Two dimensional wave motion in a compressible rotating fluid bounded internally by a radially oscillating circular cylinder

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Özet : T. V. Davies son neşriyatından birisinde (1) temel irrotasyonel hareket üzerine küçük bir pertürbasyon hareketi bindirmek suretiyle sıkıştırılabilen mütehavvil akışkan hareketine ait malûm lineerleştirme teorisinin değişik bir şeklini göstermiştir. Şimdiki yazı işte bu yeni teoriye dair bir örnek çözüm olup dahilen radial olarak titreşmekte olan dairesel bir silindir ile sınırlı bulunan ve bu silindir etrafında düzgün olarak dönen sıkıştırılabilen bir akışkan içinde meydana gelen dalga hareketini incelemektedir.

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Summary : In a recent paper Davies (1) has presented an alternative theory to the usual linearised theory of unsteady compressible flow by superimposing a small perturbation upon the correct basic irrotational flow pattern. The present paper is concerned with a particular example of this theory in which a rotating compressible fluid is bounded internally by a radialy oscillating circular cylinder.

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1. Introduction.

In a recent paper T. V. Davies (1) has attempted to improve the linear perturbation theory for unsteady motion of a compressible fluid in two dimensions, by substituting for the linear basic flow the correct irrotational flow pattern.

He has considered the flow to be composed of a steady basic flow, assumed to be compressible but irrotational, together with an unsteady perturbation flow which is small compared with the basic flow.

We shall first of all state the results he has obtained very briefly, then pass on to the particular problem under consideration. If dashes are used to represent the general expressions for velocity components, pressure and density, the equations of motion may be written

$$\frac{\partial u'}{\partial t} - v'\,\xi' = -\frac{\partial \chi'}{\partial x}, \quad \frac{\partial v'}{\partial t} + u'\,\xi' = -\frac{\partial \chi'}{\partial y} \tag{1.1}$$

where

$$\chi' = \int \frac{dp'}{\rho'} + \frac{1}{2}(u'^2 + v'^2) + \Omega, \tag{1.2}$$

and the equation of continuity is

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho'\,u') + \frac{\partial}{\partial y}(\rho'\,v') = 0. \tag{1.3}$$

Now, denoting the basic flow by capitals, we assume that

$$u' = U + u, \quad v' = V + v, \quad \rho' = P + \rho, \quad p' = \Pi + p, \quad \chi' = \chi_0 + \chi_1. \tag{1.4}$$

Since the basic flow is irrotational and steady it follows that χ_0 is an absolute constant,

$$\int \frac{d\Pi}{P} + \frac{1}{2}(U^2 + V^2) + \Omega = \chi_0 = \text{const.}, \tag{1.5}$$

and two functions φ and ψ may be introduced of the nature of velocity potential and stream functions, such that

$$U = \frac{\partial \varphi}{\partial x}, \quad V = \frac{\partial \varphi}{\partial y}, \quad PU = P_0 \frac{\partial \psi}{\partial y}, \quad PV = -P_0 \frac{\partial \psi}{\partial x}, \tag{1.6}$$

where P_0 will represent the density at a stagnation point.

Substituting the expressions (1.4) into the general equations of motion (1.1) and using the results previously stated we obtain the equations governing the perturbation flow,

$$\frac{\partial u}{\partial t} - V\zeta = -\frac{\partial \chi_1}{\partial x}, \quad \frac{\partial v}{\partial t} + U\zeta = -\frac{\partial \chi_1}{\partial y}, \tag{1.7}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

and the unsteady motion is assumed to be sufficiently small to neglect the products ζu and ζv.

In place of the components (u, v) we now introduce the perturbation velocity components (u_1, v_1), the former being along the direction of the main velocity $c = (U^2 + V^2)^{\frac{1}{2}}$, the latter perpendicular to c in the direction θ increasing, and also we introduce new independent variables $(\varphi. \psi)$ in place of (x, y), the transformation being effected by the relations (1.6).

Davies has expressed u_1, v_1, χ_1, p and ρ in terms of a single function α, which is a function of φ, ψ and t. Thus

$$u_1 = c\frac{\partial\alpha}{\partial\varphi}, \tag{1.8}$$

$$\chi_1 = -\frac{\partial\alpha}{\partial t}, \tag{1.9}$$

$$v_1 = \frac{P c}{P_0}\left(\frac{\partial\alpha}{\partial\psi} + \beta\right), \tag{1.10}$$

$$p = -P\left(\frac{\partial\alpha}{\partial t} + c^2\frac{\partial\alpha}{\partial\varphi}\right), \tag{1.11}$$

$$\rho = -\frac{P}{a^2}\left(\frac{\partial\alpha}{\partial t} + c^2\frac{\partial\alpha}{\partial\varphi}\right), \tag{1.12}$$

where β is any solution of the differentiol equation

$$\frac{\partial\beta}{\partial t} + c^2\frac{\partial\beta}{\partial\varphi} = 0, \tag{1.13}$$

and a is the local velocity of sound. Assuming that the adiabatic relation holds good in the basic flow, i.e.

$$\Pi = \varkappa P^{\gamma}, \tag{1.14}$$

we have

$$a^2 = \frac{d\Pi}{dP} = \varkappa\,\gamma\,P^{\gamma-1}, \tag{1 15}$$

It is necessary to write down the following relations for future reference, which can be found in any text book on compressible fluids :

$$a_0^2 = \varkappa\,\gamma\,P_0^{\gamma-1}, \tag{1.16}$$

$$c_m^2 = \frac{2}{\gamma - 1} a_0^2 \,(= 5\, a_0^2 \text{ for air}), \tag{1.17}$$

$$\frac{c^2}{c_m^2} = \tau, \tag{1.18}$$

$$\frac{a^2}{a_0^2} = 1 - \tau = \left(\frac{P}{P_0}\right)^{\gamma - 1} \text{ by (1.15) and (1.16)}, \tag{1.19}$$

where a_0 is the velocity of sound at a stagnation point. c_m is the maximum fluid velocity which can be attained when $P = 0$, and τ is a non-dimensional variable lying in the range $0 \leqq \tau \leqq 1$.

