Some Remarks on the Contraction of Groups

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Özet.— Daha önce tarif edilmiş olan «gurupların büzülmesi» (contraction of groups) metoduyla G gibi bir Lie gurubundan başka bir K gurubunun nasıl elde edileceği gösterilmişti. Geçenlerde Prof. Cahit Arf, bu metotla ilgili olarak şu problemi ortaya attı: Acaba her iki G ve K gurupları da alt gurup olarak ihtiva eden daha geniş bir L gurubu bulunabilir mi? Bu yazıda, verilen G gurubunun yapı sabitleri (structure constants) basit bir bağıntıyı sağladıkları takdirde böyle bir L gurubunun mahallî olarak teşkil edilebileceği gösterilmektedir. Önce büzülme işleminin infinitezimal operatörlere dayanmıyan, fakat bunlar yerine bağlı gurup (adjoint group) kavramından faydalanan yeni bir tarifi veriliyor. Bu tarif, eski tarifle eşdeğerliği gösterildikten sonra, eldeki probleme tatbik ediliyor. Sonda, misal olarak, homogen Lorentz gurubu ile homogen Galile gurubunu alt guruplar olarak ihtiva eden bir gurup, infinitezimal operatörleri cinsinden verilmiştir.

Introduction. In a previous article published elsewhere [1], the operation of "contraction of a Lie group with respect to any of its subgroups„ was defined. If this Lie group G is of dimension n and its subgroup S of dimension m then by contraction with respect to S one obtains another Lie group K of dimension n, which contains in addition to the same subgroup S, an $(n - m)$-dimensional abelian invariant subgroup; the subgroup S is isomorphic with the factor group of this invariant subgroup. Contraction was carried out by means of a singular transformation on the infinitesimal operators of the original group G, which changes its structure. Prof. Cahit ARF suggested recently that it might be possible, at least in certain cases, to find a larger group L which would contain as subgroups both G and K, the original and contracted groups.

In this paper, we want to show that it is possible indeed to construct a local group with the desired property, in the case that the structure constants of the original group G satisfy a simple relation, namely if $c^{i_1}_{j_1 k_2} = 0$ for all $i_1, j_1 = 1,2...,$ m and $k_2 = m+1,\ m+2,..., n$, where the subscript 1 refers to those infinitesimal operators that form in the infinitesimal group the subgroup corresponding to S and the subscript 2 to those operators that do not belong to this infinitesimal subgroup. Since this condition is very simple, one can give immediately the infinitesimal group corresponding to L and then check that the structure constants satisfy the Jacobi relations. However we shall follow a roundabout way and first give a definition of the contracted group which does not involve the infinitesimal operators but instead makes use of the adjoint group. It is hoped that this new definition may prove valuable in investigating also some other questions on the relationship between G and K such as for instance the problem of obtaining sufficient conditions for a group to be derivable from another by contraction.

After reviewing briefly the old definition in terms of infinitesimal operators, we shall give the new definition and show the equivalence of the two. Then we shall apply this new definition to the problem in hand and obtain directly a large local group L in finite form, in the case when the above-mentioned condition is satisfied. We shall also derive the structure constants of L. As an example, the large group which contains as subgroups the homogeneous Lorentz and Galilei groups is given at the end in terms of its infinitesimal group. We remark here that throughout the article only local groups will be considered.

Definition of the contracted group by means of the infinitesimal operators. Let G be an arbitrary local Lie group of dimension n, I_i its infinitesimal operators ($i = 1, 2,...,n$), and S a subgroup of G of dimension m. Assume the operators so chosen that $I_1, I_2,..., I_m$ fall within the infinitesimal subgroup corresponding to S. There would then exist the following commutation relations between the operators:

$$(1)\quad \begin{array}{ll} (a) & [I_{i_1}, I_{j_1}] = c^{k_1}_{i_1 j_1} I_{k_1} \\ (b) & [I_{i_1}, I_{j_2}] = c^{k_1}_{i_1 j_2} I_{k_1} + c^{k_2}_{i_1 j_2} I_{k_2} \\ (c) & [I_{i_2}, I_{j_2}] = c^{k_1}_{i_2 j_2} I_{k_1} + c^{k_2}_{i_2 j_2} I_{k_2} \end{array} \left(\begin{array}{l} i_1, j_1, k_1 = 1,2,...,m \\ i_2, j_2, k_2 = m+1, m+2,..,n \end{array} \right)$$

where the c's are the structure constants for the group G, the indices with subscript 1 refer to operators of the subgroup S and the indices with subscript 2 refer to those outside it.

