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On the Generalized Tricomi's Equation

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The relation between the Σ - monogenic functions and the solutions of the elliptic partial differential equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{k}{y} \frac{\partial U}{\partial y} = 0$$

is known [6]; the object of this paper is to investigate the properties of the *generalized Tricomi's Equation*

$$(*) \qquad \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^{\,2}} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) U = 0$$

and its solutions which will be called as Σ - harmonic functions. Our study is composed of three sections.

In section I we have investigated some properties of the solutions of the equation (*).

In section II the well - known Lord Kelvin's theorem and Almansi's expansion theorem have been established for Σ-harmonic and Σ-polyharmonic functions.

In section III we have proved Brill's theorem for Σ - polyharmonic functions which was previously given for the harmonic functions.

INTRODUCTION

Some partial differential equations may have different properties in the plane. As an example, the equation

$$(1) \qquad y \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

is elliptic for $y > 0$, parabolic for $y = 0$, and hyperbolic for $y < 0$. In the case of elliptic type, putting

$$\xi = \frac{2}{3} y^{3/2}, \qquad \eta = x$$

we transform equation (1) into

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{1}{3\xi} \frac{\partial U}{\partial \xi} = 0 \tag{2}$$

which is known as Tricomi's equation [1].

The equation (2) and it's various general forms have been the bases of many researches; it will be explained whenever we will refer to it.

The object of this paper is to investigate the equation

$$\frac{\partial^2 U}{\partial x_1^2} + \dots + \frac{\partial^2 U}{\partial x_n^2} + \frac{k_1}{x_1} \frac{\partial U}{\partial x_1} + \dots + \frac{k_n}{x_n} \frac{\partial U}{\partial x_n} = 0$$

or briefly

$$\sum_{i=1}^{n} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) = 0 \tag{3}$$

which will be called as "*Generalized Tricomi's Equation*" and its solutions, "*Σ - harmonic functions*" [2], [3], [4], where U is $U = U(x_1, x_2, \dots, x_n)$ and k_i are the constants which are called the *indices* of the equation.

I. SOME PROPERTIES OF THE SOLUTIONS OF EQUATION (3)

a) **Correspondence principle**: Let us consider the following transformation.

$$U(x_1, x_2, \dots, x_n) = U_1(x_1, x_2, \dots, x_n) \prod_{j=l}^{n} x_j^{1-k_j} \tag{4}$$

Differentiating (4) we have

$$\frac{\partial U}{\partial x_i} = \frac{\partial U_1}{\partial x_i} \prod_{j=l}^{n} x_j^{1-k_j} \quad ; \quad i < l$$

$$\frac{\partial U^2}{\partial x_i^2} = \frac{\partial^2 U}{\partial x_i^2} \prod_{j=l}^{n} x_j^{1-k_j} \quad ; \quad i < l$$

$$\frac{\partial U}{\partial x_i} = \left(\frac{\partial U_1}{\partial x_i} + \frac{1-k_i}{x_i} U_1 \right) \prod_{j=l}^{n} x_j^{1-k_j} \quad ; \quad i \geqq l$$

$$\frac{\partial^2 U}{\partial x_i^2} = \left[\frac{\partial^2 U}{\partial x_i^2} + \frac{2(1-k_i)}{x_i} \frac{\partial U_1}{\partial x_j} + \frac{(1-k_i)^2}{x_i^2} U_1 - \frac{1-k_i}{x_i^2} U_1 \right] \prod_{j=l}^{n} x_j^{1-k_j} \quad ; \quad i \geqq l$$

and so

$$\sum_{i=1}^{n} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) = \left[\sum_{i=1}^{l-1} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U_1}{\partial x_i} \right) + \sum_{i=l}^{n} \left(\frac{\partial^2 U_1}{\partial x_i^2} + \frac{2-k_i}{x_i} \frac{\partial U_1}{\partial x_i} \right) \right] \prod_{j=l}^{n} x_j^{1-k_j}$$

or

$$\sum_{i=1}^{n} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) = \left[\sum_{i=1}^{n} \left(\frac{\partial^2 U_1}{\partial x_i^2} + \frac{p_i}{x_i} \frac{\partial U_1}{\partial x_i} \right) \right] \prod_{j=1}^{n} x_j^{1-k_j}$$

where

$$p_i = \begin{cases} k_i \; ; \; i < l \\ 2 - k_i \; ; \; i \geqq l \; . \end{cases}$$