After the transformation to the new coordinates φ and ψ, ζ becomes

$$\zeta = c \left\{ u_1 \frac{\partial \theta}{\partial \varphi} + \frac{\partial v_1}{\partial \varphi} - \frac{P}{P_0} \frac{\partial u_1}{\partial \psi} + \frac{P v_1}{P_0} \frac{\partial \theta}{\partial \psi} \right\} = \frac{P c^2}{P_0} \frac{\partial \beta}{\partial \varphi}, \tag{1.20}$$

and the equation of continuity is

$$\frac{\partial}{\partial t}\left(\frac{\rho}{c^2 P}\right) + \frac{\partial}{\partial \varphi}\left(\frac{u_1}{c}\right) + \frac{1}{P_0} \frac{\partial}{\partial \psi}\left(\frac{v_1 P}{c}\right) + \frac{\partial}{\partial \varphi}\left(\frac{\rho}{P}\right) = 0. \tag{1.21}$$

After substituting for u_1, v_1, ρ their expressions in terms of α this becomes

$$- \frac{1}{a^2 c^2} \frac{\partial^2 \alpha}{\partial t^2} - \frac{2}{a^2} \frac{\partial^2 \alpha}{\partial t\, \partial \varphi} - \frac{\partial \alpha}{\partial t} \frac{\partial}{\partial \varphi}\left(\frac{1}{a^2}\right) + \frac{\partial}{\partial \varphi}\left\{\left(1 - \frac{c^2}{a^2}\right) \frac{\partial \alpha}{\partial \varphi}\right\}$$
$$+ \frac{\partial}{\partial \psi}\left\{ \frac{P^2}{P_0^2} \left(\frac{\partial \alpha}{\partial \psi} + \beta\right)\right\} = 0, \tag{1.22}$$

where β is a solution of the equation (1.13)

2. Special case when the basic flow is circular and bounded internally by an oscillating cylinder.

In plane polar coordinates the radial velocity $\varphi_r = 0$, and the transverse velocity $\frac{1}{r} \varphi_\theta = c$. Now if we assume

$$\varphi = R\, c_m\, \theta, \tag{2.1}$$

we have

$$c = \frac{R\, c_m}{r}, \tag{2.2}$$

where R is a constant of dimensions length. When $r = R$, c attains its maximum value, i e. $c = c_m$; herce R is a limiting radius for the motion.

The reciprocals of the hodograph or Chapligin equations (2) may be written as

$$\frac{\partial}{\partial\varphi}(Pc) + \frac{P^2 c}{P_0}\frac{\partial\theta}{\partial\psi} = 0, \quad P\frac{\partial c}{\partial\psi} - P_0 c\frac{\partial\theta}{\partial\varphi} = 0. \tag{2.3}$$

Since $\frac{\partial\theta}{\partial\psi} = 0$, we have $\frac{\partial}{\partial\varphi}(Pc) = 0$, i. e. Pc is a function of ψ only, and the relation between these quantities is

$$P\frac{\partial c}{\partial\psi} = \frac{P_0 c}{R\, c_m}, \tag{2.4}$$

which defines c in terms of ψ.

From (2.1) and (2.4) we obtain the operational relations

$$\frac{\partial}{\partial\varphi} = \frac{1}{R\, c_m}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial\psi} = \frac{c P_0}{P R\, c_m}\frac{\partial}{\partial c}. \tag{2.5}$$

Substituting these in the general differential equation (1.22) we obtain

$$-\frac{1}{a^2 c^2}\frac{\partial^2\alpha}{\partial t^2} - \frac{2}{a^2 R\, c_m}\frac{\partial^2\alpha}{\partial t\,\partial\theta} - \frac{1}{R\, c_m}\frac{\partial\alpha}{\partial t}\frac{\partial}{\partial\theta}\left(\frac{1}{a^2}\right)$$
$$+\frac{1}{R^2 c_m^2}\frac{\partial}{\partial\theta}\left\{\left(1-\frac{c^2}{a^2}\right)\frac{\partial\alpha}{\partial\theta}\right\} + \frac{P_0 c}{R P c_m}\frac{\partial}{\partial c}\left\{\frac{P^2}{P_0^2}\left(\frac{P_0 c}{P R\, c_m}\frac{\partial\alpha}{\partial c} + \beta\right)\right\} = 0. \tag{2.6}$$

If we assume that the boundary of the cylinder is oscillating radially with frequency n, the motion at the boundary will be

$$r = r_0 + \delta_0 \cos(nt + \varepsilon), \quad (r_0 > R) \tag{2.7}$$

where r_0 is the mean radius of the cylinder and ε is an arbitrary constant. Hereafter we shall be concerned chiefly with the outward propagation of this disturbance. Suppose therefore that we determine a solution for α of the form.

$$\alpha = \alpha(c) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} nt, \tag{2.8}$$

where $\alpha(c)$ is independent of θ. The function $\alpha(c)$ being independent of θ, (2.6) becomes (a also being independent of θ)

$$\frac{n^2}{a^2 c^2} \alpha(c) \frac{\cos}{\sin} nt + \frac{P_0 c}{PR\, c_m} \frac{\partial}{\partial c} \left\{ \frac{Pc}{P_0 R\, c_m} \frac{d\alpha}{dc} \frac{\cos}{\sin} nt + \frac{P^2 \beta}{P_0^2} \right\} = 0. \quad (2.9)$$

To determine a solution af type (2.8) we must choose β independent of θ in the form

$$\beta = \beta(c) \frac{\cos}{\sin} nt,$$

and from (1.13) it follows that β is identically zero, so that (2.9) becomes

$$\frac{n^2 \alpha}{a^2 c^2} + \frac{P_0 c}{(R\, c_m)^2 P} \frac{\partial}{\partial c} \left\{ \frac{P c}{P_0} \frac{d\alpha}{dc} \right\} = 0,$$

and the unsteady motian developed in the fluid is then irrotational (see 1.20). If we transform to the variable τ defined in (1.18) this equation may be written

$$(1-\tau)\, \tau^2 \frac{d^2 \alpha}{d\tau^2} + \tau \left\{ 1 - (1+\beta_0)\, \tau \right\} \frac{d\alpha}{d\tau} + \frac{\lambda^2 \alpha}{\tau} = 0, \quad (2.10)$$

where $$\beta_0 = \frac{1}{\gamma - 1} = \frac{5}{2} \quad \text{for air,}$$

and $$\lambda = \frac{R\, n}{2\, a_0} \quad (2.11)$$

is a positive quantity.

The unsteady perturbation components for this motion (u_1, v_1) can be derived from (1.8) and (1.10) by using the transformation (2.5) which gives

$$u_1 = c \frac{\partial \alpha}{\partial \varphi} = \frac{c}{R\, c_m} \frac{\partial \alpha}{\partial \theta} = 0, \quad (2.12)$$

so that the perturbation component along the circles is zero, and the radial component is

$$v_1 = \frac{P c}{P_0} \left(\frac{\partial \alpha}{\partial \psi} + \beta \right) = \frac{P c}{P_0} \frac{P_0 c}{PR\, c_m} \frac{\partial \alpha}{\partial c} = \frac{c^2}{R\, c_m} \frac{d\alpha}{dc} \left(= - \frac{d\alpha}{dr} \right) (2.13)$$

α being defined by (2.8).

