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RESULTS ON SOME PLANE NETS

by

Pervin Yazgan(*)

Özet

Bu makalede, Σ-monojenik fonksiyonlardan [II] (**) bazılarının reel ve imaginer kısımları keyfî sabitlere eşitlenmek suretiyle elde edilen ortogonal düzlemsel ağlar ile ilgili iki teorem ispat edildi. Keza hidrodinamik ve elastisitede [I], [III] önemli olan bazı hususi eğriler incelenerek bulunan sonuçlar sıralandı.

Yazar, tezi veren ve onu değerli tavsiyeleri ile yöneten Prof. S. Süray'a teşekkür etmeyi borç bilir.

Summary

We have proved two theorems on the orthogonal plane nets which are derived from some Σ-monogenic functions [II] (***) by equating the real and imaginary parts of the functions to arbitrary constants. And we have also studied some special cases which are important for hydrodynamics and elastisy [I], [III].

The writer is deeply indebted to prof. S. Süray for his direction and advices and whishes to take this opportunity to express her gratitude.

RESULTS ON SOME PLANE NETS

İntroduction. İt ıs evident that the real and imaginary parts of an analytic funcition $U_{(x,y)} + iV_{(x,y)}$ of a complex

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** Köşeli parantez içindeki Romen rakamları çalışmanın sonundaki literatüre aittir.

*** The Roman numbers in brackets indicates the bibliography given as references at the end of the paper.

variable satisfy Cauchy-Riemann partial differential equations

$$U_x = V_y \, , \; U_y = - V_x$$

and that the family of the curves $U_{(x,y)} =$ const., $V_{(x,y)} =$ const. form an isometric net.

İn various branches of applied mechanics and particularly in hydrodynamics and elastisy, one frequently comes across with equations similar to Cauchy-Riemann equations as well as the Cauchy-Riemann equations themselves.

While trying to find the different solutions for these types of equations, Lipman Bers and Abe Gelbart have defined a class of functions, which they called Σ-monogenic, and have established their properties by using a method similar to that in the analytic functions. In their second paper they have built their theory on analytical basis while in the first they had established it from a practical point of view.

What we have done, in this paper, is to bring out some properties of the plane nets which are derived from some Σ-monogenic functions by equating the real and imaginary parts of these functions to constants.

İn the frist part of this paper, by considering an orthogonal system of co-ordinates $U_{(x,y)} =$ const., $V_{(x,y)} =$ const., a theorem, which states the necessary and sufficient conditions for the expression $U_{(x,y)} + iV_{(x,y)}$ to be a Σ-monogenic function, has been established.

İn the second part, a transformation has been defined so that a net derived from some Σ-monogenic functions can be trnasformed into an isometric net by means of it, and the result has been expressed as a second theorem.

İn the third part of this paper a special class of the plane nets which arise from the movement of a fluid in rotation has been examined and the results have been classified.

I. As can be seen easily, the necessary and sufficient conditions for a plane net $U_{(x,y)} =$ const., $V_{(x,y)} =$ const. to be an orthogonal net are.

$$(1) \qquad U_x = \lambda\ (x,y)\ V_y$$
$$U_y = -\lambda\ (x,y)\ V_x$$

where $\lambda(x,y)$ is an arbitrary function. In general, the net $U(x,y)=$const. $V(x,y)=$const., derived from the equations

$$(2) \qquad \sigma_1(x)\ U_x = \tau_1(y) V_y$$
$$\sigma_2(x)\ U_y = -\tau_2(y) V_x$$

which are the fundamental relations in defining Σ–monogenic functions, are not an orthogonal net. This net is orthogonal if and only if the equations (1) are satisfied, that is

$$(3) \qquad \frac{\tau_1(y)}{\tau_2(y)} = \frac{\sigma_1(x)}{\sigma_2(x)} = \text{const.}$$

We will consider the case in wich the arbitrary constant on the right hand side is one.

Let us consider the symbolic and auxiliary matrix

$$(4) \qquad \Sigma = \begin{Vmatrix} \sigma(x) & \tau(y) \\ \sigma(x) & \tau(y) \end{Vmatrix}$$

which corresponds to the equations

$$(5) \qquad \sigma(x)\ U_x = \tau(y)\ V_y$$
$$\sigma(x)\ U_y = -\tau(y)\ V_x$$

These equations are a special form of (2) and satisfy (1). We are also considering the net $U(x,y)$ =const., $V(x,)$=const. derived from (5). As it is known

$$(6) \qquad ds^2 = E(U,V)dU^2 + G(U,V)\ dV^2$$

represents the square of the element of arc with respect to an orthogonal system of co–ordinates $U(x,y)$=const., $V(x,y)$ =const. in a plane. We shall now state the following theorem.