It is obvious that, if the function $U\ (x_1, x_2, \ldots, x_n)$ is a solution of the equation (3), then $U_1\ (x_1, x_2, \ldots, x_n)$ is a solution of the equation with the indices $k_1, k_2, \ldots, k_{l-1}, 2 - k_l, \ldots, 2 - k_n$. This property is expressed by

(5) $U\{k_1, k_2, \ldots k_n\} = U_1\{k_1, k_2, \ldots, k_{l-1}, 2-k_l, \ldots, 2-k_n\} \prod_{j=l}^{n} x_j^{1-k_j}$

The property that will be obtained has been previously used for very special types of Tricomi's equation [5], [6],[7]. First let us prove the uniqueness of the factor $\prod_{j=l}^{n} x_j^{1-k_j}$ in the transformation (4). In fact, if we consider

$$U(x_1, x_2, \ldots, x_n) = U_1(x_1, x_2, \ldots, x_n) \prod_{j=l}^{n} x_j^{m_j}$$

instead of (4), then we obtain

$$\sum_{i=1}^{n} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) = \left[\sum_{i=1}^{l-1} \left(\frac{\partial^2 U_1}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U_1}{\partial x_i} \right) + \right.$$

$$\left. + \sum_{i=l}^{n} \left(\frac{\partial^2 U_1}{\partial x_i^2} + \frac{k_i + 2m_i}{x_i} \frac{\partial U_1}{\partial x_i} \right) + \sum_{i=l}^{n} \frac{m_i (k_i + m_i - 1)}{x_i^2} U_1 \right] \prod_{j=l}^{n} x_j^{m_j}$$

If one requires the proceeding equation to be a Tricomi's equation then the condition

$$m_i (k_i + m_i - 1) = 0$$

must be satisfied. As $m_i = 0$ does not give anything new we conclude that

$$m_i = 1 - k_i$$

This proves the uniqueness of the factor $\prod_{j=l}^{n} x_j^{1-k_j}$ of (4).

Conclusion: If some of the indices of the equation (3), for example $k_1, k_2, \ldots, k_{l-1}$ are equal to zero, (5) can be written in the form

(6) $U\{0,0,\ldots,0,k_l, \ldots, k_n\} = U_1\{0,0,\ldots,0,2-k_l,\ldots,2-k_n\} \prod_{j=l}^{n} x_j^{1-k_j}$.

In addition to this, considering $k_l = k_{l+1} = \ldots = k_n = 2$ we get

$$U\{0,0,\ldots,0,2,\ldots,2\} = U_1\{0,0,\ldots,0\} \prod_{j=l}^{n} x_j^{-1} .$$

So the following statement is established:

If $V(x_1, x_2, \ldots, x_n)$ is a solution of Laplace's equation then

$$U(x_1, x_2, \ldots, x_n) = V(x_1, x_2, \ldots, x_n) \prod_{j=l}^{n} x_j^{-1}$$ *is a solution of the equation*

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \ldots + \frac{\partial^2 U}{\partial x_n^2} + \frac{2}{x_l}\frac{\partial U}{\partial x_l} + \ldots + \frac{2}{x_n}\frac{\partial U}{\partial x_n} = 0$$

b) A remarkable solution: Now we look for a solution of the form

$$U(x_1, x_2, \ldots, x_n) = r^m$$

where

$$r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$$

and m is a constant to be determined. Using the first and second derivatives of U, we obtain

$$\sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i}\frac{\partial}{\partial x_i}\right)U = m(m-2+n+\sum_{i=1}^{n} k_j)r^{m-2} = 0 .$$