3. The form of the general solution for α. Characteristic solutions.

Equation (2.10) may be transformed using

$$\tau^{1/2} = \frac{2\lambda}{z}, \tag{3.1}$$

and it then assumes the form

$$(z^2 - 4\lambda^2)\frac{d^2\alpha}{dz^2} + \left\{2 + \frac{4\lambda^2(2\beta_0 - 1)}{z}\right\}\frac{d\alpha}{dz} + z^2\alpha = 0. \tag{3.2}$$

If we write $b = (2\beta_0 - 1)\lambda^2 = 4\lambda^2$ for air, and assume

$$\alpha = \sum_{s=-\infty}^{\infty} A_{2s} J_{\nu+2s}(z), \tag{3.3}$$

we obtain

$$\sum_{-\infty}^{\infty}(\nu+2s)^2 A_{2s} J_{\nu+2s} - \lambda^2 \sum_{-\infty}^{\infty} A_{2s}\left\{J_{\nu+2s-2} + J_{\nu+2s+2} - 2J_{\nu+2s}\right\}$$
$$+ b\sum_{-\infty}^{\infty} A_{2s}\left\{\frac{J_{\nu+2s-2}}{\nu+2s-1} + \frac{2J_{\nu+2s}}{(\nu+2s)^2-1} - \frac{J_{\nu+2s+2}}{\nu+2s+1}\right\} \equiv 0. \; (*)$$

Equating to zero the coefficient of the term in $J_{\nu+2s}$, we obtain a recurrence formula for the A_{2s} coefficients, i.e.

$$\left\{2\lambda^2 + (\nu+2s)^2 + \frac{2b}{(\nu+2s)^2-1}\right\}A_{2s} - \lambda^2(A_{2s-2} + A_{2s+2})$$
$$+ b\left\{\frac{A_{2s+2}}{\nu+2s+1} - \frac{A_{2s-2}}{\nu+2s-1}\right\} = 0,$$

or

$$-\lambda^2 A_{2s-2}\left(1 + \frac{4}{\nu+2s-1}\right) + A_{2s}\left\{2\lambda^2 + (\nu+2s)^2 + \frac{8\lambda^2}{(\nu+2s)^2-1}\right\}$$
$$+ \lambda^2 A_{2s+2}\left(-1 + \frac{4}{\nu+2s+1}\right) = 0, \tag{3.4}$$

where we have now taken $b = 4\lambda^2$. Here we have to assume that ν does not take the values $\pm 1, \pm 3, \pm 5, \cdots$

If we take $s = \cdots, -2, -1, 0, 1, 2, 3, \cdots$ the set of relations (3.4) becomes

(*) The second sign inside the bracket was wrong in the original paper (1) as Mr Davies himself has been pointed out.

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\lambda^2 A_{-6}\left(1+\frac{4}{\nu-5}\right)+A_{-4}\left\{2\lambda^2+(\nu-4)^2+\frac{8\lambda^2}{(\nu-4)^2-1}\right\}+\lambda^2 A_{-2}\left(-1+\frac{4}{\nu-3}\right)=0, \quad \text{(a)}$$

$$-\lambda^2 A_{-4}\left(1+\frac{4}{\nu-3}\right)+A_{-2}\left\{2\lambda^2+(\nu-2)^2+\frac{8\lambda^2}{(\nu-2)^2-1}\right\}+\lambda^2 A_0\left(-1+\frac{4}{\nu-1}\right)=0, \quad \text{(b)}$$

$$-\lambda^2 A_{-2}\left(1+\frac{4}{\nu-1}\right)+A_0\left\{2\lambda_2+\nu^2+\frac{8\lambda^2}{\nu^2-1}\right\}+\lambda^2 A_2\left(-1+\frac{4}{\nu+1}\right)=0, \quad \text{(c)}$$

$$-\lambda^2 A_0\left(1+\frac{4}{\nu+1}\right)+A_2\left\{2\lambda^2+(\nu+2)^2+\frac{8\lambda^2}{(\nu+2)^2-1}\right\}+\lambda^2 A_4\left(-1+\frac{4}{\nu+3}\right)=0, \quad \text{(d)}$$

$$-\lambda^2 A_2\left(1+\frac{4}{\nu+3}\right)+A_4\left\{2\lambda^2+(\nu+4)^2+\frac{8\lambda^2}{(\nu+4)^2-1}\right\}+\lambda^2 A_6\left(-1+\frac{4}{\nu+5}\right)=0, \quad \text{(e)}$$

$$-\lambda^2 A_4\left(1+\frac{4}{\nu+5}\right)+A_6\left\{2\lambda^2+(\nu+6)^2+\frac{8\lambda^2}{(\nu+6)^2-1}\right\}+\lambda^2 A_8\left(-1+\frac{4}{\nu+7}\right)=0, \quad \text{(f)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3.5)$$

This infinite set of relations wil define an infinite determinant in ν whose vanishing will lead to a set of characteristic solutions for ν. Convergence of the infinite series (3.3) will be assured, since for large values of s equation (3.4) becomes

$$-\lambda^2(A_{2s-2}+A_{2s+2})+A_{2s}\left\{2\lambda^2+(\nu+2s)^2\right\}=0,$$

which is similar to that for the Mathieu coefficients, hence we can accept the convergence as a result of Mathieu function theory.

In order to investigate the infinite determinant of the coefficients A_{2s} we can approach the problem as in Hill's determinant. To ensure convergence of the infinite determinant we divide throughout the s th row by the coefficient

$$2\lambda^2 + (\nu + 2s)^2 + \frac{8\lambda^2}{(\nu+2s)^2-1}$$

of A_{2s} in (3.4), so that (3.4) becomes

$$\frac{-\lambda^2\left(1+\dfrac{4}{\nu+2s-1}\right)A_{2s-2}}{2\lambda^2+(\nu+2s)^2+\dfrac{8\lambda^2}{(\nu+2s)^2-1}} + A_{2s} + \frac{\lambda^2\left(-1+\dfrac{4}{\nu+2s+1}\right)A_{2s+2}}{2\lambda^2+(\nu+2s)^2+\dfrac{8\lambda^2}{(\nu+2s)^2-1}} = 0$$

suppose we write

$$\varepsilon_{2s}(\nu) = \frac{-\lambda^2\left(1+\dfrac{4}{\nu+2s-1}\right)}{2\lambda^2+(\nu+2s)^2+\dfrac{8\lambda^2}{(\nu+2s)^2-1}}, \tag{3.6}$$

$$\eta_{2s}(\nu) = \frac{\lambda^2\left(-1+\dfrac{4}{\nu+2s+1}\right)}{2\lambda^2+(\nu+2s)^2+\dfrac{8\lambda^2}{(\nu+2s)^2-1}}, \tag{3.7}$$

then

$$\varepsilon_{2s} A_{2s-2} + A_{2s} + \eta_{2s} A_{2s+2} = 0, \tag{3.8}$$

$$\Delta(\nu) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon_{-6} & 1 & \eta_{-6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{-4} & 1 & \eta_{-4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_{-2} & 1 & \eta_{-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_0 & 1 & \eta_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 & 1 & \eta_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon_4 & 1 & \eta_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon_6 & 1 & \eta_6 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{3.9}$$

