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# Note On a Generalization of Schwarz' Lemma.


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Özet : Schwarz Lemmasının bir Teşmili Hakkında Not. — L. Ahlfors Klasik Schwarz Lemmasını $|z| < 1$ den bir Riemann yüzeyine analitik fonksiyonlara teşmil etti. Bilahare çok yakın bir zamanda Z. Nehari, aynı Lemmayı Birim dairesinden Birim dairesine üniform olmıyan analitik fonksiyonlara teşmil etti. Bu teşmilde müellif Ahlfors'un metodunu kullanmaktadır. Biz ise aynı metodla Schwarz Lemmasını Birim dairesinden bir Riemann yüzeyine üniform olmıyan analitik fonksiyonlara teşmil etmiş oluyoruz,


1. In the treatment of the extremal problem within the family of normalized analytic functions generated by the conformal maps of the equiangular Schwarz' triangles lying in the unit circle onto a straight equilateral triangle, let us say [1], we stated a generalization of Schwarz' Lemma of which we gave a direct proof for the particular case at hand (1).

Here we wish to present a formal proof which is based essentially on a method of proof due to L. Ahlfors [2] and Z. Nehari [3].

We first recall preliminary notions and notations which will be used in the sequel.

A topologically invariant definition of an abstract Riemann surface is as follows :

(1) We profit of the occasion to correct two misprints in [1], i. e., p. 249, line 33, in place of "inequality" read "equality". p. 250, line 21, in place of "inequality" read "equality".

Let $\mathfrak{W}$ be a connected topological Hausdorff space such that every point $\mathfrak{w}$ is contained in at least one neighborhood $N_{\mathfrak{w}}$ homeomorph to the unit circle $|\omega| < 1$. If $h_{\mathfrak{w}}$ is a topological mapping of $N_{\mathfrak{w}}$ onto $|\omega| < 1$ we may require also the condition $h_{\mathfrak{w}}(\mathfrak{w}) = 0$. $\omega$ is called a local parameter for $\mathfrak{w}$. Let $N_{\mathfrak{w}}, N_{\mathfrak{w}'}$ be two overlapping neighborhoods and $\omega, \omega'$ the corresponding local parameters. Denote by $N_{\mathfrak{w},\mathfrak{w}'}$ the intersection $N_{\mathfrak{w}} \cap N_{\mathfrak{w}'}$ and by $N_{\omega} = h_{\mathfrak{w}}(N_{\mathfrak{w},\mathfrak{w}'})$, $N_{\omega'} = h_{\mathfrak{w}'}(N_{\mathfrak{w},\mathfrak{w}'})$ its image in $|\omega| < 1$, $|\omega'| < 1$ respectively. If $h_{\mathfrak{w}}$ and $h_{\mathfrak{w}'}$ are such that $N_{\omega'} = h_{\mathfrak{w}'} h_{\mathfrak{w}}^{-1}(N_{\omega})$ is directly conformal then $\mathfrak{W}$ is called a Riemann surface.

At every point $\mathfrak{w}$ of $\mathfrak{W}$ we introduce a Riemannian metric of the form

$$ds = \lambda \, | d\omega | \tag{1}$$

where $\lambda$ is defined in $|\omega| < 1$, is positive and of class $C_2$. $\lambda$ depends on the choice of the local parameter but $ds$ does not.

Finally the Gaussian curvature of the metric (1) is given by

$$K = -(\Delta \log \lambda)/\lambda^2 \tag{2}$$

Again (2) is independent of the choice of $\omega$. If $K \leq 0$, then $\Delta \log \lambda \geq 0$ and therefore $\log \lambda$ is subharmonic. Conversely if $\log \lambda$ is subharmonic then $\Delta \log \lambda \geq 0$ and therefore $K \leq 0$.

We shall consider negative curvatures with the upper bound equal to $-4$. In this case $\lambda$ satisfies the condition

$$\Delta \log \lambda \geq 4\lambda^2$$

or upon setting $u = \log \lambda$

$$\Delta u \geq 4e^{2u} \tag{3}$$

2. Let $\mathfrak{W}$ be a Riemann surface, $\mathfrak{a}$ a point on $\mathfrak{W}$ and $a$ its projection on the $w$-plane, then by definition

$$w - a = c_n \omega^n + \cdots, \quad c_n \neq 0, \quad n \geq 1$$

If $n = 1$ we say that $\mathfrak{a}$ is a regular point. If $n > 1$ we say that $\mathfrak{a}$ is a branch point of order $n - 1$, $n$ is the number of sheets containing $\mathfrak{a}$. In the latter case $\omega$ is called a local uniformizing parameter for $\mathfrak{a}$.

We now state the theorem to be proved.

Theorem. Let $\mathfrak{w} = f(z)$ be analytic in the unit circle $|z| < 1$, save branch points of finite order. Suppose $f'(z) < \infty$ in $|z| < 1$. If the metric $ds$ of the Riemann surface $\mathfrak{M}$ generated by $\mathfrak{w}$ has a negative curvature $\leq -4$ at every point, then

$$ds \leq d\sigma$$

where $d\sigma$ is the well known hyperbolic metric of the unit circle.