In order that r^m to be a solution, neglecting the case $m = 0$ which gives a constant, we find the condition

$$m = -(n - 2 + \sum_{i=1}^{n} k_i) .$$

That is, the equation (3) has a solution of the form

$$U = r^{-(n-2+\sum_{i=1}^{n} k_i)} ; \tag{7}$$

in other words (7) is a Σ-harmonic function with n variables.

The solution (7) of the equation (3) has the following properties:

1. If all the indices k_i are equal to zero, then this solution is reduced to the solution of the Laplace's equation with n independent variables.

2. Also the solution (7) is the generalized form of the solution of Tricomi's equation which was given by Weinstein in the case of two variables [6]. In fact, if we take $n = 2$ $k_1 \neq 0$, $k_2 = 0$ then we obtain $U = r^{-k_1}$.

3. If the indices k_i vary such that the sum $\sum_{i=1}^{n} k_i$ is constant, the solution (7) does not change at all.

The last property may take an important part in establishing the generalized potential theory for the equation (3) which was first studied by Weinstein.

c) **Solutions by the method of separation of variables:** Let the equation (3) be in the form

$$\frac{\partial^2 U}{\partial x_1^2} + \ldots + \frac{\partial^2 U}{\partial x_n^2} + \frac{k_l}{x_l}\frac{\partial U}{\partial x_l} + \ldots + \frac{k_n}{x_n}\frac{\partial U}{\partial x_n} = 0 . \tag{3'}$$

It is clear that choosing the equation in this form we will not spoil the generality of the problem, and the reason to chose it so can be easily understood in the following. To find a solution of the equation (3′) in the form

$$U = X_1(x_1) \,.\, X_2(x_2) \,. \,\ldots\, . \, X_n(x_n)$$

replacing U in the equation (3′), we have

$$\frac{X_1''}{X_1} + \ldots + \frac{X_n''}{X_n} + \frac{k_l}{x_l}\frac{X_l'}{X_l} + \ldots + \frac{k_n}{x_n}\frac{X_n'}{X_n} = 0 . \tag{8}$$

In order to find solutions of (3′) which satisfy (8), we must have

$$
(9)\quad \begin{cases}
\dfrac{X_1''}{X_1} = \lambda_1 \\
\dots\dots\dots\dots \\
\dfrac{X''_{l-1}}{X_{l-1}} = \lambda_{l-1} \\
\dfrac{X_l''}{X_l} + \dfrac{k_l}{x_l}\,\dfrac{X_l'}{X_l} = \lambda_l \\
\dots\dots\dots\dots\dots\dots \\
\dfrac{X''_n}{X_n} + \dfrac{k_n}{x_n}\,\dfrac{X'_n}{X_n} = \lambda_n
\end{cases}
$$

where λ_i denote constants. In the case of

$$\sum_{i=1}^{n} \lambda_i = 0$$

the equation (8) and the system (9) are equivalent. So we can not expect that all of these constants to be positive or negative; suppose that

$$\lambda_1, \lambda_2, \dots, \lambda_i > 0$$

$$\lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_{l-1} < 0 .$$

$$\lambda_j = a_j^2$$

we get

$$X_j'' = a_j^2 X_j$$

or

$$X_j = A_j e^{a_j x_j} + A_j' e^{-a_j x_j} \quad ; 1 \leqq j \leqq i .$$

On the other hand if

$$\lambda_k = -\beta^2_k$$

we get

$$X_k'' = -\beta^2_k X_k$$

or

$$X_k = B_k \cos \beta_k x_k + B'_k \sin \beta_k x_k \; ; \; i+1 \leqq k \leqq l-1 .$$