Theorem. If the curves $U(x,y)$ =const.,$V(x,y)$ =const. form an orthogonal net the necessary and suffient condition for $U(x,y) + i\ V\ (x,y)$ to be a Σ–monogenic function of the form (4) is

$$(7) \qquad \frac{\partial^2}{\partial x \partial y} \left(\log \frac{E}{G}\right) = 0$$

Proof. The necessity of the condition can easily be verified. In fact, if $U(x,y) + i\, V(x,y)$ is a Σ –monogenic function of the form (4), it is obvious that the net $U(x,y) = \text{const.}$, $V(x,y) = \text{const.}$ is orthogonal, that is the element of arc can be expressed in the form (6) and that (7) is satisfied. We shall now prove the sufficiency of the condition. If

$$\frac{\partial^2}{\partial x\, \partial y} (\log \frac{E}{G}) = 0$$

then $\dfrac{E}{G} = \dfrac{A\,(x)}{B\,(y)}$. Since A (x) and B (y) are both positive we can substitute $\sigma^2(x)$ and $\tau^2(x)$ for A (x) and B(y) respectively where σ and $\tau$are arbitrary functions of their own arguments. If we substitute $\varphi^2\,(x,y)$ for E then

$$G = \varphi^2\,(x,y)\;\frac{\tau^2\,(y)}{\sigma^2\,(x)}$$

Thus

$$(8)\quad ds^2 = \frac{\varphi^2\,(x,y)}{\sigma^2\,(x)}\,[\sigma^2\,(x)\; dU^2 + \tau^2\,(y)\; dV^2]$$

Substituting $dU = U_x\, dx + U_y\, dy$ and $dV = V_x\, dx + V_y\, dy$ in (8) weget

$$(9)\ ds^2 = dx^2 + dy^2 = \frac{\varphi^2\,(x,y)}{\sigma^2(x)} \{[\sigma^2\,(x)U_x^2 + \tau^2(y)V_x^2]\, dx^2$$

$$+ [\sigma^2\,(x)\; U_x^2 + \tau^2\,(y)\; V_y^2]\, dy^2$$

$$+2\,[\sigma^2\,(x)U_xU_y + \tau^2\,(y)\; V_xV_y]\, dxdy\}$$

Equating the coeficients of dx^2 and dy^2 in both sides of this equality we get

$$(10)\qquad \sigma^2\,(x)\; U_x^2 + \tau^2\,(y)\; V_x^2 = \frac{\sigma^2\,(x)}{\varphi^2\,(x,y)}$$

$$(11) \qquad \sigma^2 (x)\, U_x^2 + \tau^2 (y)\, V_y^2 + \frac{\sigma^2 (x)}{\varphi^2 (x,y)}$$

$$(12) \qquad \sigma^2 (x)\, U_x U_y + \tau^2 (y)\, V_x V_y = 0$$

By calculating the value of V_x from (12) we have

$$V_x = - \frac{\sigma^2 (x)}{\tau^2 (y)} \cdot \frac{U_x U_y}{V_y}$$

Substituting thisvalue of V_x in (10) weobtain

$$U_x^2 (\tau^2 V_y^2 + \sigma^2 U_y^2) = \frac{\tau^2 V_y^2}{\varphi^2 (x,y)}$$

which is

$$(13) \qquad \sigma^2 (x)\, U_x^2 = \tau^2 (y)\, V_y^2 \text{ (by using (11))}$$

In a similar manner we can get

$$(14) \qquad \sigma^2 (x)\, U_y^2 = \tau^2 (y)\, V_x^2$$

from (10), (11) and (12). The conditions on U(x,y) and V(x,y) to satisfy (13) and (14) are

$$(15) \qquad \begin{aligned} \sigma(x)\, U_x &= \tau(y)\, V_y \\ \sigma(x)\, U_2 &= -\tau(y)\, V_x \end{aligned}$$

or

$$(16) \qquad \begin{aligned} \sigma(x)\, U_x &= -\tau(y)\, V_y \\ \sigma(x)\, U_y &= \tau (y)\, V_x \end{aligned}$$

But the above conditions also express the fact that U(x,y)+ i V(x,y) is a Σ –monogenic function of the variables (x+iy) and (x–iy) respectively. This completes the proof. From the above proceedings we obtain the relation

$$(17) \qquad \frac{ds_U}{ds_V} = \frac{\sigma (x)}{\tau (y)} \cdot \frac{dU}{dV}$$

where ds_U and ds_V are the elements of arc of the curves U(x,y) = const., V(x,y) = const. respectively.