Write $-\nu$ for ν, then

$$\left.\begin{aligned} \varepsilon_{-2s}(-\nu) &= \frac{+\lambda^2\left(-1+\dfrac{4}{\nu+2s+1}\right)}{2\lambda^2+(\nu+2s)^2+\dfrac{8\lambda^2}{(\nu+2s)^2-1}} = \eta_{2s}(\nu) \\ \eta_{-2s}(-\nu) &= \frac{\lambda^2\left(-1+\dfrac{4}{-\nu-2s+1}\right)}{2\lambda^2+(\nu+2s)^2+\dfrac{8\lambda^2}{(\nu+2s)^2-1}} = \varepsilon_{2s}(\nu) \end{aligned}\right\} \quad (3.10)$$

and the determinant is unchanged, hence $\Delta(-\nu) = \Delta(\nu)$. That is $\Delta(\nu)$ is an even function of ν. Replacing s by $s-1$ and ν by $\nu+2$,

$$\varepsilon_{2s-2}(\nu+2) = \varepsilon_{2s}(\nu), \qquad \eta_{2s-2}(\nu+2) = \eta_{2s}(\nu),$$

and thus $\Delta(\nu+2) = \Delta(\nu)$. and similarly $\Delta(\nu \mp 2m) = \Delta(\nu)$, hence $\Delta(\nu)$ has period $\nu = 2$.

The singularities of $\Delta(\nu)$ occur at simple poles of the terms ε_{2s}, η_{2s}. We note that $\nu = -2s \mp 1$ are not poles of the function, and from (3.6) and (3.7) we derive the simple poles at the zeros of the equation

$$2\lambda^2 + (\nu+2s)^2 + \frac{8\lambda^2}{(\nu+2s)^2-1} = 0,$$

which leads to the two expressions

$$\left.\begin{aligned} (\nu+2s)^2 &= \frac{1}{2}\left\{\sqrt{(4\lambda^4-28\lambda^2+1)}-(2\lambda^2-1)\right\} = \varkappa_1(\lambda) \\ (\nu+2s)^2 &= -\frac{1}{2}\left\{\sqrt{(4\lambda^4-28\lambda^2+1)}+(2\lambda^2-1)\right\} = \varkappa_2(\lambda) \end{aligned}\right\} \quad (3.11)$$

where $\varkappa_1(\lambda)$ and $\varkappa_2(\lambda)$ are

$$\left.\begin{aligned} &\text{(a) real and positive} && \text{when } 0 < \lambda^2 \leq \tfrac{1}{2}(7-4\sqrt{3}) \\ &\text{(b) complex} && \text{when } \tfrac{1}{2}(7-4\sqrt{3}) < \lambda^2 < \tfrac{1}{2}(7+4\sqrt{3}) \\ &\text{(c) real and negative} && \text{when } \tfrac{1}{2}(7+4\sqrt{3}) \leq \lambda^2 < +\infty . \end{aligned}\right\} \quad (3.12)$$

Hence the simple poles are

$$\left.\begin{array}{l}\nu = -2s \mp \sqrt{\varkappa_1(\lambda)} \\ \nu = -2s \mp \sqrt{\varkappa_2(\lambda)}\end{array}\right\} (s = -\infty, \infty) \tag{3.13}$$

It is easily shown that $\Delta(\nu)$ is an analytic function except for simple poles at the points (3.13). The function defined by

$$\chi(\nu) = A\left\{\cot\frac{1}{2}\pi(\nu-\sqrt{\varkappa_1}) - \cot\frac{1}{2}\pi(\nu+\sqrt{\varkappa_1})\right\}$$
$$+ B\left\{\cot\frac{1}{2}\pi(\nu-\sqrt{\varkappa_2}) - \cot\frac{1}{2}\pi(\nu+\sqrt{\varkappa_2})\right\} \tag{3.14}$$

is also an analytic function except for simple poles at the points (3.13) and it has period $\nu = 2$. It follows that the function

$$f(\nu) = \Delta(\nu) - \chi(\nu) \tag{3.15}$$

will be of period 2. Furthermore, if A and B are so chosen that the singularities at $\nu = \mp\sqrt{\varkappa_1}$, and $\nu = \mp\sqrt{\varkappa_2}$ cancel, then in virtue of the periodicity it follows that all the singular'ties of $f(\nu)$ disappear and $f(\nu)$ will be an analytic function The singularities at $\nu = \mp\sqrt{\varkappa_1}$, and $\nu = \mp\sqrt{\varkappa_2}$ will cancel provided that the residues of $\Delta(\nu)$ and $\chi(\nu)$ at these poles are the same. Since $\Delta(\nu)$ and $\chi(\nu)$ are bounded as ν tends to infinity, it follows that $f(\nu)$ is bounded, and thus by Liouville's Theorem $f(\nu)$ is a constant. To determine this constant make $\nu \to \infty$ in (3.15), then since $\lim_{\nu\to\infty} \Delta(\nu) = 1$ and $\lim_{\nu\to\infty} \chi(\nu) = 0$ it follows that $f(\nu) \equiv 1$. Thus

$$\Delta(\nu) - 1 = \chi(\nu). \tag{3.16}$$

Since this is an identity in ν, A and B may be determined by equating the left and right hand sides at any two conveniet points in the ν-plane. These points are taken to be $\nu = 0$ and $\nu = 1$. With $\nu = 0$ we obtain the equation

$$\Delta(0) - 1 = -2A\cot\frac{1}{2}\pi\sqrt{\varkappa_1} - 2B\cot\frac{1}{2}\pi\sqrt{\varkappa_2}, \tag{3.17}$$

and when $\nu = 1$ the equation

$$\Delta(1) - 1 = 2A\tan\frac{1}{2}\pi\sqrt{\varkappa_1} + 2B\tan\frac{1}{2}\pi\sqrt{\varkappa_2}. \tag{3.18}$$