A_j, A_j' ; B_k, B'_k are arbitrary constants. Also it is necessary to find the solution of the equation with the index $l \leqq \nu \leqq n$ of the system (9). These equations can only be integrated by using the power series in a neighbourhood of a point. The solutions will depend on the variables x_ν, indices k_ν and parameters λ_ν. As it is known the solution X_ν is of the form

$$X_\nu = C_\nu \; Y_1 (x_\nu \; ; \; \lambda_\nu, k_\nu) + C_\nu' \; Y_2 (x_\nu \; ; \; \lambda_\nu, \; k_\nu)$$

where the functions Y_1 and Y_2 are the solutions of

$$\frac{X''_\nu}{X_\nu} + \frac{k_\nu}{x_\nu} \frac{X'_\nu}{X_\nu} - \lambda \nu = 0$$

in the form of power series and $C\nu$, $C'\nu$ are arbitrary constants.

So the required solution using the method of separation of variables is

$$(10) \quad U = \prod_{s=1}^{i} \{A_s \, e^{\alpha_s x_s} + A_s' \, e^{-\alpha_s x_s}\} \, . \prod_{s=i+1}^{l-1} \{ B_s \cos \beta_s x_s$$

$$+ B_s' \sin \beta_s x_s \} \, . \prod_{s=l}^{n} \{ C_s Y_1 (x_s ; \lambda_s , k_s) + C_s' Y_2 (x_s ; \lambda_s , k_s) \} \, .$$

We will not investigate the different forms of the equation (10) in the view of different indices.

The method of solution by separation of variables may imply a method of solution of the following type of generalized Tricomi's equation.

$$(11) \quad \frac{\partial^2 U}{\partial x_1^{\,2}} + \ldots + \frac{\partial^2 U}{\partial x_n^{\,2}} + \frac{k_l}{x_l} \frac{\partial U}{\partial x_l} + \ldots + \frac{k_n}{x_n} \frac{\partial U}{\partial x_n} = c \, U \, .$$

In fact, in this case we obtain

$$\frac{X_1''}{X_1} + \ldots + \frac{X_n''}{X_n} + \frac{k_l}{x_l} \frac{X_l'}{X_l} + \ldots + \frac{k_n}{x_n} \frac{X_n'}{X_n} = \lambda_1 + \ldots + \lambda_n$$

from the system (9). It is evident that the solution of system (9) with the condition

$$\sum_{i=1}^{n} \lambda_i = c$$

is the general solution of (11).

d) Homogeneous solutions: To find a solution of the equation (3) of the form

$$U = f(z_\rho),$$

where

$$z_\rho = \frac{x_1^\rho + x_2^\rho + \ldots + x_{n-1}^\rho}{x_n^\rho}$$

and ϱ is a real number; substituting U in (3) we obtain

$$\text{(12)} \qquad \varrho\,(z_{2\rho-2} + z_\rho^2)\, f''(z_\rho) + \Big[(\varrho-1)\, z_{\rho-2} + \frac{k_1 x_1^{\rho-2} + \ldots + k_{n-1} x_{n-1}^{\rho-2}}{x_n^{\rho-2}} + (\varrho+1-k_n)\, z_\rho \Big] f'(z_\rho) = 0.$$

One of the properties of the last equation is that it can be reduced to an ordinary differential equation for $\varrho = 2$. In fact, putting $\varrho = 2$ we have

$$\text{(13)} \quad 2\,(z_2 + z_2^2)\, f''(z_2) + [\, z_0 + (k_1 + k_2 + \ldots + k_{n-1}) + (3-k_n)\, z_2 \,]\, f'(z_2) = 0$$

or

$$\text{(14)} \quad 2\, z_2\,(1 + z_2)\, f''(z_2) + [\, (n-1) + (k_1 + k_2 + \ldots + k_{n-1}) + (3-k_n)\, z_2]\, f'(z_2) = 0 \,.$$