Consider now the following commutation relations between n new operators J_i, where the same constants c^k_{ij} are used.

$$(2)\quad \begin{aligned} &(a) \quad [J_{i_1}, J_{j_1}] = c^{k_1}_{i_1 j_1} \; J_{k_1} \\ &(b) \quad [J_{i_1}, J_{j_2}] = c^{k_2}_{i_1 j_2} \; J_{k_2} \\ &(c) \quad [J_{i_2}, J_{j_2}] = 0 \end{aligned} \qquad \begin{pmatrix} i_1, j_1, k_1 = 1,2,\ldots, m \\ i_2, j_2, k_2 = m+1, m+2, \ldots, n \end{pmatrix}$$

As was shown previously, the operators J_i form an infinitesimal group and the corresponding local group K is said to have been obtained from G by contraction with respect to its subgroup S. More briefly K is called the contracted group. It is seen from the equations (2) that the operators J_{i_2} form an abelian invariant subgroup A of K, that J_{i_1} form a subgroup locally isomorphic with S (which for our purposes can be considered to be identical with S) and that consequently the factor group of A is locally isomorphic with S.

Definition of the contracted group by means of the adjoint group. In order to obtain the contracted group directly in finite form we consider the following construction, pointed out by Prof. Arf. Let S be any local Lie group and A an additive vector group which may be considered as a vector space. Elements of S and A will respectively be represented by s and a. Assume that a representation R_S of the group S is given in terms of the one-to-one transformations of the vector space A. For any $a \varepsilon A$ and $s \varepsilon S$ we have

$$(3)\quad \begin{aligned} &(a) \quad R_s(a) = a' \text{ where } a' \text{ is determined uniquely by } s \text{ and } a, \\ &(b) \quad R_{s_1}\big(R_{s_2}(a)\big) = R_{s_1 s_2}(a), \\ &(c) \quad R_e(a) = a \text{ where } e \text{ is the identity element in S.} \end{aligned}$$

We construct now the set of all pairs (a, s). Two elements of this set, (a_1, s_1) and (a_2, s_2) are taken to be equal if and only if $a_1 = a_2$, $s_1 = s_2$. The product of two pairs is defined by

$$(4)\qquad (a_1, s_1)(a_2, s_2) = \big(a_1 + R_{s_1}(a_2), s_1 s_2\big).$$

If this set is to form a group under the law (4), the product must be associative. Thus we must have

$$\big((a_1, s_1)(a_2, s_2)\big)(a_3, s_3) = (a_1, s_1)\big((a_2, s_2)(a_3, s_3)\big)$$

or, using the definition of equality and the fact that the product $s_1 s_2$ is associative,

$$R_{s_1}(a_2) + R_{s_1 s_2}(a_3) = R_{s_1}\left(a_2 + R_{s_2}(a_3)\right).$$

It is seen that this condition will be satisfied if R_s is a linear representation on A; i. e. if

(5) $$R_s(a_1 + a_2) = R_s(a_1) + R_s(a_2).$$

Assuming (5) to be valid, the set of pairs (a, s) becomes a group H. The identity element of H is (o, e) where o is the identity element of A and e, the identity element of S. The inverse of (a, s) is given by

(6) $$(a, s)^{-1} = \left(-R_{s^{-1}}(a),\ s^{-1}\right).$$

We remark that it is not necessary for R to be a faithful representation of S. The elements (a, s) will form a group under (4) even if R gives only a homomorphic mapping of S.

The two properties of H that make it important for our purpose are the following: The elements (o, s) form a subgroup of H which is locally isomorphic with S, while the elements (a, e) form an invariant subgroup of H which is isomorphic with A. Furthermore S is locally isomorphic with the factor group of (a, e) in H. Therefore it appears likely that by taking S to be the same subgroup of G as considered above, letting A to be an $(n-m)$-dimensional vector space and specifying the representation R suitably one may identify H with K, the contracted group. This specification will be made below by using the concept of adjoint group.