* * *

2. Although the net whose $U(x,y) = \text{const.}$, $V(x,y) = \text{const.}$ curves derived from the equations

$$\begin{aligned} \sigma_1(x)\, U_x &= \tau_1(y)\, V_y \\ \sigma_2(x)\, U_2 &= -\tau_2(y)\, V_x \end{aligned} \tag{18}$$

is not, in general, an orthogonal net, the net derived from the equations

$$\begin{aligned} \sigma(x)\, U_x &= \tau(y)\, V_y \\ \frac{1}{\sigma(x)}\, U_y &= \frac{1}{\tau(y)}\, V_x \end{aligned} \tag{19}$$

which corresponds to the matrix

$$\left\| \begin{matrix} \sigma(x) & \tau(y) \\ \dfrac{1}{\sigma(x)} & \dfrac{1}{\tau(y)} \end{matrix} \right\| \tag{20}$$

can be transformed into an orthogonal net by a suitable trans- transformation. This is the subject of the following theorem.

Theorem. $\xi = \xi(x)$, $\eta = \eta(y)$ and $\tau(y)$, $\sigma(x)$ all being different form zero and infitiy, the transformations

$$\frac{d\xi}{dx} = \frac{1}{\sigma(x)}, \quad \frac{d\eta}{dy} = \frac{1}{\tau(y)} \tag{21}$$

transform the net $U(x,y) = \text{const.}$, $V(x,y) = \text{const.}$ derived from (19) into an isometric net in the ξ, η plane and conver sely any isometric net in the ξ, η plane can be transformed into the net $U(x,y) = \text{const.}$ $V(x,y) = \text{const.}$ in the x,y plane where $U(x,y)$ and $V(x,y)$ satisfiy the equations (19).

Proof. By applying the transformations (21) to the system (19) we get

$$\sigma(x)\, U_\xi \frac{d\xi}{dx} = \tau(y)\, V_\eta \frac{d\eta}{dy}$$

$$(22) \qquad \frac{1}{\sigma(x)} U_\eta \frac{d\eta}{dy} = -\frac{1}{\tau(y)} V_\xi \frac{d\xi}{dx}$$

and by substituting (21) in (22) it yields

$$U_\xi = V_\eta$$
$$U_\eta = - V_\xi$$

These equations show that the curves $U(\xi, \eta) =$ const. and $V(\xi, \eta) =$ const. form an isometric net. Conversely, if the net $U(\xi,\eta) =$ const, $V(\xi,\eta) =$ const is isometric the elements of arc is

$$ds^2 = \lambda (U,V) \quad (dU^2 + dV^2)$$

which implies that the functions $U (\xi \eta)$ and $V (\xi,\eta)$ must satisfy either

$$U_\xi = V_\eta$$
$$U_\eta = - V_\xi$$

or

$$U_\xi = - V_\eta$$
$$U_\eta = V_\xi$$

Application of the inverse tranformations to the above result in

$$\sigma (x) U_x = \tau (y) V_y$$

$$\frac{1}{\sigma (x)} U_y = -\frac{1}{\tau (y)} V_x$$

and

$$\sigma (x) U_x = - \tau (y) V_y$$

$$\frac{1}{\sigma (x)} U_y = \tau (y) V_x$$

respectively. The net which is obtained from these equations is the net mentioned in the theorem.

The method is illustrated by the following example.

