ON THE Σ - MONOGENIC FUNCTIONS

By

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Özet: Bu yazıda "Σ - monojenik" adı verilen bir fonksiyon ailesine ait bir kısım tarifler her hangi bir kompleks fonksiyona teşmil edilmiş ve bunların bazı uygulaması yapılmıştır [1], [2] (**).

Yazı aşağıdaki dört paragrafdan teşekkül etmektedir:

1. Σ - Kısmı türevler.

Bir Σ - monojenik fonksiyon için tarif edilmiş bulunan Σ - türev kavramı her hangi bir kompleks fonksiyon için genişletilerek Σ - kısmı türev tarif edilmiştir.

2. Σ - İntegraller.

$z = x + iy$ ve $\bar{z} = x - iy$ ifadeleri göz önüne alarak bunlara tekabül eden Σ - integraller tarif edilmiş ve şekilleri verilmiştir.

3. Yeni bir operator.

Önceki paragrafta sözü geçen kompleks değişkenler için bilinmekte olan $\frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n}$ operatönün birinci paragrafta verilen tariflere dayanılarak Σ - kısmi türevlere tekabül eden şekli tesis edilmiş ve bazı sonuçlara varılmıştır.

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(**) Köşeli parantezler içindeki rakamlar yazının sonundaki bibliyografiye aittir.

4. Bir uygulama.

Reel ve sanal kısımları ikinci mertebeden iki kısmi diferensiyel denklemin çözümlerini veren u (x, y) + iv (x, y) gibi bir Σ – monojenik fonksiyon ailesinden hareket edilerek dördüncü mertebeden bir kısmi diferensiyel denklemin çkzümleri bulunmuştur.

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Summary: In this paper we shall be concerned with the extension of some definitions of the so – called Σ – monogenic functions for a complex function of two variables.

The paper contains the four following paragraphs:

1. Partial Σ – derivatives.

The concept of Σ – derivative for a Σ – monogenic function is generalized for any complex function of two variables.

2. Σ – Integrals.

By considering the complex expressions $z = x + i y$ and $\bar{z} = x - i y$, Σ – integrals of Σ – monogenic functions which depend on z or $\bar{z}$ are defined.

3. A new operator.

The known operator $\frac{\partial^{2n}}{\partial z^n \partial \bar{z}^n}$ is extended for Σ — partial derivatives and some results are obtained.

4. An application.

The real and imaginary parts of a Σ – monogenic function u = (x,y) + i v (x,y) which are solutions of two partial differential equations of the second order, are also used to form solutions of a partial differential equation of fourth order.

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1. Partial Σ – derivatives.

We shall begin by an extension of the definition of Σ – derivative for a special class of functions defined for the first time by L. Bers and A. Gelbart [1], [2] to a complex function of two variables. Let $F(z,\bar{z}) = P(x,y) + i\,Q(x,y)$ be a complex function with P and Q the real and imaginary parts respectively. We define the partial Σ – derivatives of $F(z,\bar{z})$ with respect to z and $\bar{z}$ as follows.

$$(1)\quad \frac{\partial_\Sigma F}{\partial_\Sigma z} = \frac{\partial_\Sigma}{\partial_\Sigma z}(P+iQ) = \frac{1}{2}\left(P_x + \frac{1}{y}Q_y\right) + \frac{i}{2}\left(Q_x - yP_y\right)$$

$$(2)\quad \frac{\partial_\Sigma F}{\partial_\Sigma \bar{z}} = \frac{\partial_\Sigma}{\partial_\Sigma \bar{z}}(P+iQ) = \frac{1}{2}\left(P_x - \frac{1}{y}Q_y\right) + \frac{i}{2}\left(Q_x + yP_y\right)$$

Two immediate results follow from these definitions :

If $\dfrac{\partial_\Sigma F}{\partial_\Sigma \bar{z}} = 0$, then

$$(3)\qquad P_x = \frac{1}{y}Q_y, \qquad P_y = -\frac{1}{y}Q_x .$$

Thus $F = P + i\,Q$ is a Σ – monogenic function of the variable z.

Similarly if $\dfrac{\partial_\Sigma F}{\partial_\Sigma z} = 0$, then

$$(4)\qquad P_x = -\frac{1}{y}Q_y, \quad P_y = \frac{1}{y}Q_x ,$$

that is $F = P + i\,Q$ is a Σ – monogenic function of the variable $\bar{z}$.