Adjoint group of G. Let the local group G be given in canonical coordinates of the first kind. Then the inner automorphism $b_g(x)$ of the group defined by

$$b_g(x) = gxg^{-1}$$

can be written in coordinate form as

$$b^i_g(x) = p^i_j(g)x^j$$

where g and x are elements of G and b^i, x^j are respectively the canonical coordinates of b and x, with $i, j = 1,2,\cdots,n$.

In this way to every element g there corresponds a matrix $p^i_j(g)$ and these matrices form a representation of G since one finds that,

$$p^i_k(g_1)\ p^k_j(g_2) = p^i_j(g_1 g_2).$$

The set of matrices $p^i_j(g)$ is called the adjoint group of G (Cf. e. g. Ref: 2). The functions $p^i_j(g)$ can be calculated directly from the structure constants of G. We replace g^i by tu^i where t is a real parameter and u^i are the components of the direction vector of a one-parametric subgroup that goes through both the identity element and the element g. Then the following relations hold:

$$\frac{dp^i_j(tu)}{dt} = c^i_{mn}\ u^m p^n_j(tu). \tag{8}$$

Together with the initial conditions

$$p^i_j(0 \cdot u) = \delta^i_j \ , \tag{9}$$

the system of ordinary differential equations (8) can be integrated to give $p^i_j(g)$. (Ref : 2).

We shall consider in particular the matrices $p^i_j(s)$ that correspond to elements of the subgroup S of G. Let the elements s have the coordinates

$$s^{m+1} = s^{m+2} = \cdots = s^n = 0, \text{ with } s^1, s^2, \cdots, s^m \text{ arbitrary},$$

and mark all indices that run through 1 to m with subscript 1 and all indices that run through $m+1$ to n with subscript 2. Using these indices and the facts that $p^{i_2}_{j_1}(s) = 0$, $c^{i_2}_{j_1 k_1} = 0$ (both of which follow because the elements s form a subgroup), the equations for $p^i_j(s)$ can be separated into three sets:

$$\frac{dp^{i_1}_{j_1}(tu_1)}{dt} = c^{i_1}_{m_1 n_1} u^{m_1} p^{n_1}_{j_1}(tu_1) \tag{10}$$

$$\frac{dp^{i_1}_{j_2}(tu_1)}{dt} = c^{i_1}_{m_1 n_1} u^{m_1} p^{n_1}_{j_2}(tu_1) + c^{i_1}_{m_1 n_2} u^{m_1} p^{n_2}_{j_2}(tu_1) \tag{11}$$

$$\frac{dp^{i_2}_{j_2}(tu_1)}{dt} = c^{i_2}_{m_1 n_2} u^{m_1} p^{n_2}_{j_2}(tu_1) \tag{12}$$

These equations show that the functions $p^{i_1}_{j_1}(tu_1)$ are determined wholly by the constants $c^{i_1}_{m_1 n_1}$, and similarly $p^{i_2}_{j_2}(tu_1)$ are determined by $c^{i_2}_{m_1 n_2}$, while $p^{i_1}_{j_2}(tu_1)$ involve both $c^{i_1}_{m_1 n_1}$ and $c^{i_1}_{m_1 n_2}$. Conversely from these equations the c's can be determined uniquely as follows :

$$(13) \qquad c^{i_1}_{m_1 j_1} = \left(\frac{dp^{i_1}_{j_1}(tu_1{}^{(m_1)})}{dt}\right)_{t=0}, \quad c^{i_1}_{m_1 j_2} = \left(\frac{dp^{i_1}_{j_2}(tu_1{}^{(m_1)})}{dt}\right)_{t=0},$$

$$c^{i_2}_{m_1 j_2} = \left(\frac{dp^{i_2}_{j_2}(tu_1{}^{(m_1)})}{dt}\right)_{t=0},$$

where $\underset{1}{u}^{(m_1)}$ stands for the vector which has all its components equal to zero except for $\underset{1}{u}^{m_1} = 1$.