Example. Let us consider the matrix

$$\Sigma = \begin{Vmatrix} \dfrac{1}{x} & y^2 \\ x & \dfrac{1}{y^2} \end{Vmatrix}$$

where $\sigma(x) = \dfrac{1}{x}$, $\tau(y) = y^2$, the transformations are $\xi = \dfrac{x^2}{2}$, $\eta = -\dfrac{1}{y}$. The integration constants have been omitted. Since they only result in the translation the co-ordinates which does not effect the results. By these transformations the domain in x, y plane bounded by the curves

$$x^2 + y^2 = 1, \; y = x, \; x = \frac{1}{4} \quad \text{(fig. 1)}$$

is transformed onto the domain in ξ, η plane bounded by the curves

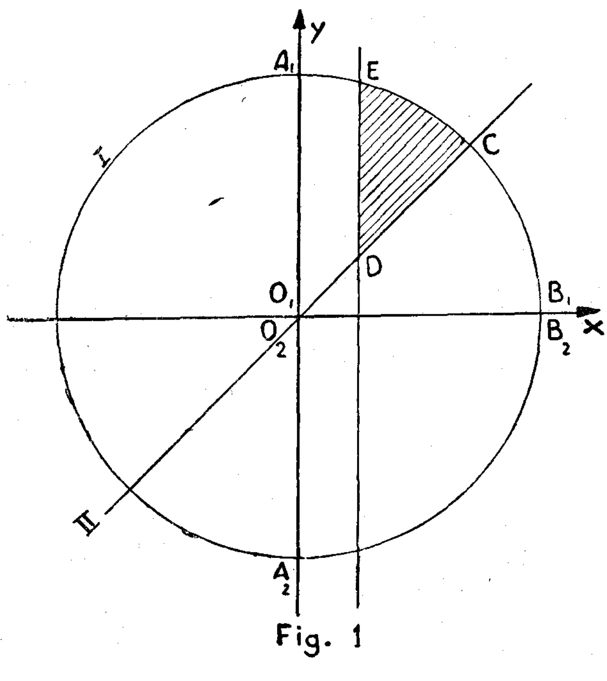

Fig. 1

$$\eta = \pm \frac{1}{\sqrt{1-2\xi}}, \quad \eta = + \frac{1}{\sqrt{2\xi}}, \; \xi = \frac{1}{32} \qquad \text{(fig. 2)}$$

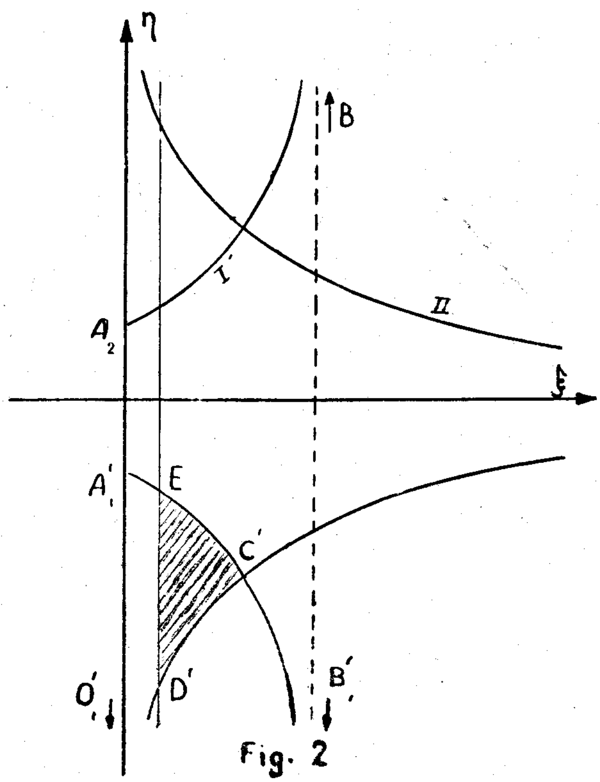

Fig. 2

Now, to every net U(x,y) = const., V(x,y) = const. inside the the domain ECD derived from (19) an isometric net can be made to correspond inside the domain E′C′D′. For example consider

$$U(x,y) = \frac{x^4}{4} - \frac{1}{y^2} \qquad V(x,y) = - \frac{x^2}{y}$$

These functions satisfy the equations (19). The net dervived from these functions is

$$U(x,y) = \frac{x^4}{4} - \frac{1}{y^2} = k \text{ or } \qquad y = \mp \frac{2}{\sqrt{x^4-4k}}$$

$$V(x,y) = -\frac{x^2}{y} = k \text{ or } \qquad x^2 = -ky$$

The variation of the curves $U(x,y) = k$ for $y > 0$ is as follows

x	$-\infty$		$-\sqrt{2\sqrt{k}}$		$\sqrt{2\sqrt{k}}$		$+\infty$
$\frac{dy}{dx}$		+	‖	//////	‖	−	
y	0	↗	$+\infty$		$+\infty$	↘	0

for $k \geq 0$

and

x	$-\infty$		0		$+\infty$
$\frac{dy}{dx}$		+	0	−	
y	0	↗	$\frac{1}{\sqrt{-k}}$	↘	

for k

the curves $V(x,y)=k$ are parabolas. If we trace thefamily of curves $U(x,y)=$ const., $V(x,y)=$ const. together we get the net corresponding to them (fig. 3).