When $\nu = 1$ it follows from (3.6) and (3.7) that

$$\varepsilon_0 = \eta_{-2} = -1, \quad \varepsilon_{-2} = \varepsilon_{-4} = \eta_0 = \eta_2 = 0,$$

and thus $\Delta(1) = 0$. Equation (3.18) now becomes

$$-1 = 2A \tan \frac{1}{2} \pi \sqrt{\varkappa_1} + 2B \tan \frac{1}{2} \pi \sqrt{\varkappa_2}, \qquad (3.19)$$

and (3.17), (3.19) lead to

$$\left.\begin{aligned} A (\cos \pi \sqrt{\varkappa_2} - \cos \pi \sqrt{\varkappa_1}) &= \frac{1}{2} \sin \pi\sqrt{\varkappa_1} [\Delta(0) \sin^2 \frac{1}{2} \pi\sqrt{\varkappa_2} - 1] \\ B (\cos \pi \sqrt{\varkappa_2} - \cos \pi\sqrt{\varkappa_1}) &= -\frac{1}{2} \sin \pi\sqrt{\varkappa_2} [\Delta(0) \sin^2 \frac{1}{2} \pi\sqrt{\varkappa_1} - 1] \end{aligned}\right\}. \qquad (3.20)$$

The equation $\Delta(\nu) = 0$, which gives the characteristic values, becomes $\chi(\nu) = -1$, and this reduces to

$$\frac{2A \sin \pi\sqrt{\varkappa_1}}{\cos \pi \sqrt{\varkappa_1} - \cos \nu \pi} + \frac{2B \sin \pi\sqrt{\varkappa_2}}{\cos \pi \sqrt{\varkappa_2} - \cos \nu \pi} = -1. \qquad (3.21)$$

By eliminating A and B between the equations (3.20) and (3.21) we obtain an equation of the second order in $\cos \nu \pi$, which after some trigonometric transformations and rearrangement can be put into the factorised form

$$(\cos \nu\pi + 1) [\cos \nu\pi - 1 + \frac{1}{2} \Delta(0) (1 - \cos \pi\sqrt{\varkappa_1} - \cos \pi\sqrt{\varkappa_2} + \cos \pi\sqrt{\varkappa_1} \cos \pi\sqrt{\varkappa_2})] = 0. \qquad (3.22)$$

Hence either

$$\cos \nu\pi = -1$$

$$\therefore \quad \nu = 2m + 1 \quad (m \text{ integer}),$$

or $$\cos \nu\pi = 1 - \frac{1}{2} \Delta(0) (1 - \cos \pi\sqrt{\varkappa_1}) (1 - \cos \pi\sqrt{\varkappa_2}). \qquad (3.23)$$

The first set of values are excluded as solutions of the problem ((see (3.4) et seq.)). In order to find the values of ν satisfying the equation (3.23) we must first of all calculate $\Delta(0)$, $\cos \pi\sqrt{\varkappa_1}$ and $\cos \pi\sqrt{\varkappa_2}$. It is evident that when $\lambda^2 > \frac{1}{2}(7 - 4\sqrt{3})$, $\varkappa_1$ and $\varkappa_2$ are either complex or negative (see 3.12); hence in this case $\cos \nu\pi$ and therefore ν is complex.

Let us first confine ourselves to the case when $\lambda^2 \leq \frac{1}{2}(7 - 4\sqrt{3})$ i.e. when ν is real. λ is positive and small and $0 < \lambda < 0.1894$. Hence we can exkress $\Delta(0)$, $\cos \pi\sqrt{\varkappa_1}$ and $\cos \pi\sqrt{\varkappa_2}$ as convergent series of positive powers of λ. The following formulae are correct to the final power included :

$$\Delta(0) = 1 - \frac{5}{4}\lambda^2 + \left(\frac{37}{8}\pi^2 - \frac{265}{6}\right)\lambda^4 - \cdots, \tag{3.24}$$

$$\cos \pi\sqrt{k_1} = -1 + 8\pi^2\lambda^4 + 128\pi^2\lambda^6 + \left(-\frac{32}{3}\pi^4 + 2368\pi^2\right)\lambda^8 + \left(\frac{1024}{3}\pi^4 + 47744\pi^2\right)\lambda^{10} + \cdots,$$

$$\cos \pi\sqrt{k_2} = 1 - 3\pi^2\lambda^2 + \left(\frac{3}{2}\pi^4 - 24\pi^2\right)\lambda^4 + \left(-\frac{3}{10}\pi^6 + 24\pi^4 - 336\pi^2\right)\lambda^6 + \left(\frac{9}{280}\pi^8 - \frac{36}{5}\pi^6 + 432\pi^4 - 5856\pi^2\right)\lambda^8 + \cdots,$$

Now the equation (3.23) becomes

$$\cos \nu\pi = 1 - 3\pi^2\lambda^2 + \left(\frac{3}{2}\pi^4 - \frac{81}{4}\pi^2\right)\lambda^4 - \left(\frac{3}{10}\pi^6 - \frac{81}{4}\pi^4 + \frac{347}{2}\pi^2\right)\lambda^6 + \cdots. \tag{3.25}$$

The solution of this equation is

$$\nu = \sqrt{6}\,\lambda\left(1 + \frac{27}{8}\lambda^2 + \frac{8917}{384}\lambda^4 + \cdots\right), \tag{3.26}$$

which is very rapidly convergent for the values of λ under consideration. There will be an infinite set of solutions for ν which make $\Delta(\nu)$ vanish, and these will be obtained from (3.26) by the addition or subtraction of an even integer.

In order to check the result (3.25) we return to the equations (3 5) and solve for ν by an alternative method (this method is possible as in Mathieu function theory, for sufficiently small values of λ). We neglect A_{-6} in (3.5a) and solve for A_{-4} in terms of A_{-2}. Then we insert this in(3.5b) and solve for A_{-2} in terms of A_0. We do the same thing with the equations (3.5c) and (3.5d) to express A_2 in term of A_0. Finally we insert A_{-2} and A_2 into the equation (3 5c) and then the coefficient of A_0 gives an approximate expression for the equation $\Delta(\nu) = 0$. This equation is

$$\frac{-\lambda^4\left(1-\dfrac{16}{(\nu-1)^2}\right)}{\dfrac{-\lambda^4\left(1-\dfrac{16}{(\nu-3)^2}\right)}{2\lambda^2+(\nu-4)^2+\dfrac{8\lambda^2}{(\nu-4)^2-1}}+2\lambda^2+(\nu-2)^2+\dfrac{8\lambda^2}{(\nu-4)^2-1}}+2\lambda^2+\nu^2$$

$$+\frac{8\lambda^2}{\nu^2-1}+\frac{-\lambda^4\left(1-\dfrac{16}{(\nu+1)^2}\right)}{\dfrac{-\lambda^4\left(1-\dfrac{16}{(\nu+3)^2}\right)}{2\lambda^2+(\nu+4)^2+\dfrac{8\lambda^2}{(\nu+4)^2-1}}+2\lambda^2+(\nu+2)^2+\dfrac{8\lambda^2}{(\nu+4)^2-1}}=0,$$

and it is easily verified that (3.26) is a solution of this equation. In fact the first three terms of (3.26) satisfies the equation up to the power λ^8.