It is easy to show that the elimination of either P or Q from (3) or (4) leads to the same pair of following partial differential equations of second order.

$$(5)\qquad P_{xx} + P_{yy} + \frac{1}{y}P_y = 0,$$

(6) $$Q_{xx} + Q_{yy} - \frac{1}{y} Q_y = 0.$$

These equations occur in Hydrodynamics, P and Q being the velocity potential and Stokes' stream function of an incompressible fluid with axial symmetry.

2. Σ – Integrals.

Let $F = P + i\,Q$ be a Σ – monogenic function of the variable z satisfying Relation (3) . Setting

$$P^* = \int_{z_0}^{z} Pdx - \frac{1}{y} Qdy\ , \qquad Q^* = \int_{z_0}^{z} Q\,dx + yPdy,$$

the comlex expression

$$F^* = P^* + i\ Q^* = \int_{z_0}^{z} Pdx - \frac{1}{y} Qdy + i \int_{z_0}^{z} Qdx + yPdy$$

is called Σ – integral of the Σ – monogenic function F. We shall write compactly

$$F^* = \int_{z_0}^{z} F\ d_{\Sigma} z.$$

Similarly, if $F = P + i\,Q$ is a Σ – monogenic function of the variable $\bar{z}$ which satifies Relations (4), we define its Σ – integral by the same way and write

$$F^{**} = P^{**} + i\ Q^{**} = \int_{\bar{z}_0}^{\bar{z}} F\ d_{\Sigma}\bar{z}$$

where

$$P^{**} = \int_{\bar z_0}^{\bar z} Pdx + \frac{1}{y} Qdy, \quad Q^{**} = \int_{\bar z_0}^{\bar z} Qdx - y\,Pdy$$

It is readily shown that F^* and F^{**} are Σ – monogenic functions of z and $\bar z$ respectively.

3. The operator

$$\partial^{2n}_{\Sigma} / \partial_{\Sigma} z^n \, \partial_{\Sigma} \bar z^n \, .$$

Using the above definitions of partial Σ – derivatives for a complex function $F = P + i\,Q$, we have

$$\frac{\partial^2_{\Sigma} F}{\partial_{\Sigma} z\, \partial_{\Sigma} \bar z} = \frac{\partial^2_{\Sigma} F}{\partial_{\Sigma} \bar z \partial_{\Sigma} z} = \frac{\partial_{\Sigma}}{\partial_{\Sigma} z} \left\{ \frac{1}{2} \left(P_x - \frac{1}{y} Q_y \right) + \frac{i}{2} \left(Q_x + y\, P_y \right) \right\}$$

$$= \frac{\partial_{\Sigma}}{\partial_{\Sigma} \bar z} \left\{ \frac{1}{2} \left(P_x + \frac{1}{y} Q_y \right) + \frac{i}{2} \left(Q_x - y\, P_y \right) \right\}$$

$$= \frac{1}{4} \left\{ P_{xx} + P_{yy} + \frac{1}{y} P_y + i \left(Q_{xx} + Q_{yy} - \frac{1}{y} Q_y \right) \right\} .$$

Repeating the same operation n times we obtain the general formula

$$\frac{\partial^{2n}_{\Sigma} F}{\partial_{\Sigma} z^n \, \partial_{\Sigma} \bar z^n} = \frac{1}{2^{2n}} \left\{ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} \right)^n P + \right.$$

$$\left. i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y} \frac{\partial}{\partial y} \right)^n Q \right\} .$$

If $\dfrac{\partial_{\Sigma}^{2n} F}{\partial_{\Sigma} z^n \, \partial_{\Sigma} \bar{z}^n} = 0$, then we have

(7) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{y} \frac{\partial}{\partial y} \right)^n P = 0,$$

(8) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y} \frac{\partial}{\partial y} \right)^n Q = 0.$$

As we have mentioned in paragraph 1, when $n = 1$ we get the well known pair of equations of Hydrodtnamics. When $n = 2$, Relation (8) becomes a partial differential equation of fourth order which occur also in Hydrodynamics, Q being the stream function for a creeping flow in the axially symmetric case [3].

4. Solutions of the equation

(9) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{y} \frac{\partial}{\partial y} \right)^2 Q = 0.$$

Let u (x,y) and v (x,y) be solutions of (5) and (6) respectively. We shall try to find solutions of (9) of the form x^m v(x,y) and y^m u (x,y) Substituting x^m v (x,y) in (9), we have

$$m(m-1)(m-2)(m-3)\, x^{m-4} v + 4\, m\, (m-1)(m-2)\, x^{m-3} v_x + 4\, m\, (m-1)\, x^{m-2} v_{xx} = 0.$$

This is satisfied by $m = 1$, thus x.v (x,y) is a solution of (9).