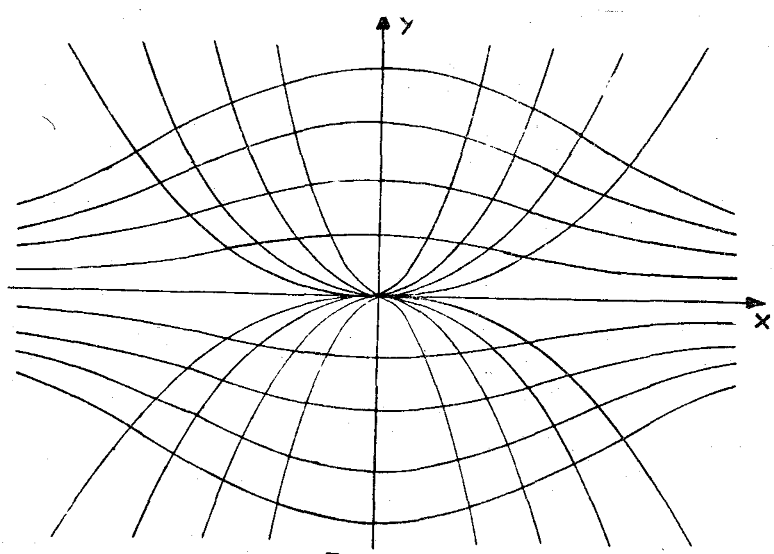

Fig. 3

On the other hand if the transformations $\xi = \frac{x^2}{2}, \eta = -\frac{1}{y}$ is applied to these functions we obtain

$$U(\xi, \eta) = \xi^2 - \eta^2$$
$$V(\xi, \eta) = 2\xi\eta$$

Thus the curves $U(\xi,\eta) = \xi^2 - \eta^2 = \text{const.}$, $V(\xi,n) = 2\xi\eta = \text{const.}$ represent two families of ortogonal hyperbolas. (fig.4) and the functions $U(\xi,\eta) = \xi^2 - \eta^2$ and $V(\xi,\eta) = 2\xi\eta$ satisfy the equation

$$U_\xi = V_\eta$$
$$U_\eta = -V_\xi$$

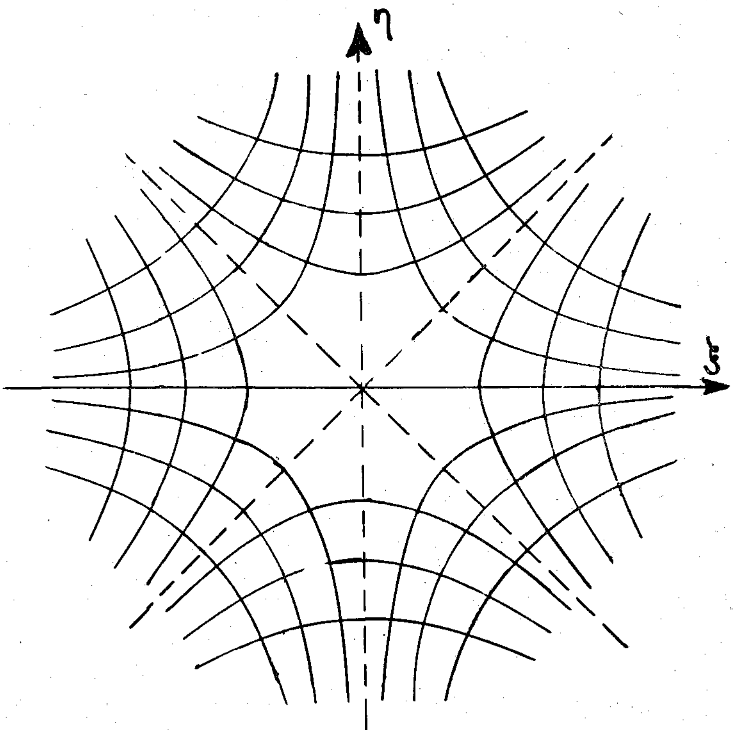

Fig. 4

After the transformations (21) have been applied to the Σ –monogenic functions of $z = x+iy$, Σ being

