aligned}F &= \alpha x + \beta y, \text{ where } \alpha^2 + \beta^2 = c_1, \\ A &= \frac{a}{2c_1}(\alpha x + \beta y)^2 + \frac{c_2}{c_1}(\alpha x + \beta y), \\ B &= \frac{b}{2c_1}(\alpha x + \beta y)^2 + \frac{c_4}{c_2}(\alpha x + \beta y),\end{aligned}\right\} \tag{9.10}$$

and c's are always related by $c_1^2 c_4 = c_2 (c_1 c_3 - c_2^2)$, whatever their values are.

$$\frac{p}{\rho} + \Omega = \nu(\beta x - \alpha y) + \frac{bc_4}{c_2}(\alpha x + \beta y) + C,$$

where C is an arbitrary constant.

If $c_1 \neq 0$, $c_2 = 0$, then $c_4 = 0$, and F, A and B are

$$\left.\begin{aligned}F &= \alpha x + \beta y, \text{ where } \alpha^2 + \beta^2 c_1, \\ A &= \frac{a}{2c_1}(\alpha x + \beta y)^2, \\ B &= \frac{b}{2c_1}(\alpha x + \beta y)^2 + \frac{c_3}{c_1}(\alpha x + \beta y).\end{aligned}\right\} \tag{9.11}$$

In this case the pressure equation is

$$\frac{p}{\rho} + \Omega = \nu\,(\beta x - \alpha y) + \frac{bc_3}{c_1}(\alpha x + \beta y) + \text{an arb. constant.}$$

If $c_1 = 0$, then $F_x^2 + F_y^2 = 0$ shows that F is constant; but we can take this constant to be equal to zero, since otherwise it will give a term depending on z only in the expression of ψ. Hence $c_2 = 0$, and the system (9.9) reduces to

$$\left.\begin{aligned} A_x^2 + A_y^2 - 2aA &= c_3 \\ A_x B_x + A_y B_y - aB - bA &= c_4, \end{aligned}\right\} \qquad (9.12)$$

where c_3, c_4 are now arbitrary.

A solution of this system is

$$\left.\begin{aligned} A &= \frac{a}{2k^2}(y + \lambda x)^2 + \frac{c_3^{1/2}}{k}(y + \lambda x), \\ B &= \frac{b}{2k^2}(y + \lambda x)^2 + \frac{c_4}{kc_3^{1/2}}(y + \lambda x), \end{aligned}\right\} \qquad (9.13)$$

where λ, k are arbitrary constants, such that $k^2 = 1 + \lambda^2$, and $c_3 \neq 0$. If $c_3 = 0$, c_4 must also be zero. The pressure equation is

$$\frac{p}{\rho} + \Omega = \text{an arbitrary costant.}$$

In every case we obtain ψ by inserting the values F, A and B into the expression (9.7).

(2) Let $\nabla_1^2 \psi$ be a function of x and z only, say

$$\nabla_1^2 \psi = h\,(x, z).$$

The equation (9.5) shows that $\psi_y = 0$. Then

$$\nabla^2 \psi = h\,(x, z) + \psi_{zz} = F\,(x, y) + G\,(z)$$

requires $F_y = 0$. Now $\nabla_1^2 F = 0$ furnishes

$$F(x) = \alpha x + \beta,$$

$$\text{and} \qquad h_{xx} + h_{zz} = 0, \qquad (9.14)$$

where α, β are arbitrary constants.

After determining a solution of (9.14), ψ is obtained by integrating the equation

$$\psi_{xx} = h,$$

with the condition

$$\psi_{xx} + \psi_{zz} = \alpha x + \beta + G(z). \tag{9.15}$$

The equations (9.1) and (9.2) are always satisfied, whatever the value of $\psi(x, z)$ is; and the stream lines in planes $z =$ const. are the straight lines $x =$ constant.

The pressure equation is

$$\frac{p}{\rho} + \Omega = -\nu\alpha y + C,$$

where C is an arbitrary constant.

For example, if h is of the form

$$h(x, z) = X + Z,$$

where $X = X(x)$, $Z = Z(z)$, (9.14) requires

$$X'' + Z'' = 0;$$

$$X'' = -Z'' = \text{a const., say } 2a.$$

Hence $X = ax^2 + bx + c$, $Z = -az^2 + b_1 z + c_1$. Now

$$\psi_{xx} = ax^2 + bx + (c + c_1) - az^2 + b_1 z,$$

$$\psi(x, z) = \frac{1}{12} ax^4 + \frac{1}{6} bx^3 + \frac{1}{2} x^2(-az^2 + b_1 z + c + c_1) + xA(z),$$

where $A(z)$ is yet arbitrary. This gives

$$\psi_{zz} = -ax^2 + xA''.$$

$$\nabla^2 \psi = \psi_{xx} + \psi_{zz} = (A'' + b)x + (c + c_1) - az^2 + b_1 z.$$

Comparing this with (9.15) we find

$$A'' + b = \alpha$$

$$c = \beta$$

$$-az^2 + b_1 z + c_1 = G(z).$$

The first of these furnishes

$$A(z) = \frac{1}{2}(\alpha - b) z^2 + mz + n.$$

Hence ψ becomes

$$\psi(x, z) = \frac{1}{12} a x^4 + \frac{1}{6} b x^3 + \frac{1}{2} x^2 (-a z^2 + b_1 z + \beta + c_1)$$
$$+ x\left[\frac{1}{2}(\alpha - b) z^2 + m z + n\right] \qquad (9.16)$$

where b, b_1, c_1, m, n are arbitrary constants.

The pressure equation is

$$\frac{p}{\rho} + \Omega = C - \nu\alpha y,$$

where C is an arbitrary constant.

If we assume the form $h = XZ$, (9.14) requires

$$\frac{X''}{X} = -\frac{Z''}{Z} = \lambda,$$

where λ is an arbitrary constant. In a similar way we obtain the solutions

$$\psi = a\, ch\,(mx + \varepsilon)\cdot \cos(mz + \delta) + x\left(\frac{1}{2}\alpha z^2 + bz + c\right), \text{ if } \lambda = m^2,$$

$$\psi = a \cos(mx + \varepsilon)\cdot ch\,(mz + \delta) + x\left(\frac{1}{2}\alpha z^2 + bz + c\right), \text{ if } \lambda = -m^2,$$

$$\psi = (ax^3 + bx^2)(cz + d) + x\left[-acz^3 + \frac{1}{2}(\alpha - 6ad) z^2 + c_1 z + c_2\right] \quad \text{if } \lambda = 0,$$

where the coefficients are arbitrary parameters.

It can easily be proved by the equations (1.6) and (1.7) that when $\psi_x = 0$ or $\psi_y = 0$ or more generally when $\psi_x/\psi_y =$ a constant, i.e. when the stream lines in planes $z =$ constant are parallel straight lines ψ_1 and ψ_2 corresponding to different values of the arbitrary parameters are both *ss.* and superposable on each other hence if the differential equation for ψ is linear and homogeneous these solutions can be added.

Thus instead of the solutions obtained above we can take the following more general expressions:

$$\left.\begin{aligned}
\psi &= \sum_m \left[a_m\, ch\,(mx+\varepsilon_m)\cdot\cos\,(mz+\delta_m) + x\left(\frac{1}{2}\,\alpha_m z^2 + b_m z + c_m\right)\right],\\
\psi &= \sum_m \left[a_m \cos\,(mx+\varepsilon_m)\cdot ch\,(mz+\delta_m) + x\left(\frac{1}{2}\,\alpha_m z^2 + b_m z + c_m\right)\right],\\
\psi &= \sum_m \Big\{(a_m x^3 + b_m x^2)\,(c_m z + d_m) + x\,[-a_m c_m z^3 + \frac{1}{2}\,(\alpha_m - 6a_m d_m)\,z^2\\
&\qquad + c_{1m}\,z + c_{2m}]\Big\}\,.
\end{aligned}\right\} \quad (9.17)$$

(3) More generally, let

$$\nabla_1^2\psi = h\,(\psi), \qquad (9{,}18)$$

where $h(\psi)$ is an arbitrary function of ψ. Then the equation (9.3), i.e.

$$\nabla^2\psi = h + \omega_{zz} = F\,(x,y) + G\,(z)$$

requires

$$\nabla_1^2\,(\nabla^2\psi) = \nabla^2\,(\nabla_1^2\psi) = \nabla^2 h = 0,$$

and

$$(\nabla^2\psi)_{xz} = (\nabla^2\psi)_{yz} = 0.$$

Hence

$$h''\,(\psi_x^2 + \psi_y^2 + \psi_z^2) + h'\cdot\nabla^2\psi = 0, \qquad (9.19)$$

and

$$\left.\begin{aligned}
h''\,\psi_x\,\psi_z + h'\,\psi_{xz} + \psi_{xzzz} = 0\\
h''\,\psi_y\,\psi_z + h'\,\psi_{yz} + \psi_{yzzz} = 0
\end{aligned}\right\}. \qquad (9.20)$$

We shall consider the following cases:

(i) $h' = 0$, $\psi_{zzz} = 0$. That is $\nabla_1^2\psi = C$ and

$$\psi = \frac{1}{2}\,z^2 F\,(x,y) + z\,A\,(x,y) + B\,(x,y).$$

$\nabla_1^2\psi = C$ requires

$$\nabla_1^2 F = 0,$$

$$\nabla_1^2 A = 0,$$

$$\nabla_1^2 B = C.$$

This is a particular case of (i) with $a = 0$, $b = C$; see (9.7) and (9.8)

(ii) $h'' = 0$, $\nabla^2 \psi = 0$. Then $h = C\psi + D$, and

$$\nabla^2 \psi = C\psi + D + \psi_{zz} = 0.$$

Both the equations (9.19) and (9.20) are satisfied. We have

$$\psi_{zz} + C\psi = -D.$$

We can take $D = 0$, since its contribution to ψ is only to add an arbitrary constant.