The matrices $p^i_j(s)$ form an n-dimensional representation of the subgroup S. Since $p^{i_2}_{j_1}(s) = 0$, this representation is reducible and $p^{i_1}_{j_1}(s)$, $p^{i_2}_{j_2}(s)$ form separately representations of S. The representation $p^{i_1}_{j_1}(s)$ which will be denoted simply by $p_1(s)$ is m-dimensional while $p^{i_2}_{j_2}(s)$, or briefly $p_2(s)$, is $(n-m)$-dimensional. Remembering that p_2 depends only on the constants $c^{i_2}_{m_1 j_2}$ we see that this representation suggests itself as a suitable R in order to identify H with K. Indeed, in the definition of (a, s), let S be the subgroup of G with respect to which contraction is made, A an $(n-m)$-dimensional vector space and $R_s = p_2(s)$ or in coordinate form,

$$(14) \qquad R^{i_2}_s(a) = p^{i_2}_{j_2}(s)\, a^{j_2},$$

where $i_2, j_2 = m+1, m+2, \ldots, n$. The representation p_2 is obviously linear in the vector space and the set (a, s) forms a group.

Furthermore, the structure constants $\bar{c}^i_{jk}$ of this group H are identical with the constants c^i_{jk} of K. It follows directly from the construction that

$$\bar{c}^{k_1}_{i_1 j_1} = c^{k_1}_{i_1 j_1}, \quad \bar{c}^{k_2}_{i_2 j_2} = c^{k_2}_{i_2 j_2} = 0, \quad \bar{c}^{k_1}_{i_1 j_2} = c^{k_1}_{i_1 j_2} = 0.$$

To obtain $\bar{c}^{k_2}_{i_1 j_2}$ we consider the part $q(s)$ of the adjoint group of H corresponding to S. For $(o, s) \in$ S, $(a, s') \in$ H, we have the relation,

$$(o, s)\,(a, s')\,(o, s^{-1}) = (p_{2s}(a), s\, s'\, s^{-1})$$

or denoting the elements simply by s, h and s^{-1},

$$(shs^{-1})^{i_2} = p^{i_2}_{j_2}(s)\, a^{j_2}.$$

Thus for the part $q(s)$ of the adjoint group of H, we have

$$q^{i_2}_{j_2}(s) = p^{i_2}_{j_2}(s)$$

and consequently from (13)

$$\bar{c}^{i_2}_{j_2 k_2} = c^{i_2}_{j_2 k_2}.$$

This completes the proof of the local isomorphism of H with K. Therefore we may, if we wish, define the contracted group as the group H. It may be interesting to point out that because (a, e) form an invariant subgroup of H, the matrices $q^{i_1}_{j_2}(s)$ vanish identically. The matrices $q(s)$ have the general form,

$$q(s) = \begin{Vmatrix} P_1(c^{i_1}_{j_1 k_1}, s) & 0 \\ 0 & P_2(c^{i_2}_{j_2 k_2}, s) \end{Vmatrix} \tag{15}$$

while $p(s)$ had the form

$$p(s) = \begin{Vmatrix} P_1(c^{i_1}_{j_1 k_1}, s) & P_{12}(c^{i_1}_{j_1 k_2}, c^{i_2}_{j_1 k_2}, s) \\ 0 & P_2(c^{i_2}_{j_2 k_2}, s) \end{Vmatrix} \tag{16}$$

where we have indicated the dependence of the functions $p(s)$ on various structure constants.

Construction of a large group L. Let us consider a similar combination of the whole local group G with an n-dimensional vector space A; i.e. let (a, g) be the elements of the new local group L, where $a \in A$ and $g \in G$. The product is defined by

$$(a_1, g_1)(a_2, g_2) = (a_1 + p_{g_1}(a_2), g_1 g_2) \tag{17}$$

where p_{g_1} stands for the adjoint group of G. In coordinate form,

$$p^i_{g_1}(a_2) = p^i_j(g_1)\, a^j_2 \quad \text{with} \quad i, j = 1,2,\dots, n. \tag{18}$$

The group L is 2n-dimensional.