Using the expression for ν we obtain

$$\left.\begin{aligned}
A_{-4}&=\frac{1}{192}\lambda^4\left(35+\frac{941}{12}\sqrt{6}\lambda+\cdots\right)A_0\,,\\
A_4&=\frac{1}{192}\lambda^4\left(35-\frac{941}{12}\sqrt{6}\lambda+\cdots\right)A_0\,,\\
A_{-2}&=\frac{1}{4}\lambda^2\left(5+9\sqrt{6}\lambda+\frac{194}{3}\lambda^2+\frac{6523}{72}\sqrt{6}\lambda^3+\cdots\right)A_0\,,\\
A_2&=\frac{1}{4}\lambda^2\left(5-9\sqrt{6}\lambda+\frac{194}{3}\lambda^2-\frac{6523}{72}\sqrt{6}\lambda^3+\cdots\right)A_0\,.
\end{aligned}\right\}\qquad(3.27)$$

We see that A_{-2} and A_2 are of order 2 in λ, A_{-4} and A_4 are of order 4, etc., and since ν changes sign when λ does, we have the following relations between the coefficients

$$\left.\begin{aligned}
A_{-2s}(-\nu)&=A_{2s}(\nu)\,,\\
A_{2s}(-\nu)&=A_{-2s}(\nu)
\end{aligned}\right\}\qquad(3.28)$$

If we neglect λ^6 the solution (3.3) reduces to

$$\alpha=\sum_{s=-2}^{2}A_{2s}(\nu)J_{\nu+2s}(z)=A_0\alpha_1$$

say, where A_i are given in terms of λ by (3.27) and by (3.28).

Now, if we change the signs of ν and z in (3.5) the system turns to itself. Hence, considering the relations (3.28), we conclude that

$$\alpha = \sum_{s=-\infty}^{\infty} A_{2s}(\nu) J_{-\nu-2s}(z) \tag{3.29}$$

is also a solution of the differential equation (2.2), and by neglecting λ^6 we obtain a second independent solution

$$\alpha = \sum_{s=-2}^{2} A_{2s}(\nu) J_{-\nu-2s}(z) = A_0 \alpha_2$$

say.

Hence the general solution is

$$\alpha = K_1 \alpha_1 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} nt + K_2 \alpha_2 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} nt \,, \tag{3.30}$$

where K_1 and K_2 are arbitrary constants. Let $\frac{A_{2s}}{A_0} = b_{2s}$, then

$$\left.\begin{aligned} \alpha_1 &= \sum_{s=-2}^{2} b_{2s} J_{\nu+2s}, \\ \alpha_2 &= \sum_{s=-2}^{2} b_{2s} J_{-\nu-2s} \end{aligned}\right\} \tag{3.31}$$

and $b_{2s} = A_{2s}/A_0$ are given by (3.27) in terms of λ.

We now have two arbitrary constants in the general solution (3.30), as expected, since the differential equation (3.2) is of the second order, and we now satisfy the boundary conditions.

The inner conditions near the oscillating cylinder is that the radial velocity of the fluid at the surface of the cylinder is the same as the radial velocity of a point on the cylinder. From (2.7) it follows that

$$\frac{dr}{dt} = -n\delta_0 \sin(nt + \varepsilon).$$

Since

$$v_1 = \frac{c^2}{Rc_m}\frac{d\alpha}{dc} = \frac{2c\tau}{Rc_m}\frac{d\alpha}{d\tau} = -\frac{2\lambda}{R}\frac{d\alpha}{dz}$$

it folows that the inner condition can be expressed by

$$\left(\frac{d\alpha}{dz}\right)_{r=r_0} = a_0\delta_0 \sin(nt+\varepsilon) \tag{3.32}$$

approximately.

Suppose the general solution (3.30) is of the form

$$\alpha = K_1\left(\sum_{-2}^{2} b_{2s} J_{\nu+2s}\right)e^{int} + K_2\left(\sum_{-2}^{2} b_{2s} J_{-\nu-2s}\right)e^{-int}, \tag{3,33}$$

and suppose also that $K_1 = k_1+ik_1'$, $K_2 = k_2+ik_2'$ are arbitrary complex numbers. Then

$$\frac{d\alpha}{dz} = K_1\, e^{int}\cdot\sum_{-2}^{2} b_{2s} J'_{\nu+2s} + K_2\, e^{-int}\cdot\sum_{-2}^{2} b_{2s} J'_{-\nu-2s}$$

$$= \frac{1}{2} K_1 e^{int} \sum_{-2}^{2} b_{2s}(J_{\nu+2s-1} - J_{\nu+2s+1})$$

$$+ \frac{1}{2} K_2\, e^{-int} \sum_{-2}^{2} b_{2s}(J_{-\nu-2s-1} - J_{-\nu-2s+1}),$$

after using the relation $2J_n' = J_{n-1} - J_{n+1}$. Only the real part of $\frac{d\alpha}{dz}$ must be retained, since the right hand side of (3.32) is real. Now (3.32) becomes

$$\frac{1}{2} R\Big\{(k_1+ik_1')e^{int}\cdot\sum_{-2}^{2} b_{2s}(J_{\nu+2s-1} - J_{\nu+2s+1})_{r=r_0}$$

$$+ (k_2+ik_2')\, e^{-int}\cdot\sum_{-2}^{2} b_{2s}(J_{-\nu-2s-1} - J_{-\nu-2s+1})_{r=r_0}\Big\}$$

$$= a_0\,\delta_0 \sin(nt+\varepsilon).$$

or

$$(\ )_1(k_1\cos nt - k_1'\sin nt) + (\ )_2(k_2\cos nt + k_2'\sin nt) = 2a_0\delta_0\sin(nt+\varepsilon)$$

where

$$\left.\begin{aligned} (\ )_1 &= \sum_{-2}^{2} b_{2s}(J_{\nu+2s-1} - J_{\nu+2s+1})_{r=r_0}, \\ (\ )_2 &= \sum_{-2}^{2} b_{2s}(J_{-\nu-2s-1} - J_{-\nu-2s+1})_{r=r_0}, \end{aligned}\right\} \tag{3.34}$$