If $C > 0$, let $C = k^2$. Then a solution of

$$\psi_{zz} + k^2 \psi = 0 \tag{9.21}$$

is
$$\psi = A(x, y) \cos kz + B(x, y) \sin kz.$$

$\nabla_1^2 \psi = C$ requires

$$\left.\begin{aligned} \nabla_1^2 A &= k^2 A \\ \nabla_1^2 B &= k^2 B \end{aligned}\right\}. \tag{9.22}$$

The equations (9.1) and (9.2) furnish

$$\left.\begin{aligned} A_x^2 + A_y^2 - k^2 A^2 &= c_1 \\ B_x^2 + B_y^2 - k^2 B^2 &= c_2 \\ A_x B_x + A_y B_y - k^2 AB &= c_3, \end{aligned}\right\} \tag{9.23}$$

where c_1, c_2, c_3 are arbitrary constants of integration.

The systems (9.22) and (9.23) determine A and B. From (9.22) we find particular solutions

$$A = a_1\, ch\,(kx + \varepsilon_1) + a_2\, ch\,(ky + \varepsilon_2),$$
$$B = b_1\, ch\,(kx + \delta_1) + b_2\, ch\,(ky + \delta_2),$$

where a_1, a_2, b_1, b_2, ε_1, ε_2, δ_1, δ_2 are arbitrary parameters.

Inserting these into the system (9.23) we obtain the relations

$$a_1 a_2 = b_1 b_2 = a_1 b_2 = a_2 b_1 = 0,$$
$$-k^2 (a_1^2 + a_2^2) = c_1,$$
$$-k^2 (b_1^2 + b_2^2) = c_2,$$
$$-k^2 [a_1 b_1\, ch\,(\varepsilon_1 - \delta_1) + a_2 b_2\, ch\,(\varepsilon_2 - \delta_2)] = c_3.$$

The last three determine the values of c_1, c_2, c_3. The first set of relations is satisfied if $a_2 = b_2 = 0$, or if $a_1 = b_1 = 0$. Hence we have the solutions

$$\left.\begin{aligned}\psi(x, z) &= a_1\, ch\,(kx+\varepsilon_1)\cos kz + b_1\, ch\,(kx+\delta_1)\sin kz\\ \psi(y, z) &= a_2\, ch\,(ky+\varepsilon_2)\cos kz + b_2\, ch\,(ky+\delta_2)\sin kz\end{aligned}\right\} \quad (8.24)$$

Since the stream lines are parallel straight lines and the equation $\psi_{zz} + k^2\psi = 0$ is linear we can take the more general expressions

$$\left.\begin{aligned}\psi(x, z) &= \sum_i \left[a_{1i}\, ch\,(kx+\varepsilon_{1i})\cos kz + b_{1i}\, ch\,(kx+\delta_{1i})\sin kz\right]\\ \psi(y, z) &= \sum_i \left[a_{2i}\, ch\,(ky+\varepsilon_{2i})\cos kz + b_{2i}\, ch\,(ky+\delta_{2i})\sin kz\right]\end{aligned}\right\} \quad (9.25)$$

The pressure equation in both cases is

$$\frac{p}{\rho} + \Omega = \text{an arbitrary constant.}$$

If $C < 0$, let $C = -k^2$ in $\psi_{zz} + C\psi = 0$, then

$$\psi_{zz} - k^2\psi = 0. \quad (9.26)$$

$$\therefore\ \psi = A(x, y)e^{kz} + B(x, y)e^{-kz}.$$

$\nabla^2\psi = 0$ requires

$$\left.\begin{aligned}\nabla_1^2 A + k^2 A &= 0\\ \nabla_1^2 B + k^2 B &= 0.\end{aligned}\right\} \quad (9.27)$$

The equations (9.1) and (9.2) furnish

$$\left.\begin{aligned}A^2_x + A^2_y + k^2A^2 &= c_1\\ B^2_x + B^2_y + k^2B^2 &= c_2,\end{aligned}\right\} \quad (9.28)$$

where c_1, c_2 are arbitrary constants.

The systems (9.27) and (9.28) determine A and B. From (9.27) we find

$$A(x, y) = a_1\cos(kx + \varepsilon_1) + a_2\cos(ky + \varepsilon_2)$$
$$B(x, y) = b_1\cos(kx + \delta_1) + b_2\cos(ky + \delta_2),$$

where a's, b's, ε's and δ's are arbitrary constants.

Inserting these into the system (9.28) we obtain the relations

$$a_1a_2 = b_1b_2 = 0.$$
$$k^2(a_1^2 + a_2^2) = c_1,$$
$$k^2(b_1^2 + b_2^2) = c_2.$$

The last two determine c_1 and c_2. The first is satisfied if

$$a_1 = b_1 = 0,$$
or
$$a_2 = b_2 = 0,$$
or
$$a_1 = b_2 = 0,$$
or
$$a_2 = b_1 = 0.$$

Thus we obtain the solutions

$$\left.\begin{aligned}\psi(y,z) &= a_2\cos(ky+\varepsilon_2)e^{kz} + b_2\cos(ky+\delta_2)e^{-kz}\\ \psi(x,z) &= a_1\cos(kx+\varepsilon_1)e^{kz} + b_1\cos(kx+\delta_1)e^{-kz}\\ \psi(x,y,z) &= a_2\cos(ky+\varepsilon_2)e^{kz} + b_1\cos(kx+\delta_1)e^{-kz}\\ \psi(x,y,z) &= a_1\cos(kx+\varepsilon_1)e^{kz} + b_2\cos(ky+\delta_2)e^{-kz}.\end{aligned}\right\} \quad (9.29)$$

These are also ss. and superposable on each other for different values of the constants, except k. For we had previously pointed out that two solutions ψ_1 and ψ_2 satisfing the relation $\nabla_1^2\psi = \lambda\psi$ are superposable on each other only if λ is the same in both. Thus instead of the solutionns (9.29) we can take the following more general ones:

$$\left.\begin{aligned}\psi(y,z) &= \sum_i [a_{2i}\cos(ky+\varepsilon_{2i})e^{kz} + b_{2i}\cos(ky+\delta_{2i})e^{-kz}],\\ \psi(x,z) &= \sum_i [a_{1i}\cos(kx+\varepsilon_{1i})e^{kz} + b_{1i}\cos(kx+\delta_{1i})e^{-kz}],\\ \psi(x,y,z) &= \sum_i [a_{2i}\cos(ky+\varepsilon_{2i})e^{kz} + b_{1i}\cos(kx+\delta_{1i})e^{-kz}],\\ \psi(x,y,z) &= \sum_i [a_{1i}\cos(kx+\varepsilon_{1i})e^{kz} + b_{2i}\cos(ky+\delta_{2i})e^{-kz}]\end{aligned}\right\} \quad (9.30$$

The more important solutions are the last two; because they contain both x and y, as well as z. Therefore u and v are different from zero. In the last one for example

$$u = -ke^{-kz}\sum_i b_{2i}\sin(ky+\delta_{2i}),$$
$$v = ke^{kz}\sum_i a_{1i}\sin(kx+\varepsilon_{1i});$$

and the pressure equation is

$$\frac{p}{\rho} + \Omega = C_1 - k^2\sum_i a_{1i}\cos(kx+\varepsilon_{1i}).\sum_i b_{2i}\cos(ky+\delta_{2i}),$$

where C_1 is an arbitrary constant.

If $C = 0$ in $\psi_{zz} + C\psi = 0$, we obtain a particular case of (1), with $F = a = b = 0$.

(iii) Neither h'' nor h' is zero

Multiplying the first of the equations (9.20) by ψ_y and the second by $-\psi_x$ and adding. we find

$$h' \cdot \frac{D(\psi_z, \psi)}{D(x, y)} + \frac{D(\psi_{zzz}, \psi)}{D(x, y)} = 0.$$

This requires $\psi_z =$ a function of ψ, say

$$\psi_z = f(\psi). \tag{9.31}$$
$$\psi_{zz} = ff',$$

where (′) means a differentiation with respect to ψ. Now (9.20) is satisfied if

$$h''f + h'f' + \frac{1}{2}[f(f^2)'']' = 0,$$
$$2h'f + f(f^2)'' = c, \tag{9.32}$$

where c is a contant of integration.

The equation (9.19) becomes

$$h''(\psi^2_x + \psi^2_y + f^2) + h'(h + ff') = 0$$

which shows that $\psi^2_x + \psi^2_y$ must be a function of ψ also, say

$$\psi^2_x + \psi^2_y = g(\psi). \tag{9.33}$$

Hence

$$h''(g + f^2) + h'(h + ff') = 0. \tag{9.34}$$

We have yet to satisfy the equations (9.2). Since $Q_{xz} = Q_{yz} = 0$, we obtain

$$f(g' - 2h) = a, \tag{9.35}$$

where a is a constant of integration.

Hence we have the following simultaneous equations :

$$h''(g + f^2) + h'(h + ff') = 0, \tag{9.34}$$
$$2h'f + f(f^2)'' = c, \tag{9.32}$$
$$f(g' - 2h) = a \tag{9.35}$$

to determine h, g, and f as functions of ψ.

If $a = 0$, either $f = 0$ or $g' = 2h$. The first possibility makes $\psi_z = 0$. i. e. $\psi = \psi(x, y)$. The last two equations are satisfied, and the first becomes

$$h''g + hh' = 0.$$
$$\therefore \nabla_1^2 h = 0, \quad \text{where} \quad \nabla_1^2 \psi = h(\psi).$$

Then ψ satisfies the equations (3.1). Hence this determines all ss. steady plane motions studied in section 3.

Now take the other possibility:

$$2h = g'.$$

By eliminating h from (9.34) we find

$$2gf'' + f'g' - fg'' = 0,$$

which is satisfied by

$$g = bf^2, \tag{9.36}$$

where b is an arbitrary positive constant different from zero. For, if $b = 0$, then $g = \psi^2_x + \psi^2_y = 0$, i e. both $u = \psi_y = 0$, and $v = -\psi_x = 0$. Hence from (9.36) and $2h = g'$ we find

$$h = bff'. \tag{9.37}$$

Now if we use the relations (9 36) and (9.37), then both the equations (9.32) and (9.34) are satisfied by

$$f(f^2)'' = 2m, \tag{9.38}$$

where $2m = c/(b + 1)$, it is arbitrary since c and b is arbitrary. (9.38) determines f as a function of ψ.

m can not be zero. For otherwise we should have

$$(f^2)'' = \frac{2}{b}\, h' = 0 \qquad \text{by (9.37)},$$

which is not true, since at the beginning we have assumed $h'' \neq 0$. The equation (9.39) can be written in the form

Since ψ is absent, put $f' = p$ and $f'' = p\dfrac{dp}{df}$, then we have

$$f^2p\,dp + p^2f\,df = m\,df,$$

or

$$f^2p^2 = 2mf + n:$$

$$\therefore \quad p = \frac{df}{d\psi} = \frac{1}{f}\,(2mf + n)^{1/2}.$$

By integrating again we obtain

$$(2mf + n)^{3/2} - 3n(2mf + n)^{1/2} = 3m\psi + l, \tag{9.39}$$

where n, l are arbitrary constants.