(14) is an ordinary differential equation of second-order; for every solution f of this equation, (3) has a solution of the form

$$U = f(z_2) = f\left(\frac{x_1^2 + \ldots + x_{n-1}^2}{x_n^2} \right).$$

Another solution of (14) can be obtained in the case of two independent variables for $\varrho = 1$. In order to have an ordinary differential equation from (12), the indices $k_1, k_2, \ldots, k_{n-1}$ must necessarily vanish, but in the case of two independent variables we can have a solution for arbitrary indices when $\varrho = 1$; for,

$$z_1 = \frac{x_1}{x_2}$$

and we have from (12)

$$z_1 (1+z_1^2) f'' (z_1) + [k_1 + (2-k_2) z_1^2] f' (z_1) = 0 .$$

It is obvious that the solution which is obtained for $\varrho = 2$ is also valid in the case of two independent variables.

e) **Product of Solutions :** Let

$$U_1 = U_1 (x_1, x_2, \ldots, x_n) , \quad U_2 = U_2 (x_1, x_2, \ldots, x_n)$$

be the two solutions of the equation (3). The following property will be proved:

If U_1, U_2 are the two solutions of the equation (3), *the necessary and sufficient condition that the product $V = U_1 . U_2$ be a solution of the same equation is that U_1 = constant, U_2 = constant be orthogonal.*

In fact substituting the derivatives of V in (3) we get

$$(15) \quad \sum_{i=1}^{n} \left(\frac{\partial^2 V}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial V}{\partial x_i} \right) = \sum_{i=1}^{n} \left[U_2 \left(\frac{\partial^2 U_1}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U_1}{\partial x_i} \right) + U_1 \left(\frac{\partial^2 U_2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U_2}{\partial x_i} \right) + 2 \frac{\partial U_1}{\partial x_i} \frac{\partial U_2}{\partial x_i} \right] .$$

Since U_1, U_2 are solutions by hypothesis, if V is also a solution we must have

$$\sum_{i=1}^{n} \frac{\partial U_1}{\partial x_i} . \frac{\partial U_2}{\partial x_i} = 0,$$

that is, $U_1 = constant$, $U_2 = constant$ are orthogonal. These are the necessary condition of the proof. The sufficient condition can be seen from (15).

II. LORD KELVIN'S THEOREM AND ALMANSI'S EXPANSION FOR Σ - HARMONIC FUNCTIONS

a) Lord Kelvin's theorem: It is known that if $U(x_1, x_2, \ldots, x_n)$ is a harmonic function, the function V which is defined as

$$(16) \qquad V(x_1, x_2, \ldots, x_n) = r^{-(n-2)} \; U\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \ldots, \frac{x_n}{r^2}\right),$$

$$r^2 = \sum_{i=1}^{n} x_i^2$$

is also a harmonic function [9], [10]. The relation (16) denotes Lord Kelvin's theorem. In the following this theorem will be extended to the Σ - harmonic functions.

Choosing a function V of the form

$$V(x_1, x_2, \ldots, x_n) = r^m \; U(\xi_1, \xi_2, \ldots, \xi_n)$$

where

$$\xi_i = \frac{x_i}{r^2}$$

and let

$$\Delta\Sigma = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i}\right)$$

and $U = U(x_1, x_2, \ldots, x_n)$ be a Σ - harmonic function, then V must satisfy $\Delta\Sigma \, V = 0$. Now our problem is reduced to determine the power m of the function $V = r^m U$. Calculating the derivatives we obtain

$$\frac{\partial V}{\partial x_i} = mr^{m-2} x_i U + r^m \left\{ \sum_{j=1}^{n} \frac{\partial U}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} \right\},$$

$$\frac{\partial^2 V}{\partial x_i^2} = m(m-2) r^{m-4} x_i^2 U + mr^{m-2} U$$

$$+ 2 mr^{m-2} x_i \left\{ \sum_{j=1}^{n} \frac{\partial U}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_i} \right\}$$