Repeating a similar calculation with the expression y^m u (x,y), we find that Equation (9) is satisfied if we take $m = 2$. Then we can can write

(10) $$Q = y^2 u(x,y) + x\, v(x,y)\,.$$

It should be remarqued that the functions u (x,y) and v (x,y) in (10) are not nesscessarily the real and imaginary parts of a Σ – monogenic function, but they are submitted only to be solutions of the equations (5) and (6) respectively.

By means of Σ – monogenic functions L. Bers and A. Gelbart formed remarquable expressions in polar wcoordinates for u (x,y) and v (x,y) [1], we shall use some of them in (10). In polar coordinates $x = \varrho \cos \theta$, $y = \varrho \sin \theta$ Equation (9) takes the form

$$(11) \quad \left\{ \frac{\partial^2}{\partial \varrho^2} + \frac{\sin \theta}{\varrho^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right\}^2 Q(\varrho, \theta) = 0.$$

We consider first the Σ – monogenic function [1]

$$(12) \qquad Z^{(n)} = \varrho^n P_n (\cos \theta) + \frac{i}{n+1} \varrho^{n-1} P_n' (\cos \theta)$$

where n is a pozitif integer, P_n and P'_n represent the known Legendre polynomial and its derivative. In (12) $u = \varrho^n P_n (\cos \theta)$,

$$v = \frac{\varrho^{n-1}}{n+1} P' (\cos \theta), \text{ then}$$

$$y^2 u = \varrho^{n+2} \sin^2 \theta P_n (\cos \theta),$$

$$x v = \frac{\varrho^n}{n+1} \cos \theta P_n' (\cos \theta)$$

and

$$Q = \varrho \sin^2 \theta P_n (\cos \theta) + \frac{\varrho^n}{n+1} \cos \theta P_n' (\cos \theta)$$

are the solutions of (9) or (11).

Taking now the Σ – monogenic function

$$Z^{(-n)} = \varrho^{-n-1} P_n \;(\cos\theta) - i \frac{\varrho^{-n} \sin^2\theta}{n} P_n' \;(\cos\theta),$$

we see that

$$y^2 u = \varrho^{-n+1} \sin^2\theta \; P_n (\cos\theta),$$

$$x\, v = -\frac{\varrho^{-n+1}}{n} \sin^2\theta \cos\theta \; P_n' \;(\cos\theta)$$

and

$$Q = \varrho^{-n+1} \sin^2\theta \; P_n \;(\cos\theta) - \frac{\varrho^{-n+1}}{n} \sin^2\theta \cos\theta \; P_n' (\cos\theta)$$

are also solutions of (9) or (11).

Finally we consider the form of the Σ – monogenic exponential and trigonometric functions written in rectangular coordinates

$$E\;(\alpha, z) = e^{\alpha x} \left[J_0\;(\alpha y) + i\; y\; J_1\;(\alpha y) \right],$$
$$S\;(\alpha, z) = \sin \alpha\, x\, J_0 (i\alpha y) + y \cos \alpha x\, J_1 (i\alpha y),$$
$$C\;(\alpha, z) = \cos \alpha\, x\, J_0 (i\alpha y) - y \sin \alpha\, x\, J_1 (i\alpha y)$$

where J_0, J_1 are the Bessel functions and α is a real parameter. From (13) we obtain for the equation (9) the following solutions

$$y^2 u = y^2 e^{\alpha x} J_0 (\alpha y), \qquad x\, v = x\, y\, e^{\alpha x} J_1 (\alpha y);$$
$$y^2 u = y^2 \sin \alpha\, x\, J_0 (i\alpha y), \qquad x\, v = i\, x\, y \cos \alpha\, x\, J_1 (i\alpha y);$$
$$y^2 u = y^2 \cos \alpha\, x\, J_0 (i\alpha y), \qquad x\, v = i\, x\, y \sin \alpha\, x\, J_1 (i\alpha y)$$

and therefore

$$Q = y\, e^{\alpha x} \left[y\, J_0 (\alpha y) + x\, J_1 (\alpha y) \right],$$
$$Q = y \left[y \sin \alpha x\, J_0 (i\alpha y) + i\, x \cos \alpha\, x\, J_1 (i\alpha y) \right],$$
$$Q = y \left[y \cos \alpha\, x\, J_0 (i\alpha y) - i\, x \sin \alpha\, x\, J_1 (i\alpha y) \right].$$

The preceeing solutions of the equations (9) or (11) can be used for some boundary value problems in Hydrodtnamics.

References

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