Differentiate both sides with respect to z, and put $\psi_z = f$. After rearranging the terms we find

$$2m(2mf + n)^{-1/2}\,df = dz\,,$$

or by integrating

$$2(2mf + n)^{1/2} = z + A(x, y)\,,$$

where $A(x,y)$ is an arbitrary function of x and y.

Now by eliminating $(2mf + n)^{1/2}$ between this and (9.39) we obtain

$$2m\psi = \frac{1}{12}(z + A)^3 - (z + A), \tag{9.40}$$

the arbitrary additive constant l is neglected. Hence

$$2m \cdot \nabla_1^2\psi = \frac{1}{2}(z + A)(A_x^2 + A_y^2) + \left[\frac{1}{4}(z + A)^2 - n\right]\nabla_1^2 A$$
$$= \frac{1}{2}b(z + A) \quad \text{by (9.37)},$$

requires

$$\nabla_1^2 A = 0\,,$$
$$A_x^2 + A_x^2 = b.$$

These show that A is of the form $A = \alpha x + \beta y + \gamma$ where $\alpha^2 + \beta^2 = b$. Hence (9.40) becomes

$$2m\psi = \frac{1}{12}(\alpha x + \beta y + z + \gamma)^3 - n(\alpha x + \beta y + z + \gamma). \tag{9.40}$$

The stream lines in planes $z =$ constant are straight lines parallel to $\alpha x + \beta y =$ constant. The vorticity components are

$$\xi = \frac{\alpha}{4m}(\alpha x + \beta y + z + \gamma), \quad \eta = \frac{\beta}{4m}(\alpha x + \beta y + z + \gamma),$$
$$\zeta = -\frac{b}{4m}(\alpha x + \beta y + z + \gamma).$$

Hence vorticity is constant on each stream line.

Finally the pressure equation is

$$\frac{p}{\rho} + \Omega = \frac{\nu}{4m}(b + 1)(\beta x - \alpha y) + C,$$

where C is an arbitrary constant.

10. Solution in the non-steady case.

The ss. condition is given by the equations

$$\left.\begin{aligned}\frac{1}{2}(\psi^2_x+\psi^2_y)_y-\psi_y\cdot\nabla^2_1\psi=Q_y,\\ \frac{1}{2}(\psi^2_x+\psi^2_y)_x-\psi_x\cdot\nabla^2_1\psi=Q_x,\end{aligned}\right\}\qquad(8.4)$$

and the consistency equation is

$$\nu\nabla^2\psi-\psi_t=F(x,y,t)+G(z,t),\qquad(8.5)$$

where $Q(x, y, t)$ is an arbitrary function of x, y and t; $\nabla^2_1 F=0$, and $G(z, t)$ is arbitrary.

The pressure equation is

$$\frac{p}{\rho}+\Omega=P(x,y,t)-Q(x,y,t),\qquad(8.7)$$

where P is the harmonic conjugate of F given in (8.5), and Q is defined by (8.4).

If we choose $\nabla^2\psi$ as in (8.5), the remaining equations to be satisfied are (8.4), and in all cases $Q_{xy}=Q_{yx}$ requires

$$\frac{D(\psi,\nabla^2_1\psi)}{D(x,y)}=0.\qquad(9.5)$$

We shall consider the following cases:

(1) $\nabla^2_1\psi=$ a function of z and t only, say $\nabla^2_1\psi=h(z,t)$;

(2) $\nabla^2_1\psi=$ a function of x, z and t at most, i.e. $\nabla^2_1\psi=h(x,z,t)$;

(3) $\nabla^2_1\psi=$ a function of ψ, i. e. $\nabla^2_1\psi=h(\psi)$.

To obtain a solution we shall start from the equation (9.5) and satisfy it in the ways indicated above. Then we shall determine the unknown function h by (8.5) and ψ by (8.4).

(1) Let $\nabla^2_1\psi$ be a function of z and t only, say $\nabla^2_1\psi=h(z,t)$.

Applying the operator ∇^2_1 to both sides of (8.5) we find

$$\nu\nabla^2 h-h_t=0,\qquad(10.1)$$
$$\therefore\ \nu h_{zz}-h_t=0,$$

This is the diffusion equation in one dimension. Its solution depends on the boundary conditions of the problem. If we assume, for example, that $\nabla^2_1\psi=h$, i e. the component of vorticity in the direction of z-axis is constant when $t\to\infty$ we take

$$h(z,t)=e^{-\nu\lambda^2 t}a\cos(\lambda z+\varepsilon)+b,\qquad(10.1)$$

where $a, b, \lambda, \varepsilon$ are arbitrary constants. Note that by (10.1) h can not be a function of t only. Now the equation

$$\nu\nabla^2\psi - \psi_t = F(x, y, t) + G(z, t)$$

can be written as

$$\nu\psi_{zz} - \psi_t = F + G - \nu h.$$

We can take

$$G(z, t) = \nu h(z, t) = \nu a\, e^{-\nu\lambda^2 t}\cos(\lambda z + \varepsilon) + \nu b,$$

since $G(z, t)$ is arbitrary. Hence

$$\nu\psi_{zz} - \psi_t = F(x, y, t).$$

A solution of this equation is

$$\psi = e^{-\nu\lambda^2 t} A(x, y)\cos(\lambda z + \varepsilon) + \varphi(x, y, t), \tag{10.3}$$

where $A(x, y)$ is arbitrary, and φ is a particular solution, i. e.

$$\nu\varphi_{zz} - \varphi_t = -\varphi_t = F(x, y, t).$$

Now $\quad \nabla_1^2\psi = e^{-\nu\lambda^2 t}\Delta_1^2 A\cos(\lambda z + \varepsilon) + \nabla_1^2\varphi,$

$$= e^{-\nu\lambda^2 t} a\cos(\lambda z + \varepsilon) + b \qquad \text{by (10.2).}$$

$$\therefore \quad \nabla_1^2 A = a, \qquad \nabla_1^2\varphi = b. \tag{10.4}$$

The ss. condition (8.4) furnishes

$$\left.\begin{aligned} A^2{}_x + A^2{}_y - 2aA &= c_1, \\ A_x\varphi_x + A_y\varphi_y - a\varphi - bA &= c_2(t), \end{aligned}\right\} \tag{10.5}$$

where c_1 is an arbitrary absolute constant, and $c_2(t)$ is arbitrary.

The equations (10.4) and (10.5) determine $A(x, y)$ and $\varphi(x,y,t)$. These equations are just the same as those in (9.8) and (9.12). The only difference is that in (9.12) c_3 and c_4 were absolute constants, but in (10.5) c_3 is replaced by c_1 which is an absolute constant again, and c_4 is replaced by $c_2(t)$, an arbitrary function of time. Hence in a similar way we obtain the solutions

$$A = \frac{a}{2k^2}(y + \alpha x)^2 + \frac{\sqrt{c_1}}{k}(y + \alpha x),$$

$$\varphi = \frac{b}{2k^2}(y + \alpha x)^2 + \frac{c_2(t)}{k\sqrt{c}}(y + \alpha x);$$

where α, k are are arbitrary constants, such that $k^2 = 1 + \alpha^2$, and $c_1 \neq 0$. If $c_1 = 0$, $c_2(t)$ must also be zero. Hence ψ is

$$\psi = \left[\frac{a}{2k^2}(y + \alpha x)^2 + \frac{\sqrt{c_1}}{k}(y + \alpha x)\right] \cdot e^{-\nu\lambda^2 t}\cos(\lambda + \varepsilon)$$
$$+ \frac{b}{2k^2}(y + \alpha x)^2 + \frac{c_2(t)}{k\sqrt{c_1}}(y + \alpha x). \tag{10.6}$$

The stream lines are straight lines parallel to $y + \alpha x =$ constant. The pressure equation is

$$\frac{p}{\rho} + \Omega = \frac{c_2'}{k\sqrt{c_1}}(x - \alpha y) - \frac{c_2^2}{2c_1}.$$

(2) $\nabla_1^2\psi =$ a function of x, z and t only, say $\nabla_1^2\psi = h(x, z, t)$. (9.5) shows that $\psi_y = 0$, then according to the equation

$$\nu\nabla^2\psi - \psi_t = F(x, y, t) + G(z, t).$$

$F_y = 0$. Now the equation (10.1) requires

$$\nu(h_{xx} + h_{zz}) - h_t = 0.$$

A solution which is constant when $t \to \infty$ is

$$h = e^{-\nu\lambda^2 t}[a\cos(\lambda x + \varepsilon) + b\cos(\lambda z + \delta)] + 2c,$$

where $a, b, c, \varepsilon, \delta, \lambda$ are arbitrary real constants. Hence integrating the equation $\nabla_1^2\psi = \psi_{xx} = h$ twice with respect to x, we obtain

$$\psi = e^{-\nu\lambda^2 t}\left[-\frac{a}{\lambda^2}\cos(\lambda x + \varepsilon) + \frac{1}{2}bx^2\cos(\lambda z + \delta) + cx^2\right.$$
$$+ x\varphi(z, t) + \chi(z, t). \tag{10.7}$$

This makes

$$\nu\nabla^2\psi - \psi_t = \nu b e^{-\nu\lambda^2 t}\cos(\lambda z + \delta) + 2\nu c + (\nu\varphi_{zz} - \varphi_t)x$$
$$+ (\nu\chi_{zz} - \chi_t), \tag{10.7'}$$

which must be of the form $F(x, t) + G(z, t)$. This is possible only if $\nu\varphi_{zz} - \varphi_t$ and $\nu\chi_{zz} - \chi_t$ are functions of t only, say

$$\left.\begin{aligned} \nu\varphi_{zz} - \varphi_t &= c_1(t) \\ \nu\chi_{zz} - \chi_t &= c_2(t). \end{aligned}\right\} \tag{10.8}$$

Hence ψ is defined by the equations (10.7) and (10.8). The stream lines of the motion are always straight lines parallel to y-axis, whatever the value of $\psi(x, z, t)$. Note that when one of the variables x and y is absent in the stream function the ss. conditions (8.4) and (9.5) are satisfied automatically. Then there remains to satisfy the consistency equation (8.5) which becomes if for example $\psi_y = 0$,

$$\nu(\psi_{xx} + \psi_{zz}) - \psi_t = F(x, t) + G(z, t),$$

where $\nabla_1^2 F = F_{xx} = 0$.