This has to be true for all values of nt, hence the coefficients of cos nt and sin nt must be equal on both sides ; this gives

$$k_1 (\ )_1 + k_2 (\ )_2 = 2\, a_0\, \delta_0 \sin \varepsilon\,,$$

$$- k_1' (\ )_1 + k_2' (\ )_2 = 2\, a_0\, \delta_0 \cos \varepsilon\,.$$

Thus we can express k_2 and k_2' in terms of k_1 and k_1'

$$k_2 = \frac{2a_0\,\delta_0 \sin \varepsilon - k_1 (\ )_1}{(\ )_2}\,, \quad k_2' = \frac{2\,a_0\,\delta_0 \cos \varepsilon + k_1' (\ )_1}{(\ )_2}\,.$$

After k_2 and k_2' are eliminated, the general solution (3.33) becomes

$$\alpha = \mathrm{R} \left\{ (k_1 + ik_1')\, e^{int} \cdot \sum_{-2}^{2} b_{2s} \mathrm{J}_{\nu+2s} + \frac{2a_0\delta_0 (\sin \varepsilon + i \cos \varepsilon) - (\ )_1 (k_1 - ik_1')}{(\ )_2}\, e^{-int} \sum_{-2}^{2} b_{2s} \mathrm{J}_{-\nu-2s} \right\}. \quad (3.35)$$

This satisfies the inner boundary condition. The outer condition is the existence of diverging waves at points distant from the cylindir. Since we are considering the motion at points far from the cylinder we can use the asymptotic expansions for the Bessel functions involved. We use the formula

$$\mathrm{J}_n(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos \left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right). \quad (3.36)$$

Then we obtain the relations

$$\sum_{-2}^{2} b_{2s}\, \mathrm{J}_{\nu+2s}(z) = (1 - b_2 - b_{-2} + b_4 + b_{-4}) \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right),$$

and

$$\sum_{s=-2}^{2} b_{2s} \mathrm{J}_{-\nu-2s}(z) = (1 - b_2 - b_{-2} + b_4 + b_{-4}) \cos\left(z + \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

After inserting these into the equation (3.35) and transforming the cosines into exponentials it can be written

$$\alpha = (1 - b_2 - b_{-2} + b_4 + b_{-4}) \left(\frac{2}{\pi z}\right)^{1/2} \mathrm{R} \left\{ (k_1 + ik_1') \left[e^{-i\left(\frac{\nu\pi}{2} + \frac{\pi}{4}\right)} \cdot e^{i(z+nt)} + e^{i\left(\frac{\nu\pi}{2} + \frac{\pi}{4}\right)} \cdot e^{-i(z-nt)} \right] + \frac{2\,a_0\,\delta_0 (\sin \varepsilon + i \cos \varepsilon) - (\ )_1 (k_1 - i k_1')}{(\ )_2} \left[e^{-i\left(\frac{\nu\pi}{2} - \frac{\pi}{4}\right)} \cdot e^{-i(z+nt)} + e^{i\left(\frac{\nu\pi}{2} - \frac{\pi}{4}\right)} \cdot e^{i(z-nt)} \right] \right\}.$$

In order to get rid of the terms in $(z+nt)$ which represent converging waves we choose

$$(k_1 + i k_1') e^{-i\left(\frac{\nu\pi}{2}+\frac{\pi}{4}\right)}$$

$$= -\frac{2a_0\delta_0 (\sin\varepsilon + i\cos\varepsilon) - (\ )_1(k_1 - i k_1)}{(\ )_2} e^{-i\left(\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}$$

$$= \frac{2a_0\delta_0}{(\ )_1^2 - (\ )_2^2} K \quad (3.37)$$

say, where K is a constant and then (3.35) becomes

$$\alpha = \frac{2a_0\delta_0(1 - b_2 - b_{-2} + b_4 + b_{-4})}{(\ )_1^2 - (\ )_2^2}\left(\frac{2}{\pi z}\right)^{1/2}$$

$$R\left\{iK\sin(z+nt) + iKe^{i\nu\pi}\cos(z-nt)\right\}.$$

If K is real we obtain the solution for large volues of z

$$\alpha = -\frac{2a_0\delta_0(1 - b_2 - b_{-2} + b_4 + b_{-4})K}{(\ )_1^2 - (\ )_2^2}\left(\frac{2}{\pi z}\right)^{1/2}\sin\nu\pi\cdot\cos(z-nt) \quad (3.38)$$

which represents diverging wave solution.

Since K is now given by

$$K = [(\ )_1(\sin\varepsilon - i\cos\varepsilon) - (\ )_2(\cos\varepsilon - i\sin\varepsilon)]\, e^{-i\left(\frac{\nu\pi}{2}+\frac{\pi}{4}\right)},$$

from (3.37) and the imaginary part of K must vanish, this determines the unknown constant ε ; we obtain

$$\tan\varepsilon = \frac{(\ )_1 - (\ )_2\tan\left(\frac{\nu\pi}{2}+\frac{\pi}{4}\right)}{(\ )_2 - (\ )_1\tan\left(\frac{\nu\pi}{2}+\frac{\pi}{4}\right)}, \quad (3.39)$$

and with this value of ε, K becomes

$$K = \frac{(\ )_1^2 - (\ )_2^2}{[(\ )_1^2 + (\ )_2^2 - 2(\ )_1(\ )_2\cos\nu\pi]^{1/2}} \quad (3.40)$$

Finally the general solution (3 35) takes the form

$$\alpha = \frac{2a_0\delta_0}{[(\ )_1^2 + (\ )_2^2 - 2(\ )_1(\ )_2\cos\nu\pi]^{1/2}}\left\{\sum_{s=-2}^{2} b_{2s}J_{\nu+2s}(z)\cdot\cos\left(nt + \frac{\nu\pi}{2}+\frac{\pi}{4}\right) + \sum_{s=-2}^{2} b_{2s}J_{-\nu-2s}(z)\cdot\sin\left(nt - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right\}, \quad (3.41)$$

where a_0 is velocity of sound at a stagnation point, δ_0 is the amplitude of the periodic motion of the surface of the cylinder, ν is given in terms of λ by (3.26), $(\ )_1$, and $(\ )_2$ are given by (3.34), $b_{2s} = A_{2s}/A_0$ by (3.27), and

$$z = \frac{2\lambda}{\tau^{1/2}} = \frac{2\lambda c_m}{c} \qquad \text{by (1.18)},$$

$$= \frac{2\lambda r}{R} \qquad \text{by (2.2)},$$

$$= \frac{n}{a_0} r \qquad \text{by (2.11)},$$

r being the distance from the axis of the cylinder.