Obviously, the elements (o, g) form a subgroup of L which is locally isomorphic with G. Similarly, if S is a subgroup of G and $s \varepsilon$ S, the elements (a, s) will form a subgroup of L, which we shall indicate simply by (A, S). Assume that the subgroup S is of dimension m and its elements have coordinates $s^{i_2} = 0$, s^{i_1} being arbitrary, where we use the same convention for the indices. Since $p^{i_2}_{j_1}(s) = 0$, the matrices $p^i_j(s)$ which correspond in the adjoint group to the subgroup S, form a reducible representation of S. If in addition we have,

$$p^{i_1}_{j_2}(s) = 0 \,,$$

then the matrices $p^i_j(s)$ are completely reducible. When the transformation $p^i_j(s)$ is applied to a^j, the components a^{j_1} and a^{j_2} transform separately among themselves as $p^{i_1}_{j_1}(s)\; a^{j_1}$ and $p^{i_2}_{j_2}(s)\; a^{j_2}$. Consequently, in this case the elements (a_1, s) and (a_2, s) form two separate subgroups of L, where a_1 and a_2 indicate respectively elements of A with coordinates $a^{i_2} = 0$, a^{i_1} arbitrary and $a^{i_1} = 0$, a^{i_2} arbitrary. Of these, the subgroup (A_2, S) formed by the elements (a_2, s) is locally isomorphic with the contracted group K, obtained from G by contraction with respect to S.

The relations (11) and (13) show that the conditions (19) are equivalent to the conditions:

$$(20) \qquad c^{i_1}_{j_1 k_2} = 0.$$

Thus, in case a local group G and its subgroup S satisfy the conditions (20), one can construct a large local group L = (A, G) which has one subgroup (o, G) locally isomorphic with G and another subgroup (A_2, S) locally isomorphic with K.

Structure Constants of the large group L. Let the structure constants of the group G be indicated by c^i_{jk} (where $i, j, k = 1, 2, \ldots, n$) and those of L by $C^\alpha_{\beta\gamma}$ (where $\alpha, \beta, \gamma = 1, 2, \ldots, 2n$). Let $\alpha_1, \beta_1, \gamma_1 = 1, 2, \ldots, n$ refer to infinitesimal operators of the subgroup (o, G) in L and $\alpha_2, \beta_2, \gamma_2 = n+1, n+2, \ldots, 2n$ to those operators outside it. As the elements (a, e) form an abelian invariant subgroup of L, we have first,

(21) $$C^{\alpha_1}_{\beta_2\gamma_2} = C^{\alpha_2}_{\beta_2\gamma_2} = 0$$

and $$\begin{pmatrix} \alpha_1, \beta_1, \gamma_1 = 1,2,\ldots,n. \\ \alpha_2, \beta_2, \gamma_2 = n+1, n+2, \ldots, 2n. \end{pmatrix}$$

(22) $$C^{\alpha_1}_{\beta_1\gamma_2} = 0 \, .$$

In order to determine the remaining constans $C^{\alpha_2}_{\beta_1\gamma_2}$, we use the following definition of the C's which brings in the commutator $q(x, y)$ of two elements x, y of L:

(23) $$q^{\alpha}(x, y) = (xyx^{-1}y^{-1})^{\alpha} = C^{\alpha}_{\beta\gamma} x^{\beta} y^{\gamma} + \varepsilon^{\alpha}(3) \, ,$$

where $\varepsilon^{\alpha}(3)$ is a quantity of the third order of magnitude with respect to the coordinates of x and y (Ref: 2).

For $x = (o, g)$, $y = (a, e)$, where g has the coordinates g^1, $g^2, \ldots, g^n$ and a has the coordinates $a^1, a^2, \ldots, a^n$, we obtain from (23),

(24) $$q^{\alpha_2}(x, y) = C^{\alpha_2}_{\beta_1\gamma_2} g^{\beta_1} a^{\gamma_2 - n} + \varepsilon^{\alpha_2}(3) \, ,$$

where $\varepsilon^{\alpha_2}(3)$ is of the third order in the coordinates g^{β_1}, $a^{\gamma_2 - n}$. On the other hand by direct computation we obtain,

$$q(x, y) = (o, g)(a, e)(o, g^{-1})(-a, e)$$
$$= (p_g(a) - a, e) \, ,$$

or

(25) $$q^{\alpha_2} = (p^{\alpha_2 - n}_{\gamma_2 - n}(g) - \delta^{\alpha_2 - n}_{\gamma_2 - n}) \, a^{\gamma_2 - n} .$$