$$\left\| \begin{matrix} \sigma(x) & \tau(y) \\ \dfrac{1}{\sigma(x)} & \dfrac{1}{\tau(y)} \end{matrix} \right\|$$

tehese functions arise as analitic fonctions of $\zeta = \xi + i\eta$. İn fact, the existence of the Σ – derivative implies that the paritial derivaties U_ξ, V_ξ, U_η, V_η are continious. They satisfy Cauchy–Riemann equations. Hence the function $U(\xi, \eta) + iV(\xi, \eta)$ is an analytic function of $\zeta = \xi + i\eta$.

* * *

3. An interesting case which has a significance in physics is the case in which the net $U(x,y) = \text{const.}$, $V(x,y) = \text{const.}$ is orthgonal and the equations

$$\sigma_1(x)\, U_x = \tau_1(y)\, V_y$$
$$\sigma_2(x)\, U_y = -\tau_2(y)\, V_x$$

are of thefollowing special form

$$U_x = \frac{1}{y} V_y$$

(23)

$$U_y = -\frac{1}{y} V_x$$

In this case, as known, the curves $U(x,y) = \text{const.}$, $V(x,y) = \text{const.}$ represent equipotantial lines and stream lines of the symetrical movements of a fluid referred to an axis. Therefore study of the properties of the curves has a special importance. But such an examination represent great difficulty not only in the generol case but also in the special case (23). However, we have considered two kinds of formal powers of these functions and we have established some properties of the curves $U(x,y) = \text{const.}$, $V(x,y) = \text{const.}$ The formal powers considered are

$$Z^n = U + iV = r^n \; P_n \; (\cos\theta) + i \frac{r^{n-1}}{n+1} y^2 \; P_n' \; (\cos\theta)$$

$$Z^{(-n)} = U^* + i\,V^* = r^{-n} P_n \; (\cos\theta) - i \frac{r^{-n-2}}{n} y^2 \; P_n' \; (\cos\theta)$$

where the variables are polar co-ordinates and P_n and P_n' are the Legendere's polinomial of the n th degree and its derivative respectively. From $U(r,\theta) = k$ it follows that

(24) $$r_U = \left[\frac{k}{Pn(\cos\,\theta)} \right]^{\frac{1}{n}}$$

and

(25) $$\frac{dr_U}{d\theta} = \frac{1}{n} \quad \left(\frac{k}{P_n}\right)^{\frac{1}{n}} \quad \frac{\sin\,\theta\; P'_n\,(\cos\,\theta)}{P_n\,(\cos\,\theta)}$$

From $V(r,\;\theta) = 1$ it follows taht

(26) $$r_v = \left[\frac{1\;(n+1)}{\sin^2\theta\; P_n(\cos)} \right]^{\frac{1}{n+1}}$$

and

(27) $$\frac{dr_v}{d\theta} = -n \left[\frac{1\;(n+1)}{\sin^2\,\theta\; P_n} \right]^{\frac{1}{n+1}} \quad \frac{\sin\,\theta\; P_n}{(\sin^2\,\theta P_n')^{n+1}}$$

If the angles between the radius vectors and the tangents to the curves $U = k$ and $V = 1$ are α and β respectively then

$$(28) \quad \mathrm{Tan}\ \alpha = \frac{n\ P_n}{\sin\ \theta\ P_n'}, \quad (29) \quad \mathrm{Tan}\ \beta = -\frac{\sin\ \theta\ P_n'}{nP_n}$$

where k, l are arbitrary constants.

From (24) and (26) it can be seen that the curves $U = k$ are symmetrical with respect to Ox and Oy axes for the even values of n and $V = l$ are symmetrical with respect to Ox and Oy axes for the odd values of. n.

The curves $U = k$ and $V = l$ are symmetrical with respect to the Ox axis for odd and even values of n respectively. The curves which correspond to the negative values of constans are symmetries of the curves which are obtained for the pozitive values of constants.

The curves $U = k$, $V = l$ have noasymptotes passing through the origin. For the odd values of n, Oy axis is always one of the asymptotes of the curves $U = k$ and Ox axis is that of the curves $V = l$.

Combining the formulae (24), (25), (26), (27) we observe that the vertices of the curves $U = k$ are on the asymptotes of $V = k$ and vice versa.