If we insert this walue of z and K into the equation (3.38) it becomes

$$\alpha = \frac{-2a_0\delta_0(1 - b_2 - b_{-2} + b_4 + b_{-4})}{[(\ )_1^2 + (\ )_2^2 - 2(\ )_1(\ )_2 \cos \nu\pi]^{1/2}} \left(\frac{2\,a_0}{n\pi r}\right)^{1/2} \sin \nu\pi \cdot \cos \frac{n}{a_0}(r - a_0 t), \quad (3.42)$$

which shows that the diverging waves far from the cylinder are propagated with velocity a_0, namely with the velocity of sound at a stagnation point, and the amplitude of the waves decreases as $r^{-\frac{1}{2}}$.

If instead of (3.36), the form

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left\{\cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) - \frac{4n^2 - 1}{8z} \sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right\}$$

is used, proceeding exactlly in the same way, we obtain

$$\alpha = \frac{2a_0\delta_0(1 - b_2 - b_{-2} + b_4 + b_{-4})\,K}{(\ )_1^2 - (\ )_2^2} \left(\frac{2}{\pi z}\right)^{1/2} \sin \nu\,\pi$$

$$\left[\frac{4\nu^2 - 1}{8z} \sin(z - nt) - \cos(z - nt)\right] \quad (3.43)$$

corresponding to (3.38), where K has the same value given by (3.40).

The solution (3.41) is valid only if $\lambda^2 \leq \frac{1}{2}(7 - 4\sqrt{3})$. When λ^2 is greater than this value the selution of the equation (3.23) for ν is complex. Then since λ^2 is not necessarily small the series (3.24) for $\Delta(0)$ will not converge at least after some critical value of λ^2. In this case, we have to express $\Delta(0)$ by an infinite series in powers of $1/\lambda^2$. The same is true for $\cos \pi \sqrt{\varkappa_1}$ and $\cos \pi \sqrt{\varkappa_2}$. Then the equation (3.23) gives ν in series form proceeding in powers of $1/\lambda^2$, but it will be complex, since $\sqrt{\varkappa_1}$ and $\sqrt{\varkappa_2}$ are complex. Then the Bessel functions become of complex order. But as we take the real part of α, these functions or, where it is necessary, the combinations of functions must be expressed in the form $a+bi$, and then the method of satisfying the boundary conditions explained above proceeds in the same way.

It is interesting to know how the amplitude of the waves depens on $\lambda = \frac{nR}{2a_0}$, i.e. on the frequency n of the oscillating cylinder. In order to obtain a dimensionless coeffient for this dependence we divide the amplitude of resulting wave by the amplitude of the motion of the cylinder, and denote its absolute value by M. Then (see 3.42)

$$M = \frac{2(1 - b_2 - b_{-2} + b_4 + b_{-4})}{[(\ )_1^2 + (\ )_2^2 - 2(\ )_1(\ )\cos\nu\pi]^{1/2}} \left(\frac{2a_0}{n\pi r}\right)^{1/2} \sin\nu\pi \quad (3.44)$$

For the sake of simplicity let us take only the middle terms in $(\ )_1$, and $(\ )_2$, namely the terms corresponding to $s=0$, and shoose a_0=33,500 cm/sec. R=10 cm, r_0= 20 cm, and r =10,000 cm. That is, we shall do our calculations for a particular point 100 metres far from the axis of the cylinder. Then $1 - b_2 - b_{-2} + b_4 + b_{-4}$ reduces to unity, and taking $\nu = \sqrt{6}\,\lambda$, M becomes

$$M = \frac{0.03568 \sin\nu\pi}{\sqrt{\lambda}\{\ \}^{1/2}}, \quad (3.45)$$

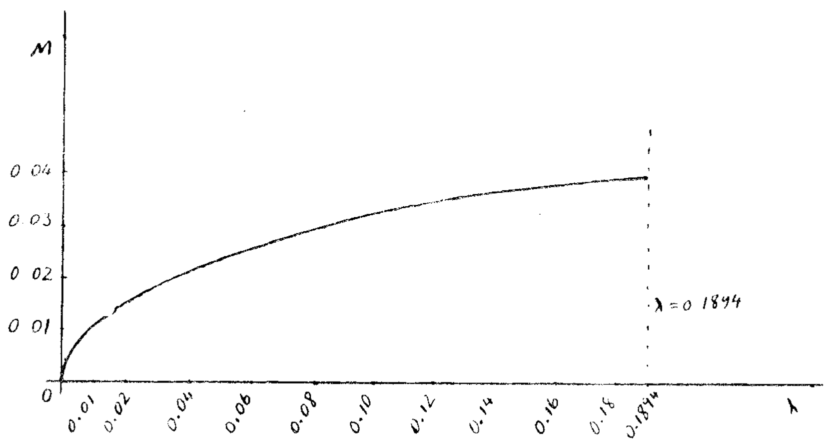
where

$$\{\ \} = (2\lambda)^{2\nu}\left[\frac{1}{2\lambda\Gamma(\nu)} - \frac{2\lambda}{\Gamma(\nu+2)}\right]^2 + (2\lambda)^{-2\nu}\left[\frac{1}{2\lambda\Gamma(-\nu)} - \frac{2\lambda}{\Gamma(-\nu+2)}\right]^2$$
$$- 2\left[\frac{1}{2\lambda\Gamma(\nu)} - \frac{2\lambda}{\Gamma(\nu+2)}\right]\left[\frac{1}{2\lambda\Gamma(-\nu)} - \frac{2\lambda}{\Gamma(-\nu+2)}\right]\cos\nu\pi$$

By some lengthy calculations we can prepare the table of values of M for different values of λ in the range $0 \leq \lambda^2 \leq \frac{1}{2}(7 - 4\sqrt{3})$ and plot a curve. When $\lambda = 0$, M vanishes since there is then no radial motion.

λ	0.001	0.01	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.1894
M	0.00354	0.0111	0.0156	0.0217	0.0260	0.0293	0.0321	0.0344	0.0365	0.0386	0.0396	0.0401

The graph is as follows :



This shows that M increases as λ or the frequency n of the periodic motion of the cylinder increases, and it reaches the value 0.0401 when $\lambda = 0.1894$. This value of λ sorresponds to $n = 1270/\text{sec}$. Hence when the frequency of the cylinderis 1270 per second the amplitude of the diverging waves at a point 100 metres distant from the axis of the cylinder is $0.0401\ \delta_0$.

References

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