Applying the operator ∇_1^2 to both sides, we find

$$\nu(h_{xx} + h_{zz}) - h_t = 0, \quad \text{where} \quad h = \nabla_1^2\psi = \psi_{xx}.$$

To determine ψ we first solve this last equation suitably, and then integrate the equation $\psi_{xx} = h$ twice with respect to x. This procedure brings some new arbitrary functions to the expression of ψ. They must be determined by taking account of the form

$$\nu(\psi_{xx} + \psi_{zz}) - \psi_t = F(x, t) + G(z, t).$$

Now the pressure equation for the solution (10.7) is given by

$$\frac{p}{\rho} + \Omega = P - Q,$$

where P is the harmonic conjugate of F, and Q is determined by (8.4). (10.7′) and (10.8) show that $F = c_1(t)x + c_2(t)$. Then $P = -c_1(t)y$, and $Q = Q(t)$ is arbitrary. Hence

$$\frac{p}{\rho} + \Omega = -c_1(t)y - Q(t).$$

(3) Let $\nabla_1^2\psi$ be a function of ψ,

$$\nabla_1^2\psi = h(\psi). \tag{10.9}$$

The equation (9.5) is satisfied, and (10.1) furnishes

$$\nu h''(\psi^2{}_x + \psi^2{}_y + \psi^2{}_z) + h'(\nu\nabla^2\psi - \psi_t) = 0. \tag{10.10}$$

This can be satisfied in different ways. We shall consider all possibilities :

(i) $h' = 0$. Then $h = \nabla_1^2\psi = C = a$ constant. Hence this becomes a particular case of (1) with $a = 0$, $b = C$.

(ii) $h'' = 0$, $\nu\nabla^2\psi - \psi_t = 0$ Then $h' = C$ and

$$h = \nabla_1^2\psi = C\psi, \tag{10.11}$$

where C is an arbitrary parameter. We have neglected an arbitrary additive constant of integration, since its contribution to ψ will be only to add an arbitrary function of z. Now

$$\nu\nabla^2\psi - \psi_t = \nu(C\psi + \psi_{zz}) - \psi_t = 0;$$
$$\therefore \quad \nu\psi_{zz} - \psi_t = -\nu C\psi.$$

Let $\psi = \psi_0 \cdot T$, where ψ_0 is a function of x, y, z and T is a function of t only. Then this becomes

$$\nu\left(\frac{\psi_{0zz}}{\psi_0} + C\right) = \frac{T'}{T} = \text{a constant} = -\nu\lambda^2 \text{ say.}$$

We have assumed the constant to be negative. The cases where it is positive or zero can be investigated similarly,Hence

$$T \backsimeq e^{-\nu\lambda^2 t}, \tag{10.12}$$

and ψ_0 will be determined by

$$\psi_{0zz} + (C + \lambda^2)\,\psi_0 = 0. \tag{10.13}$$

Now there are three possibilities: $C + \lambda^2$ may be positive, negative or zero.

a) If $C + \lambda^2 > 0$, let $C + \lambda^2 = m^2$. Then (10.13) gives [1]

$$\psi_0 = A(x, y) \cos mz + B(x, y) \sin mz,$$

and $\nabla_1^2 \psi_0 = C\psi_0$ furnishes

$$\left.\begin{aligned} \nabla_1^2 A &= CA \\ \nabla_1^2 B &= CB. \end{aligned}\right\} \tag{10.14}$$

The ss. condition (8 4) gives other three equations:

$$\left.\begin{aligned} A^2{}_x + A^2{}_y - CA^2 &= c_1 \\ B^2{}_x + B^2{}_y - CB^2 &= c_2 \\ A_x B_x + A_y B_y - CAB &= c_3. \end{aligned}\right\} \tag{10.15}$$

C itself may be positive. negative and zero If $C > 0$ let $C = k^2$, then the equation (10.14) and (10.15) become just the same as the equations (8 22) and (9.23). In a similar way we find the solutions:

$$\psi_0(x, z) = a_1 ch(kx + \varepsilon_1) \cos mz + b_1 ch(kx + \delta_1) \sin mz,$$
$$\psi_0(y, z) = a_2 ch(ky + \varepsilon_2) \cos mz + b_2 ch(ky + \delta_2) \sin mz,$$

corresponding to (9.24), or

$$\left.\begin{aligned} \psi(x,z,t) &= \sum_m [a_{1m} ch(kx + \varepsilon_{1m}) \cos mz + b_{1m} ch(kx + \delta_{1m}) \sin mz] e^{-\nu\lambda^2 t} \\ \psi(x,z,t) &= \sum_m [a_{2m} ch(ky + \varepsilon_{2m}) \cos mz + b_{2m} ch(ky + \delta_{2m}) \sin mz] e^{-\nu\lambda^2 t} \end{aligned}\right\} \tag{10.16}$$

corresponding to (9.25), where $\lambda^2 = m^2 - k^2$.

If $C < 0$, let $C = -k^2$. Then A and B satisfy the systems (9.27) and (9.27), and we find the solutions

$$\left.\begin{aligned}\psi(x,z,t) &= \sum_m [a_{1m}\cos(kx+\varepsilon_{1m})\cos mz + b_{1m}\cos(kx+\delta_{1m})\sin mz]e^{-\nu\lambda^2 t}\\ \psi(y,z,t) &= \sum_m [a_{2m}\cos(ky+\varepsilon_{2m})\cos mz + b_{2m}\cos(ky+\delta_{2m})\sin mz]e^{-\nu\lambda^2 t}\end{aligned}\right\} \quad (10.17)$$

where $\lambda^2 = m^2 + k^2$. For these two sets of solutions the pressure is given by

$$\frac{p}{\rho} + \Omega = \text{arbitrary function of } t.$$

If $C = 0$, we find the case (i), since then $h' = 0$.

b) If $C + \lambda^2 < 0$, let $C + \lambda^2 = -m^2$. Then C must always be negative. Take $C = -m^2 - \lambda^2 = -k^2$. The equation (10.13) for ψ_0 becomes

$$\psi_{0zx} - m^2\psi_0 = 0,$$

$$\therefore \quad \psi_0 = A(x,y)\, e^{mz} + B(x,y)e^{-mz}.$$

A and B are the the solutions of the equations (9.27) and (9.28). Corresponding to (9.30) we obtain the solutions:

$$\left.\begin{aligned}\psi(y,z,t) &= \sum_m [a_{2m}\cos(ky+\varepsilon_{2m})e^{mz} + b_{2m}\cos(ky+\delta_{2m})e^{-mz}]\, e^{-\nu\lambda^2 t},\\ \psi(x,z,t)n &= \sum_m [a_{1m}\cos(kx+\varepsilon_{1m})e^{mz} + b_{1m}\cos(kx+\delta_{1m})e^{-mz}]\, e^{-\nu\lambda^2 t},\\ \psi(x,y,z,t) &= \sum_m [a_{2m}\cos(ky+\varepsilon_{2m})e^{mz} + b_{1m}\cos(kx+\delta_{1m})e^{-mz}]\, e^{-\nu\lambda^2 t},\\ \psi(x,y,z,t) &= \sum_m [a_{1m}\cos(kx+\varepsilon_{1m})e^{mz} + b_{2m}\cos(ky+\delta_{2m})e^{-mz}]\, e^{-\nu\lambda^2 t},\end{aligned}\right\} \quad (10.18)$$

where $\lambda^2 = k^2 - m^2$.

c) If $C + \lambda^2 = 0$, then $C = -\lambda^2$ and (10.13) gives

$$\psi_0 = A(x,y)\, z + B(x,y),$$

$$\therefore \quad \psi = (Az + B)\, e^{-\nu\lambda^2 t}.$$

$\nabla_1^2\psi = -\lambda\psi^2$ requires

$$\nabla_1^2 A = -\lambda^2 A,$$

$$\nabla_1^2 B = -\lambda^2 B,$$

and ss. condition (8.4) gives

$$A^2_x + A^2_y + \lambda^2 A^2 = c_1,$$
$$A_x B_x + A_y B_y + \lambda^2 AB = c_2,$$

where c_1, c_2 are arbitrary parameters. From the first set of equations we find particular solutions of the form

$$A(x, y) = a_1 \cos(\lambda x + \varepsilon_1) + a_2 \cos(\lambda y + \varepsilon_2),$$
$$B(x, y) = b_1 \cos(\lambda x + \delta_1) + b_2 \cos(\lambda y + \delta_2),$$

where a, b, ε, δ are constants of integration. Now the second set requires

$$a_1 a_2 = 0,$$
$$a_1 b_2 + a_2 b_1 = 0,$$
$$\lambda^2 (a_1^2 + a_2^2) = c_1,$$
$$\lambda^2 [a_1 b_1 \cos(\varepsilon_1 - \delta_1) + a_2 b_2 \cos(\varepsilon_2 - \delta_2)] = c_2.$$