Proof: Let $\mathfrak{Z}$ and $\mathfrak{W}$ be the two conformally equivalent Riemann surfaces with respect to $\mathfrak{w} = f(\mathfrak{z})$, the latter being single valued on the finitely many sheeted Riemann surface $\mathfrak{Z}$ lying over the unit circle $|z| < 1$. We introduce the metric $ds = \lambda \, | dw |$ where $w$ is the projection of $\mathfrak{w}$(1), $\lambda > 0$ is of Class $C_2$ and $ds$ has a curvature $\leq -4$ at every point of $\mathfrak{W}$. We have

$$ds = \lambda_z \, | dz | = \lambda \, | dw |$$

and

$$\lambda_z = \lambda \, | f'(z) |$$

Then everywhere on the Riemann surface $\mathfrak{Z}$, $u = \log \lambda_z$ satisfies the inequality (3) with the exception of the points at which $f'(z) = 0$.

Furthermore (3) holds at the branch points at which $f'(z) \neq 0$. For, at such points $f'(z) < \infty$, so that $f(z)$ *remains conformal.*

We wish to compare the metric $ds$ with the hyperbolic metric

$$d\sigma = | dz | / 1 - | z |^2$$

when these metrics are carried over the Riemann surface $\mathfrak{Z}$.

For an arbitrary $R < 1$, we set

$$v = \log \frac{R}{R^2 - | z |^2} \quad , \quad | z | < R$$

We have

$$\Delta v = 4e^{2v} \tag{4}$$

Combining (3) and (4) we get

$$\Delta(u - v) \geq 4(e^{2u} - e^{2v}) \tag{5}$$

(1) In other words the variable w is used as a local parameter for some neighborhood of each point $\mathfrak{w}$.

(5) holds at every point of the Riemann surface $\mathfrak{Z}_R \subset \mathfrak{Z}$ lying over the circle $|z| < R$, except at the points for which $f'(z) = 0$. As we know (5) holds also at the branch points at which $f'(z) \neq 0$.

Let $E$ denote the open point set in $\mathfrak{Z}_R$ for which

$$u > v$$

It is clear that $E$ cannot contain any point of $\mathfrak{Z}_R$ at which $f'(z) = 0$. Hence by (5)

$$\Delta(u - v) > 0 \tag{6}$$

holds in $E$. We conclude that $u - v$ is subharmonic in $E$ except at the branch points of $\mathfrak{Z}_R$ inside $E$.

Hence $u - v$ can have no maximum at any interior point of $E$ which is not a branch point. Let us verify that $u - v$ can have no local maximum at these branch points either. $\zeta$ being a local uniformizing parameter for a branch point $\mathfrak{a}$ in $E$, we have

$$z - a = c_n \zeta^n + \cdots \quad n \geqq 2,\ c_n \neq 0$$

It is well known that

$$\Delta_\zeta(u - v) = \Delta_z (u - v) \, | dz/d\zeta |^2$$

By (6) it follows that in the neighborhood of $\zeta = 0$

$$\Delta_\zeta(u - v) \geqq 0$$

On the other hand $u - v$ as a function of $\zeta$ being single valued and of Class $C_2$ in the neighborhood of $\zeta = 0$ is subharmonic there. Consequently it can have no local maximum at $\zeta = 0$. Hence $u - v$ considered as a function of $z$ can have no loal maximum at $\mathfrak{a}$. The conclusion is that $u - v$ can have no maximum at any interior point of $E$ without exception. Hence $u - v$ must approach its least upper bound on a sequence tending to the boundary of $E$. But $E$ can have no boundary point on $|z| = R$ since $v$ becomes positively infinite as $z$ tends to this circumference, while $f'(z)$ is finite. For a boundary point of $E$ inside $|z| = R$, since $u - v$ is continuous for all $|z| < R$, we must have $u - v = 0$ that is, $u - v$ cannot be maximum at such boundary points either. Thus a contradiction is reached thereby implying that the set $E$ must be empty. Hence $u \leqq v$ must hold for all points $|z| < R$. As $R$ is arbitrary it must hold for $R \to 1$.

Consequently we have for all $| z | < 1$

$$\lambda_z \leq \frac{1}{1 - | z |^2}$$

Multiplying both sides by $| dz |$ we get finally

$$ds \leq d\sigma$$

## Literature

[1] C. Uluçay, On The Bloch-Landau Constants, Communications de la Faculté des Sciences de l'Université d'Ankara, Série A, Tome VII, Fasc. 1,1955

[2] L. Ahlfors, An extension of Schwarz's Lemma, Transactions of the Amer, Math. Soc. vol. 43, 1938.

[3] Z. Nehari, A generalization of Schwarz' Lemma, Duke Math. Journal vol. 14, 1947.

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