From (28) and (29) we also observe that for fixed n and θ the values of tan α and tan β remain unchanged. Hence the curves $U = k$ and $V = l$ are homothetic with respect to the origin with the homothety ratios $\left[\frac{k_1}{k_2}\right]^{\frac{1}{n}}$ and $\left[\frac{l_1}{l_2}\right]^{\frac{1}{n+1}}$ respectively.

The element of arc in the U,V plane, where $U = U(r, \theta)$, $V = V(r, \theta)$ is

$$ds^2 = \frac{1}{r^{2n}\,(n^2\ P_n^2 + \sin^2\theta\, P_n'^2)}\,(r^2 \sin^2\ \theta\ dU^2 + dV^2)$$

and

$$\frac{E}{G} = r^2\ \sin^2\ \theta = y^2$$

For V = const. we have

$$ds_U = \frac{r \sin \theta \, dU}{r^n (n^2 P_n^2 + \sin^2 \theta P_n'^2)^{1/2}}$$

For U = const. we have

$$ds_V = \frac{dV}{r^n (n^2 P_n^2 + \sin^2 \theta P_n'^2)^{1/2}}$$

Hence

$$\frac{ds_V}{ds_U} = \frac{1}{y} \frac{dV}{dU}$$

Starting from $Z^{(-n)}$ ve have the curves $U^* = k$ and $V^* = 1$. We shall give similar results for these f curves For the curves $U^* = k$

(30) $$r_U^* = \left[\frac{P_n (\cos \theta)}{k} \right]^{\frac{1}{n+1}}$$

and

(31) $$\frac{dr_U^*}{d\theta} = - \frac{1}{(n+1)k} \left[\frac{k}{P_n \cos \theta} \right]^{\frac{n}{n+1}} P_n' (\cos \theta) \sin \theta$$

For the curves $V^* = 1$

(32) $$r_V^* = \left(\frac{\sin^2 \theta P_n'}{n+1} \right)^{\frac{1}{n}}$$

and

(33) $$\frac{dr_V^*}{d\theta} = (n+1) \left[\frac{n+1}{\sin^2 \theta P_n'} \right]^{\frac{n-1}{n}} \sin \theta \, P_n,$$

If γ, δ are the angles betveen the radius vectors and the tangents to the curves $U^* = k$, $V^* = 1$ respectively. Then

(34) $$\tan\gamma = -\frac{(n+1)\ P_n}{\sin\ \theta\ P_n'}$$

(35) $$\tan\ \delta = \frac{\sin\ \theta\ P_n'}{(n+1)\ P_n}$$

the curves $U^* = k$ and $V^* = 1$ are symmtrical with respect to Ox and Oy axes for the even and odd values of n respectively. The curves $U^* = k$ and $V^* = 1$ are symmetrical with respect to Ox axis for the odd and even values of n respectively. The curves which corrspond to the negative values of k and 1 are the symmetries of the curves obtained with respect to oy axis for the positive values of k and 1. The curves $U^* = k$, $V^* = 1$ are closed curves. They pass through the origin n times. The straight line $\theta =$ const. which makes $r_U^* = o$ pass through the vertices of the curves $V^* = 1$ and vice versa. From (34 and (35) it follows that the the curves $U^* = k$ and $V^* = 1$ are homothetic with respect to the origin with the homotyhety ratio

$$\left(\frac{k_1}{k_2}\right)^{\frac{1}{n+1}} \quad \text{and} \quad \left(\frac{l_1}{l_2}\right)^{\frac{1}{n}}$$

respectively. For the curves U=const., V=const., =const., $U^* =$ const., V^* = const. which pass through the same point, there are the relations

$$\tan\alpha.\tan\delta = \frac{n}{n+1}$$

$$\tan\beta.\tan\gamma = = \frac{n+1}{n}$$

which express that the product of these tangents at a point is independent from the co-ordinates of the point. On the other hand, if we make following substitutions

$$\text{Arg } Z^n = \Theta, \qquad \text{Arg } Z^{(-n)} = \Theta^*$$

and consider the relations

$$\tan \Theta = \frac{V}{U}, \quad \tan \Theta^* = \frac{V^*}{U^*}$$

then

$$\tan \Theta^* . \text{ Cotg } \Theta = \frac{n+1}{n}$$

that is at any point, Arg Z^{-n} : Arg Z^n is independent of the co-ordinates of the point.

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