The last two determine the constants c_1, c_2; the first two are satisfied if either

$$a_2 = b_2 = 0,$$

or

$$a_1 = b_1 = 0,$$

or

$$a_1 = a_2 = 0.$$

The last possibility can not be accepted, because then ψ can not depend on z. The first and the second furnish the solutions

$$\left.\begin{aligned} \psi &= [a_1 \cos(\lambda x + \varepsilon_1) z + b_1 \cos(\lambda x + \delta_1)] e^{-\nu\lambda^2 t}, \\ \psi &= [a_2 \cos(\lambda y + \varepsilon_2) z + b_2 \cos(\lambda y + \delta_2)] e^{-\nu\lambda^2 t}, \end{aligned}\right\} \quad (10.19)$$

Since the stream lines are parallel straight lines and since the equation (10.13) is linear and homogeneous, we can take as the more general solutions

$$\left.\begin{aligned} \psi &= \sum_\lambda [a_{1\lambda} \cos(\lambda x + \varepsilon_{1\lambda}) z + b_{1\lambda} \cos(\lambda x + \delta_{1\lambda})] e^{-\nu\lambda^2 t}, \\ \psi &= \sum_\lambda [a_{2\lambda} \cos(\lambda y + \varepsilon_{2\lambda}) z + b_{2\lambda} \cos(\lambda y + \delta_{2\lambda})] e^{-\nu\lambda^2 t}, \end{aligned}\right\} (10.20)$$

The pressure equation is

$$\frac{p}{\rho} + \Omega = \text{an arbitrary function of } t.$$

(iii) In the equation (10.10) assume that neither h'' nor h' is zero. The equation is integrable if and only if

$$\psi^2_x + \psi^2_y + \psi^2_z = g(\psi), \quad (10.21)$$

and

$$\nu \nabla^2 \psi - \psi_t = \nu p(\psi),$$

where g and p are arbitrary functions of ψ.

Now (8.5) requires $\nabla_1^2 p = 0$, $p_{xz} = p_{yz} = 0$, i. e.

$$p''(\psi_x^2 + \psi_y^2) + p'h = 0, \tag{10.23}$$

$$\left.\begin{aligned} p''\psi_x\psi_z + p'\psi_{xz} &= 0, \\ p''\psi_y\psi_z + p'\psi_{yz} &= 0, \end{aligned}\right\} \tag{10.24}$$

(8.4) requires $Q_{xz} = Q_{yz} = 0$, i. e.

$$\left.\begin{aligned} \frac{1}{2}(g'' - 2h')\psi_x\psi_z + \frac{1}{2}(g' - 2h)\psi_{xz} - \psi_{zz}\psi_{zx} - \psi_z\psi_{zzx} &= 0 \\ \frac{1}{2}(g'' - 2h')\psi_y\psi_z + \frac{1}{2}(g' - 2h)\psi_{yz} - \psi_{zz}\psi_{zy} - \psi_z\psi_{zzn} &= 0. \end{aligned}\right\} \tag{10.25}$$

In addition to these we have the equation (10.10), which now becomes

$$h''g + h'p = 0. \tag{10.26}$$

From (10.24) we find

$$p' \cdot \frac{D(\psi_z, \psi)}{D(x, y)} = 0 \tag{10.27}$$

by multiplying the first equation by ψ_y the second by $-\psi_x$ and adding. By the same way from (10.25) we obtain

$$\left[\frac{1}{2}(g' - 2h) - \psi_{zz}\right]\frac{D(\psi_z, \psi)}{D(x, y)} - \psi_z \frac{D(\psi_{zz}, \psi)}{D(x, y)} = 0. \tag{10.28}$$

Both the equations (10.27) and (10.28) are satisfied if ψ_z is a function of ψ. Let

$$\psi_z = f(\psi). \tag{10.29}$$

Then (10.23) may be written in the form

$$p''(g - f^2) + p'h = 0. \tag{10.30}$$

The equations (10.24) are satisfied if

$$\begin{aligned} p''f + p'f' &= 0, \\ \therefore\quad p'f &= c, \end{aligned} \tag{10.31}$$

where c is *a* constant of integration.

The equations (10.25) require

$$\frac{1}{2}(g'' - 2h')f + \frac{1}{2}(g' - 2h)f' - ff' \cdot f' - f(ff')' = 0.$$

$$\therefore\quad (g' - 2h)f - 2f^2f' = c_1, \tag{10.32}$$

where c_1 is a constant of integration.

In addition to these we can find other two equations: The first of which is obtained by applying the operator ∇_1^2 to both sides of (10.29).

$$\nabla_1^2(\psi_z) = (\nabla_1^2\psi)_z = h_z = \nabla_1^2 f,$$
$$\therefore\ f''(g - f^2) + f'h - fh' = 0. \tag{10.33}$$

The second additional equations from (10.22). It can be written in the form

$$\psi_t = \nu(\nabla^2\psi - p),$$
$$\therefore\ \psi_t = \nu(h + ff' - p) \tag{10.34}$$

by (10.9) and (10.29).

Now from $\psi_{tz} = \psi_{zt}$ we find

$$(h + ff' - p)'f = f'(h + ff' - p),$$

or by integrating once

$$h + ff' - p = af, \tag{10.35}$$

where a is an arbitrary constant. Thus (10.34) becomes

$$\psi_t = \nu af. \tag{10.34'}$$

Hence we have the following similtaneous equations

$$h''g + h'p = 0, \tag{10.26}$$
$$p''(g - f^2) + p'h = 0, \tag{10.30}$$
$$p'f = c, \tag{10.31}$$
$$(g' - 2h)f - 2f^2f' = c_1, \tag{10.32}$$
$$(g - f^2)f'' + f'h - fh' = 0, \tag{10.33}$$

and
$$h - p + ff' = af, \tag{10.35}$$

to determine the functions p, g, h and f in terms of ψ. Only four of these equations are independent.

Solution: Eliminate $g - f^2$ between the equations (10.30) and (10.33). We find

$$p''(f'h - fh') - f''p'h = 0,$$
$$\therefore\ \frac{p''}{p'}\left(\frac{f'}{f} - \frac{h'}{h}\right) - \frac{f''}{f} = 0.$$

By using (10.31) this becomes

$$\frac{h'}{h} = \frac{f''}{f'} + \frac{f'}{f}.$$

$$\therefore \quad h = bff', \tag{10.36}$$

where b is an arbitrary constant.

Inserting the value of h from (10.36) and the values of p', p'' from (10.31) into the equation (10.30), we obtain

$$(g - f^2)\, f' - bf^2f' = 0.$$

$f' \neq 0$ since otherwise h becomes zero, hence

$$g = (b + 1)f^2, \tag{10.37}$$

where b is different from -1, for if $b = -1$, $g = \psi^2_x + \psi^2_y + \psi_z^2 = 0$, and ψ becomes only a function of t, which can not be the stream function of any motion.

These values of h and g satisfy the equations (10 30), (10.33) and (10 82), and show that $c_1 = 0$. (10.35) determines p as a function of f:

$$p = (b + 1)ff' - af. \tag{10.38}$$

There remains only to satisfy (10.26) and (10.31). They furnish respectively

$$\begin{gathered}(b + 1)f(f^2)''' + (f^2)''[(b + 1)f' - a] = 0,\\ [(b + 1)(f^2)'' - 2af\,)]f = 2c,\end{gathered} \tag{10.39}$$

where the first equation is a consequence of the second one.

We first try tp solve (10.39), which determines $f = \psi_z$ as a function of ψ. Then the equations (10.36), (10 37) and (10.37) and (10.38) give the values of h, g and p in terms of ψ. Finally ψ will be determined as a function of x, y, z and t, by using the definitions of h, g, p and f.

Neither b nor $b + 1$ is zero. For $b = 0$ makes $h = 0$, and $b + 1 = 0$ makes $g = 0$, but both are impossible since $h' \neq 0$ and ψ is not a function of t only. In fact $b + 1$ is always positive,

If $a = 0$, the equation (10.34′) shows that $\psi_t = 0$. This gives the steady case, which has been discussed in section 9. Hence a must also be different from zero.

c is arbitrary. It may be or may not be zero. We shall consider the case $c = 0$ later on.

The equation (10.39 can be written in the form

$$\psi_{zzz} - k\psi_{zz} = s, \tag{10.40}$$

since $\psi_z = f$, $\psi_{zz} = ff'$, and $\psi_{zzz} = f(ff')'$, where $k = a/(b + 1)$, and $s = c/(b+1)$. By integrating three times with respect to z, we find

$$\psi = \frac{1}{k^2} A e^{kz} + Bz + C - \frac{s}{2k} z^2, \qquad (10.01)$$

where A, B, C are functions of x, y and t.

The equation (10.34') requires

$$A_t = \nu akA,$$
$$B_t = -\frac{\nu as}{k} = -\nu c,$$
$$C_t = \nu aB.$$

$$\therefore \left. \begin{aligned} A &= A_0 e^{\nu akt}, \\ B &= B_0 - \nu ct, \\ C &= C_0 + \nu aB_0 t - \frac{1}{2} \nu^2 act^2 ; \end{aligned} \right\} \qquad (10.42)$$

where A_0, B_0, C_0 depend only on x and y

The equations (10.36) and (10.37) furnish

$$\left. \begin{aligned} \nabla_1^2 A_0 &= bk^2 A_0, \\ \nabla_1^2 B_0 &= 0, \\ \nabla_1^2 C_0 &= -\frac{bs}{k} ; \end{aligned} \right\} \qquad (10.43)$$

and

$$\left. \begin{aligned} A^2_{0x} + A^2_{0y} &= bk^2 A^2_0, \\ A_{0x}B_{0x} + A_{0y}B_{0y} &= -bsA_0, \\ A_{0x}C_{0x} + A_{0y}C_{0y} &= bkA_0B_0, \\ B_{0x}C_{0x} + B_{0y}C_{0y} &= -\frac{bs}{k} B_0, \\ B^2_{0x} + B^2_{0y} &= \frac{bs^2}{k^2}, \\ C^2_{0x} + C^2_{0y} &= bB^2_0. \end{aligned} \right\} \qquad (10.44)$$

These two sets of equations determine A_0, B_0 and C_0 completely. Before solving them we must consider the cases $c = 0$ $c \neq 0$ respectively.