Now, it follows from the definition of p^i_j that

(26) $$p^i_j(g) = \delta^i_j + c^i_{kj} g^k + \varepsilon^i_j(2) \, ,$$

where $\varepsilon^i_j(2)$ is a quantity of the second order in the coordinates g^i. Replacing i, j, k by $\alpha_2 - n$, $\gamma_2 - n$, β_1 respectively and substituting (26) into (25) we obtain

(27) $$q^{\alpha_2} = c^{\alpha_2 - n}_{\beta_1\gamma_2 - n} g^{\beta_1} a^{\gamma_2 - n} + \varepsilon^{\alpha_2}(3) \, ,$$

where $\varepsilon^{\alpha_2}(3)$ is of the third order in the coordinates g^{β}, a^{γ}. Comparing the equations (24) and (27) we have the result

(28) $$C^{\alpha_2}_{\beta_1\gamma_2} = c^{\alpha_2 - n}_{\beta_1\gamma_2 - n} \quad \text{where} \quad \begin{matrix} \beta_1 = 1,2,\ldots, n, \\ \alpha_2, \gamma_2 = n+1, n+2, \ldots, 2n. \end{matrix}$$

In particular we have

(29) $$C^{\alpha_2}_{\beta_1\beta_1+n} = c^{\alpha_2-n}_{\beta_1\beta_1} = 0,$$

and

(30) $$C^{\alpha_2}_{\beta_1\gamma_1+n} = C^{\alpha_2}_{\beta_1+n\gamma_1} = c^{\alpha_2-n}_{\beta_1\gamma_1}.$$

An example. In nearly all the examples of contraction considered in Ref. 1 the conditions (20) are satisfied. Therefore a group L can be constructed for nearly each case. We select among these, as an example, the contraction of the homogeneous Lorentz group with respect to the subgroup of rotations in space by which operation one obtains the homogeneous Galilei group. In the Lorentz group let the infinitesimal operator for spatial rotation in the kl-plane be represented by I_{kl} (where $k, l = 1,2,3$ and $I_{kl} = -I_{lk}$ by definition) and that for the Lorentz transformation in the k-direction by I_{k4} ($k = 1,2,3$). There are six independent operators. Their commutation relations may be given in the following compact form [3]:

(31) $[I_{kl}, I_{mn}] = g_{lm}I_{kn} - g_{km}I_{ln} + g_{kn}I_{lm} - g_{ln}I_{km}$ $(k, l, m, n = 1,2,3)$,

(32) $[I_{l4}, I_{mn}] = g_{lm}I_{n4} - g_{ln}I_{m4}$ $(l, m, n = 1,2,3)$,

(33) $[I_{l4}, I_{n4}] = -I_{ln}$ $(l, n = 1,2,3)$,

where $g_{kl} = 0$ for $k \neq l$ and $g_{11} = g_{22} = g_{33} = -1$. The corresponding large group L will have in addition to these operators I_{kl}, I_{k4} for G, six other operators for its abelian subgroup. In analogy with the I's, we represent these operators by J_{kl}, J_{k4} (where $J_{kl} = -J_{lk}$). Taking account of the expressions (21) (22) and (28) for the structure constans of L, one can write immediately the commutation relations which involve the J's. They are:

(34) $$[I_{kl}, J_{mn}] = g_{lm}J_{kn} - g_{km}J_{ln} + g_{kn}J_{lm} - g_{ln}J_{km}$$

(35) $$[I_{l4}, J_{mn}] = g_{lm}J_{n4} - g_{ln}J_{m4}$$

(36) $$[J_{l4}, I_{mn}] = g_{lm}J_{n4} - g_{ln}J_{m4}$$

(37) $$[J_{l4}, J_{n4}] = -J_{ln}$$

and

(38) $$[J_{kl}, J_{mn}] = 0$$

(39) $[J_{kl}, J_{m4}] = 0$

(40) $[J_{k4}, J_{l4}] = 0.$

The relations (31) — (40) determine the group locally. The subgroup with the operators I_{kl}, I_{k4} is locally isomorphic with the Lorentz group, as seen from (31)-(33), while the subgroup with the operators I_{kl}, J_{k4} is locally isomorphic with the Galilei group, as seen from (31), (36), (40).

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