If $c = 0$, then $s = 0$. Now the equations

$$\nabla_1^2 B_0 = 0$$

and $$B^2_{0x} + B^2_{0y} = 0$$

show that B_0 is a constant, say

Then the remaining equations become

$$\nabla_1^2 A_0 = bk^2 A_0,$$
$$\nabla_1^2 C_0 = 0,$$
$$A^2_{ox} A + A^2_{oy} C_{oy} = bkmA_0,$$
$$C^2_{oy} + C^2_{oy} = bm^2.$$

The second and the last one show that C_0 is linear in x and y. Let

$$C_0(x,y) = \alpha x + \beta y,$$

where $\alpha^2 + \beta^2 = bm^2$. Now the equation $A_{ox}C_{ox} + A_{oy}C_{oy} = bkmA_0$ gives

$$A_0 = \varphi(\beta x - \alpha y).\ \exp\left[\frac{k}{m}(\alpha x + \beta y)\right],$$

where φ is an arbitrary function, bu $A^2_{ox} + A^2_{oy} = bk^2 A_o^2$ requires $\varphi =$ constant, say n. Hence

$$A_0 = n.\ \exp\left[\frac{k}{m}(\alpha x + \beta y)\right].$$

With these values of A_0, B_0, and C_0 all the equations are satisfied when $c = 0$. Hence, by neglecting the terms which depend only on z and t, w find

$$\psi = \frac{n}{k^2}\ \exp\left[\frac{k}{m}(\alpha x + \beta y + mz + \nu\, amt)\right] + \alpha x + \beta y; \qquad (10.45)$$

where $k = a/(b+1)$, and m, n are arbitrary constants, such that $m \neq 0$. If $m = 0$, then $B_0 = 0$, $C_0 = a$ const. and

$$A_0 = n_1 \cdot \exp\left[l(\alpha_1 x + \beta_1 y)\right],$$

where α_1. β_1, l, n_1 are parameters, such that $l^2(\alpha_1^2 + \beta_1^2) = bk^2$. Hence

$$\psi = \frac{n_1}{k^2}\ \exp\left[l(\alpha_1 x + \beta_1 y) + kz + \nu\, akt\right]. \qquad (10.46)$$

If $c \neq 0$, then $s \neq 0$. By a similar way we obtain the values

$$A_0 = n.\ \exp\left[-\frac{k^2}{s}(\alpha x + \beta y)\right],$$
$$B_0 = \alpha x + \beta y,$$
$$C_0 = -\frac{k}{2s}(\alpha x + \beta y)^2,$$

from the system (10.43) and (10.44) where α, β, n are arbitrary constant such that $\alpha^2 + \beta^2 = bs^2/k^2$, and an arbitrary additive constant in the expression of C_0 has been neglected. Hence ψ is

$$\psi = \frac{n}{k^2} \exp\left[-\frac{k^2}{s}(\alpha x + \beta y) + k(z + \nu at)\right]$$
$$+ (\alpha x + \beta y) z - \frac{k}{2s}(\alpha x + \beta y)(\alpha x + \beta y - 2\nu ct), \qquad (10.47)$$

terms depending only on z and t have been neglected.

The stream lines in planes $z =$ constant for the solutions (10.45), (10.46) and (10.47) are straight lines (parallel to $\alpha x + \beta y$ $=$ constant). Finally the pressure equation for (10.46) is or to $\alpha_1 x + \beta_1 y =$ const.

$$\frac{p}{\rho} + \Omega = \nu a(\alpha_1 y - \beta_1 x) + m(t),$$

and for the others it is

$$\frac{p}{\rho} + \Omega = \nu a(\alpha y - \beta x) + m(t),$$

where $m(t)$ is arbitrary.

CHAPTER IV.

Three dimensional Ss. Motions (Continued)

11. Ss. pseudo-plane motions of the second kind.

In pseudo-plane motions of the kind the velocity components are of the form

$$u = u\,(x, y, t), \qquad v = v\,(x, y, t), \qquad w = w\,(x, y, t),$$

i. e. they do not depend on z.

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Hence there is a velocity potential $\psi\,(x, y, t)$ such that

$$u = \psi_y\,, \qquad v = -\,\psi_x\,, \tag{11.1}$$

and ψ is determined except for an arbitrary additive function of t.

The vorticity components are

$$\xi = w_y\,, \qquad \eta = -\,w_x\,, \qquad \xi = -\,\nabla_1^2\psi.$$

Ss. condition (1.8) give two equations:

$$\frac{D\,(\psi\,, \nabla_1^2\psi)}{D\,(x,y)} = 0, \tag{11.2}$$

$$\frac{D\,(\psi\,, w)}{D(x,y)} = 0. \tag{11.3}$$

The consistency equation (1.9) gives other two equations:

$$\nu\,\nabla_1^2\,(\nabla_1^2\varphi) - (\nabla_1^2\psi)_t = 0, \tag{11.4}$$

$$\nu\,\nabla_1^2 w - w_t = c\,(t), \tag{11.5}$$

where ∇_1^2 is Laplacian differential operator in two-dimensions and $c\,(t)$ is an arbitrary function of time.

Equations (11.2) and (11.4) show that ψ is the stream function of a ss. plane motion (See the equations 2.4 and 2.5).

After determining ψ from (11.2) and (11.4), the equations (11.3) and (11.5) determine w.

The quantity

$$H = \frac{p}{\rho} + \Omega + \frac{1}{2}(u^2 + v^2 + w^2)$$

is determined by the equation (1.3), which is equivalent to the system

$H_x = -\psi_{yt} + \psi_x \cdot \nabla_1^2\psi + \nu\,(\nabla_1^2\psi)_y + ww_x,$

$H_y = \psi_{xt} + \psi_y \cdot \nabla_1^2\psi - \nu\,(\nabla_1^2\psi)_x + ww_y,$

$H_z = -w_t + \nu\,\nabla_1^2 w + \psi_x w_y - \psi_y w_x = c(t),$ by (11.3) and (11.5).

The first three terms in the expressions of H_x and H_y are the same as those of the expressions H_x, H_y in (2.6). Hence if H_1 is the corresponding value of H in the case of pure plane motion we obtain the relations

$$H_x = H_{1x} + ww_x,$$
$$H_y = H_{1y} + ww_y,$$
$$H_z = c(t).$$
$$\therefore \quad H = H_1 + \frac{1}{2} w^2 + c(t)\, z;$$

and

$$\frac{p}{\rho} + \Omega = c(t)\, z + \varphi(x, y, t), \tag{11.6}$$

where $\varphi(x, y, t) = H_1 - \frac{1}{2}(u^2 + v^2)$ is not anything else than $\frac{p}{\rho} + \Omega$ in the case of pure plane motion.

In cylindrical coordinates the velocity components of a pseudo-plane motion of the second kind are of the form

$$v_1 = v_1(r, \theta, t), \quad v_2 = v_2(r, \theta, t), \quad v_3 = v_3(r, \theta, t),$$

i.e. they do not depend on z.

The continuity equation is

$$\frac{\partial(rv_1)}{\partial r} + \frac{\partial v_2}{\partial \theta} = 0.$$

Hence there is a stream function $\psi(r,\theta,t)$ such that

$$v_1 = \frac{1}{r}\psi_\theta, \qquad v_2 = -\psi_r, \tag{11.7}$$

and ψ is determined except for an arbitrary additive function of t.

The vorticity components are

$$\zeta_1 = \frac{1}{r}\frac{\partial u_3}{\partial\theta}, \quad \zeta_2 = -\frac{\partial u_3}{\partial r}, \quad \zeta_3 = -\nabla_1^2\psi,$$

where $$\nabla_1^2\psi = \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{r^2}\psi\theta\theta.$$

Ss. condition (1.8) gives two equations:

$$\frac{D(\psi, \nabla_1^2\psi)}{D(r,\theta)} = 0, \tag{11.8}$$

$$\frac{D(\psi, v_3)}{D(r,\theta)} = 0. \tag{11.9}$$

The consistency equation (1.9) also gives two equations:

$$\nu\nabla_1^2(\nabla_1^2\psi) - (\nabla_1^2\psi)_t = 0, \tag{11.10}$$

$$\nu\nabla_1^2 v_3 - v_{3t} = c_1(t), \tag{11.11}$$

where $c_1(t)$ is arbitrary. And finally

$$\frac{p}{\rho} + \Omega = c_1(t)z + \varphi_1(r, \theta, t), \tag{11.12}$$

where φ_1 is the value of $\frac{p}{\rho} + \Omega$ in the case of pure plane motion.

Prof. Ratip Berker says [2] that the motions of this class seems to be the result of a superposition of

1°) a linear flow in straight lines parallel to OZ, i.e. $(0,0,w)$ on

2°) a pure plane motion in planes parallel to XOY, i.e. $(u,v,0)$.

R. Berker's equations for pseudo-plane motions of the second kind (*) are [See the equations (32.1) and (32.2) of the same memoir]

$$\nu\Delta_4 + \frac{D(\psi, \Delta_2\psi)}{D(x,y)} - (\Delta_2\psi)'t = 0, \tag{A}$$

$$\nu\Delta_2 w + \frac{D(\psi, w)}{D(x,y)} - w'_t = K(t), \tag{B}$$

(*) R. Berker has not considered the as. property. Therefore his equations (A) and (B) are more general than ours.

where Δ_2 and Δ_4 are the operators

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Delta_4 = \Delta_2 \cdot \Delta_2.$$

The equation (A) shows that ψ is the stream function of a pure plane motion in plane parallel to XOY plane. In order that $(0,0,w)$ may represent an actual motion it must satisfy the equations of motion, which requires

$$\nu \Delta_2 w - w'_t = K(t), \tag{C}$$

and after that the as. condition

$$\frac{D(\psi, w)}{D(x,y)} = 0 \tag{D}$$

is satisfied automatically. Hence in order that R. Berker's guess may be true the equation (B) must be separable into (C) and (D).

As for our own equations, (11.2) and (11.4) show that ψ is the stream function of a ss. plane motion in planes parallel to XOY plane. (11.5) indicates that $(0,0,w)$ satisfies the equation of motion, it is the same as (C). And lastly (11.3) shows that the motion $(0,0,w)$ is superposable on $(u,v,0)$ the plane motion of which the stream function is ψ. (11.3) is just the equation (D) above; it is the ss. condition.

The equations (11.2) - (11.5) show that if $c(t) = 0$ equations (11.3) and (11.5) are satisfied by

$$w = k \cdot \nabla_1^2 \psi, \tag{11.13}$$

where k is a contant. If $c(t) \neq 0$, then the solution takes the form

$$w = k \cdot \nabla_1^2 \psi + l(t), \tag{11.14}$$

where $l'(t) = -c(t)$, and since $c(t)$ is arbitrary $l(t)$ is also arbitrary. Nevertheless we can find other expressions for w, by solving tha equations (11.3) and (11.5) directly.

Solution: In order to find ss. pseudo-plane motions of the second kind we have to solve the system of equations (11.2) — (11.5). As we have stated before ψ is determined from the equations (11.2) and (11.4); it is the stream function of a ss. plane motion. After ψ is determined the equations (11.3) and (11.5) determine w. If both ψ and w are independent of time the resulting motion is steady; otherwise it is non-steady.

In chapter I. we have shown all ss. plane motions steady or not. By using them we can construct a ss. pseude-plane motion of the second kind corresponding to each one. We shall give some examples;

1°) Let $\nabla_1^2\psi = K$, where K is an absolute constant.

The equations (11.2) and(11.4)are satisfied whether ψ is independent of t or not. If one of the variables x and y is absent in the expression of ψ, the equation (11.3) shows that the same variable is absent in the expression of w. For example if $\psi_y=0$, then $w_y = 0$, and (11.5) furnishes

$$\nu\, w_{xx} - w_t = c(t).$$

A particular solution of this equation is

$$w = e^{-\nu\lambda^2 t} a \cos(\lambda x + \varepsilon) + b(t),$$

or a more general solution is

$$w = \sum_\lambda e^{-\nu\lambda^2 t} a_\lambda \cos(\lambda x + \varepsilon_\lambda) + b(t),$$

where λ, a_λ, ε_λ are arbitrary parameters and $b'(t) = -c(t)$.

Since $\psi_y = 0$, $\nabla_1^2\psi = \psi_{xx} = K$,

$$\therefore\ \psi = \frac{1}{2} K x^2 + l(t)\, x + m(t) \qquad (11.15)$$

where $l(t)$ and $m(t)$ are arbitrary. Hence the velocity components are

$$\left.\begin{aligned} u &= 0, \\ v &= -Kx - l(t), \\ w &= \sum_\lambda a_\lambda \cos(\lambda x + \varepsilon_\lambda) e^{-\nu\lambda^2 t} + b(t). \end{aligned}\right\} \qquad (11.16)$$

The pressure equation (11.6) becomes

$$\frac{p}{\rho} + \Omega = -b'(t)z + l'(t)y.$$

If both $\psi_x \neq 0$ and $\psi_y \neq 0$ the equation (11.3) is satisfied by

$$w = f(\psi), \qquad (11.17)$$

where $f(\psi)$ is arbitrary. Then (11.5) furnishes

$$\nu f''(\psi_x^2 + \psi_y^2) + f'(\nu K - \psi_t) = c(t). \qquad (11.18)$$

This is integrable if

$$\psi_x^2 + \psi_y^2 = g(\psi)$$

and
$$\psi_t = \nu h\,(\psi),$$

and this implies that $c(t)$ is a constant, say νc.

Now we have the simultaneous equations

$$\psi_x^2 + \psi_y^2 = g(\psi),$$
$$\nabla_1^2\psi = K,$$
$$\psi_t = \nu h\,(\psi)$$

to determine ψ. After ψ is determined (11.17) and (11.18) give w.

Let $\psi = \psi\,(s,t)$, where $s = s\,(x,y)$. Then

$$\psi_x^2 + \psi_y^2 = \psi_s^2(s_x^2 + s_y^2) = g\,(\psi),$$

$$\nabla_1^2\psi = \psi_{ss}(s_x^2 + s_y^2) + \psi_s \cdot \nabla_1^2 s = K.$$

The last equation is satisfied if

$$\nabla_1^2 s = 0,$$
$$s_x^2 + s_y^2 = G(s).$$

Kampé de Fériet (6,) has shown that $G(s)$ is of the form

$$G(s) = s_x^2 + s_y^2 = a^2 e^{ks}.$$

where a, k are arbitrary constants.

Now a can not be zero, since this requires $s_x = s_y = 0$. If $a \neq 0$, $k = 0$, then $s_x^2 + s_y^2 = a^2$ shows that s is a linear function of x and y. But by a suitable transformation we can always express the solution in the form

$$s = ax.$$

Hence ψ is a function of x and t, and we find the solution (11.15) if $a = 1$.

Now
$$g(\psi) = \psi_x^2 + \psi_y^2 = (Kx + l)^2$$
$$= 2K\psi \text{ if } m(t) = l^2/2K, \tag{11.19}$$

and
$$\nu h(\psi) = \psi_t = xl' + \frac{l\,l'}{K} = \frac{l'}{K}(2K\psi)^{1/2},$$

which shows that $l' = a$ constant, say $\nu\lambda$ i.e. $l(t) = \nu\lambda t$. Thus (11.18) becomes

$$2K^2\psi \cdot f'' + [K^2 - \lambda(2K\psi)^{1/2}]\,f' = Kc.$$

The solution of this equation is

$$f = w = -\frac{c}{\lambda}(2K\psi)^{1/2} + \frac{nK^2}{\lambda} e^{\frac{\lambda}{K^2}(2K\psi)^{1/2}} + n_1,$$

or by putting the value of $2K\psi$ from (11.19) we find

$$W = -\frac{c}{\lambda}(Kx + \nu\lambda t) + \frac{nK^2}{\lambda} e^{\frac{\lambda}{K^2}(Kx+\nu\lambda t)} + n_1, \quad (11.20)$$

where n, n_1 arbitrary constants. Hence the velocity components are

$$\left. \begin{aligned} u &= 0, \\ v &= -Kx - \nu\lambda t, \\ W &= -\frac{c}{\lambda}(Kx + \nu\lambda t) + \frac{nK^2}{\lambda} e^{\frac{\lambda}{K^2}(Kx+\nu\lambda t)} + n_1. \end{aligned} \right\} \quad (11.12)$$

If $a \neq 0$, $k \neq 0$ in the equation

$$s_x^2 + s_y^2 = a^2 e^{ks},$$

we find (See 6)

$$s = -\frac{2}{k}\log r - \frac{1}{k}\log\frac{|ak|}{2},$$

where $r^2 = x^2 + y^2$. In this case the equation

$$\psi_{ss}(s_x^2 + s_y^2) + \psi_s \cdot \nabla_1^2 s = K$$

becomes

$$a^2 e^{ks} \cdot \psi_{ss} = K$$

$$\therefore \psi = \frac{K}{k^2 a^2} e^{-ks} + c_1(t)\, s + c_2(t),$$

or
$$\psi = \frac{1}{4}Kr^2 - \frac{2c_1(t)}{k}\log\frac{|ak|}{2} r + c_2(t). \quad (11.22)$$

The equation (11.18) furnishes

$$\nu f'' \cdot \psi_r^2 + (\nu K - \psi_t) f' = c(t). \quad (11.23)$$

It is integrable if

$$\psi_r^2 = g(\psi),$$
$$\psi_t = \nu h(\psi),$$

and
$$c(t) = \nu c.$$

From (11.22) we find

$$\psi_r^2 = \frac{1}{4} K^2 r^2 - \frac{2Kc_1}{k} + \frac{4c_1^2}{k^2}\frac{1}{r^2},$$
$$= K\psi \text{ if } c_1(t) = c_2(t) = 0.$$

Then $\psi_t = 0$, i.e. $h(\psi) = 0$. Hence (11,23) becomes

$$K\psi \cdot f'' + Kf' = c,$$

$$\therefore \quad \psi f' = \frac{c}{K}\psi + m,$$

$$\therefore \quad f = \frac{c}{K}\psi + m \log \psi + m,$$

or inserting the value of $\psi = \frac{1}{4} Kr^2$ we obtain

$$w = f(\psi) = \frac{c}{4} r^2 + 2m \log r + m \log \frac{K}{4} + n, \qquad (11.24)$$

where m, n are arbitrary constants.

Now the velocity components cylindrical coordinates are

$$\left. \begin{aligned} v_1 &= 0, \\ v_2 &= -\frac{1}{2} Kr, \\ v_3 &= \frac{1}{4} cr^2 + 2m \log r + m \log \frac{K}{4} + n. \end{aligned} \right\} \qquad (11.25)$$

And finally

$$\frac{p}{\rho} + \Omega = \frac{1}{5} K^2 r^2 + cz,$$

where $\frac{1}{8} K^2 r^2$ is the same as the value of $\frac{p}{\rho} + \Omega$ in the case of pure plane motion, i.e. when $v_3 = 0$.

The solution (11.25) represents a steady ss. motion of an incompressible viscous fluid in a finite region which excludes the z-axis. If the region includes the z-axis we must take $m = 0$ in the solution.

2°) Let ψ be the function (3.3):

$$\psi = Ay^3 + By^2 + Cy, \qquad (3.3)$$

where A, B, C are constants.

(11.3) shows that w is independent of x. If $w_t = 0$, then 11.5 becomes

$$\nu w_{yy} = c, \quad \text{where } c \text{ is constant}$$

This gives $$w(y) = A_1 y^2 + B_1 y + C_1,$$

where A_1, B_1, C_1 are arbitrary constants, such that $2A_1\nu = c$. The velocity components and the pressure equation are respectively

$$\left.\begin{array}{l} u = 3Ay^2 + 2By + C, \\ v = 0, \\ w = A_1 y^2 + B_1 y + C_1; \end{array}\right\} \qquad (11.26)$$

and

$$\frac{p}{\rho} + \Omega = 6A\nu x + cz.$$

If $w_t \neq 0$, the equation (11.5) becomes

$$\nu w_{yy} - w_t = c(t).$$

A solution of this equation is

$$w(y,t) = V(y,t) + b(t),$$

where $b'(t) = -c(t)$, and V is any solution of the homogeneous equation $\nu w_{yy} - w_t = 0$. Two different values of V were given by (4.3) and (4.4) formerly. Hence we have the solution

$$\left.\begin{array}{l} u = 3Ay^2 + 2By + C, \\ v = 0, \\ w = V(y,t) + b(t); \end{array}\right\} \qquad (11.\ 7)$$

and $$\frac{p}{\rho} + \Omega = 6A\nu x - b'(t)z.$$

The stream lines of the motions represented by (11.26) and (11.27) are parallel straight lines in planes parallel to XOZ. Their slope w/u is a function of y. These are in fact pseudo-plane motions of the *first kind*, since the stream lines are plane curves.

3°) Let ψ be the function (3.4):

$$\psi = Ar^2 \log r + B \log r + Cr^2, \qquad (3.4)$$

where A, B, C are arbitrary constants.

(11.9) shows that v_3 is independent of θ. If $v_{3t} = 0$, then the equation (11.11) becomes

$$\nu\left(v_{3rr} + \frac{1}{r} v_{3r}\right) = c_1,$$

where c_1 is a constant. This gives

$$v_3 = A_1 r^2 + B_1 \log r + C_1,$$

where A_1, B_1, C_1 are arbitrary constants, such that $4A_1\nu = c_1$. Hence the velocity components and the pressure equation in cylindrical coordinates are respectively

$$\left.\begin{aligned} v_1 &= 0, \\ v_2 &= -2Ar \log r - \frac{B}{r} - (A + 2C)r, \\ v_3 &= A_1 r^2 + B_1 \log r + C_1. \end{aligned}\right\} \tag{11 28}$$

If $v_{3t} \neq 0$, then the equation (11.11) becomes

$$\nu\left(v_{3rr} + \frac{1}{r} v_{3r}\right) - v_{3t} = c_1(t).$$

A solution of this equation is

$$v_3 = V(r,t) + b(t),$$

where $b'(t) = -c_1(t)$, and V is any solution of the homogeneous equation $\nu(v_{3rr} + \frac{1}{r} v_{3r}) - v_{3t} = 0$. Two different values of V were given by (4.6) and (4.7) formerly. Hence we have the solution

$$\left.\begin{aligned} v_1 &= 0, \\ v_2 &= -2Ar \log r - \frac{B}{r} - (A + 2C)r, \\ v_3 &= V(r,t) + b(t); \end{aligned}\right\} \tag{11.29}$$

and

$$\frac{p}{\rho} + \Omega = \varphi_1(r,\theta) - b'(t)\, z,$$

where φ is the value of $\frac{p}{\rho} + \Omega$ in the case of pure plane motion i.e. when $v_3 = 0$.

Then stream lines of the motions represented by (11.28) and (11.29) are spirals traced on coaxial circular cylinders with their axes coinciding with OZ. The solutions (11.26) and (11.28) are contained among R. Berker's solutions [2, p. 84].

4°) Let ψ be the function (4.2):

$$\psi(y,t) = V(y,t) + \alpha y^3 + \beta y^2 + \gamma y, \tag{4.2}$$

where α, β are arbitrary constants, γ is an arbitrary function of t, and $V(r,t)$ is any solution of

$$\nu V_{yy} - V_t = 0.$$

Now taking the value of V from (4.3) and using the form (11.13) we find

$$w = K \cdot \nabla_1^2 \psi = K(\nabla_1^2 V + 6\alpha y + 2\beta)$$
$$= -K \cdot \sum_k k^2 A_k \cos(ky + \varepsilon_k) e^{-\nu k^2 t} + 2K(3\alpha y + \beta),$$

where K is an arbitrary constant. Hence we have the solution

$$\left.\begin{aligned} u &= -\sum_k kA_k \sin(ky+\varepsilon_k)e^{-\nu k^2 t} + 3\alpha y^2 + 2\beta y + \gamma(t), \\ v &= 0, \\ w &= -K \cdot \sum_k k^2 A_k \cos(ky+\varepsilon_k)e^{-\nu k^2 t} + 2K(3\alpha y + \beta); \end{aligned}\right\} (11.30)$$

and

$$\frac{p}{\rho} + \Omega = (\gamma' - 6\alpha\nu)\, x + C(t),$$

where $C(t)$ is arbitrary. The stream lines of the motion (11.30) are parallel straight lines in planes parallel to XOZ. Their slope w/u is a functon of y and t. Hence this is also a pseudo-plane of the *first kind* since the stream lines are plane curves.

5°) Let ψ be the function (4.5):

$$\psi(r,t) = V(r,t) + (\alpha r^2 + \beta) \log r + \gamma r^2, \qquad (4.5)$$

where α, γ are constants. β is an arbitrary function of time, and $V(r,t)$ is any solution of

$$\nu(\psi_{rr} + \frac{1}{r}\psi_r) - \psi_t = 0.$$

Taking V from (4.6) and using the form (11.14), i.e.

$$v_3 = w = K \cdot \nabla_1^2 \psi + l(t),$$

we obtain the solution

$$\left.\begin{aligned} v_1 &= 0, \\ v_2 &= \sum_k k[A_k J_1(kr) + B_k Y_1(kr)]\, e^{-\nu k^2 t} - 2\alpha r \log r - \frac{\beta}{r} - (\alpha + 2\gamma) r \\ v_3 &= -K \cdot \sum_k k^2 [A_k J_0(kr) + B_k Y_0(kr)] e^{-\nu k^2 t} + 4\alpha K \log r + l(t), \end{aligned}\right\} (11.31)$$

and $$\frac{p}{\rho} + \Omega = \varphi_1(r,\theta,t) - l'(t)z;$$

where J_n, Y_n are Bessel's functions of the first and second kind respectively of order n, $l'(t) = -c_1(t)$ and φ_1 is the value of $p/\rho + \Omega$ in the case of pure plane motion. If in (11 31) we take $\alpha=\gamma=0$ we obtain R. Berker's solution (32.9) as a particular case (2).

The stream line of the motion (11.31) are spirals traced on coaxial circular cylinders having their axes coincident with OZ.

6°) Let ψ be the solution (4.10):

$$\psi(x\ y,t) = e^{-\nu\lambda^2 t}\cdot\sum_m [A_m \cos(\lambda x+\varepsilon_m) + B_m \cos(\lambda y + \delta_m)]. \quad (4.10)$$

$$\therefore\quad \nabla_1^2\psi = -\lambda^2\psi.$$

Now using the form (11.13), we find

$$w = K\cdot\nabla_1^2\psi = -K\lambda^2\psi,$$
$$= -K\lambda^2 e^{-\nu\lambda^2 t}\cdot\sum_m [A_m \cos(\lambda x + \varepsilon_m) + B_m \cos(\lambda y + \delta_m)].$$

Hence we obtain the motion of which the velocity components are

$$\left.\begin{aligned} u &= -\lambda e^{-\nu\lambda^2 t}\cdot\sum_m B_m \sin(\lambda y + \delta_m), \\ v &= \lambda e^{-\nu\lambda^2 t}\cdot\sum_m A_m \sin(\lambda x + \varepsilon_m), \\ w &= -K\lambda^2 e^{-\nu\lambda^2 t}\cdot\sum_m [A_m \cos(\lambda x+\varepsilon_m) + B_m \cos(\lambda y+\delta_m)]. \end{aligned}\right\} (11.32)$$

The vorticity components become

$$\xi = -K\lambda^2 u, \quad \eta = -K\lambda^2 v, \quad \zeta = -\frac{1}{K} w.$$

Hence if $K = -1/\lambda$ the solution (11.32) satisfies the relation

$$\nabla \times U = \lambda U,$$

where U is the velocity vector. This states that at every point of the fluid the vorticity vector has the same direction as the velocity vector and is proportional to it, i.e. equal vorticity lines are coincident with the stream lines. Motions of this type were first studied by Beltrami (1889), then by M. Caldonazzo (1926), V. Trkal (1926), and R. Ballabh (1,1940) gave general expressions for u, v, w satisfying the relation $\nabla \times U = \lambda U$.

Finally, the function $H = \frac{p}{\rho} + \Omega + \frac{1}{2}(u^2 + v^2 + w^2)$ for (11.32) is given by

$$H = -\frac{1}{2}\lambda^2\psi^2 ,$$

which is the same as the value of H in thə case of pure plane motion (4.10).

REFERENCES

[1] BALLABH (R.)— Superposable fluid motions. Proceedings Benares Math. Soc. II, New series, p. 78, 1940.

[2] BERKER (R.) — Sur quelques cas d'integration des équations du mouvement d'un fluide visqueux incompressible. Pris-Lille, 1936.

[3] CRUDELI (U.)— Sui moti di un liquido viscoso (omogeneo) simmetrici rispetto ad un asse. Rend. Acc. Lincei, 6° s., t, 5, pp. 500-504, 1927.

[4] CRUDELI (U.) — Una nuova categoria di moti stazionari dei liquidi pesanti viscosi eniro tobi cilindrici (rotondi) verticali. Rend. Acc. Lincei, 6° s., t. 5, pp. 783-789, 1927.

[5] CRUDELI (U.) — Sopra una catagoria di moti stazionari dei liquidi pesanti viscosi entro tubi cilindrici (rotondi) verticali. Rend. Acc. Lincei, 6° s., t. 6, pp. 397-401, 1927.

[6] KAMPE DE FERIET (J.)— Sur quelques cas d'integration des équations du mouvement plan d'un fluide visqueux incompressible. Annales de la soc. Sc. de Bruxelles, t. 50 A pp. 77-80, 1930.

[7] KAMPE DE FERIET (J.)— Détermination des mouvements plans d'un fluide visqueux incompressible où la tourbillon est constant le long des lignes de courant. Comptes Rendus du 9° Congrès Int. des Math., Zurich, 1932, t. 2, pp. 298-299.

[8] LAMB (H.)— Hydrodynamics 6° ed., Cambridge, 1932.

[9] STRANG (J. A.)— Superposable fluid motions. Comm. de la Fac. des Sc. de l'Université d'Ankara, t. 1, pp. 1 - 32, 1948.

[10] ERGUN (A. N.)— Some cases of superposable fluid motions. Comm. de la Fac. des Sc. de l'Université d'Ankara, t. 2, pp. 48-88, 1949.

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