$$+ r^m \left\{ \sum_{j,k=1}^{n} \frac{\partial_2 U}{\partial \xi_j \partial \xi_k} \frac{\partial \xi_j}{\partial x_i} \frac{\partial \xi_k}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial U}{\partial \xi_j} \frac{\partial^2 \xi_j}{\partial x_i^2} \right\};$$

$$\sum_{i+1}^{n} \left(\frac{\partial^2 V}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial V}{\partial x_i} \right) = r^m \left\{ \sum_{j=1}^{n} \frac{\partial^2 U}{\partial \xi_j^2} \left[\sum_{i=1}^{n} \left(\frac{\partial \xi_j}{\partial x_i} \right)^2 \right] \right.$$

$$+ \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \frac{\partial^2 U}{\partial \xi_j \partial \xi_k} \left[\sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i} \frac{\partial \xi_k}{\partial x_i} \right]$$

$$\left. + \sum_{j=1}^{n} \frac{\partial U}{\partial \xi_j} \left[\sum_{i=1}^{n} \left(\frac{\partial^2 \xi_j}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial \xi_j}{\partial x_i} \right) \right] \right\}$$

$$+ 2 m r^{m-2} \left\{ \sum_{j=1}^{n} \frac{\partial U}{\partial \xi_j} \left[\sum_{i=1}^{n} x_i \frac{\partial \xi_j}{\partial x_i} \right] \right\}$$

$$+ m \left(m-2+n+ \sum_{i=1}^{n} k_i \right) r^{m-2} U = 0 .$$

The derivatives of ξ_i give the following values:

$$\sum_{i=1}^{n} \left(\frac{\partial \xi_i}{\partial x_i} \right)^2 = r^{-4}, \quad \sum_{i=1}^{n} \frac{\partial \xi_j}{\partial x_i} \frac{\partial \xi_k}{\partial x_i} = 0, \quad \sum_{i=1}^{n} x_i \frac{\partial \xi_j}{\partial x_i} = -x_j r^{-2},$$

$$\sum_{i=1}^{n} \left(\frac{\partial^2 \xi_j}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial \xi_j}{\partial x_i} \right) = -2 \left(n-2 + \sum_{i=1}^{n} k_i \right) x_j r^{-4} + \frac{k_j}{x_j} r^{-2} .$$

Substituting these in the above equation we have

$$\sum_{i=1}^{n} \left(\frac{\partial^2 V}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial V}{\partial x_i} \right) = r^{m-4} \left\{ \sum_{j=1}^{n} \left(\frac{\partial^2 U}{\partial \xi_j^2} + \frac{k_i}{\xi_j} \frac{\partial U}{\partial \xi_j} \right) \right.$$

$$- 2 \left(n-2+ \sum_{i=1}^{n} k_i\right) \left[\sum_{j=1}^{n} x_j \frac{\partial U}{\partial \xi_j} \right] - 2m \left[\sum_{j=1}^{n} x_j \frac{\partial U}{\partial \xi_j} \right] \Big\}$$

$$+ m \left(m-2+\mathrm{n}+ \sum_{i=1}^{n} k_i \right) r^{m-2} \; U = 0$$

or

$$(17) \quad \Delta_\Sigma V = \left(m-2+n+ \sum_{i=1}^{n} k_i\right) r^{m-2} \left\{m \, U-2 \sum_{j=1}^{n} \xi_j \frac{\partial U}{\partial \xi_j} \right\} = 0 \; .$$

As we can not impose further conditions on U,

$$m = - \left(n-2+ \sum_{i=1}^{n} k_i\right)$$

is deduced. So we have established Lord Kelvin's theorem for Σ - harmonic functions in the form

$$V(x_1, x_2, \ldots, x_n) = r^{-(n-2+ \sum_{i=1}^{n} k_i)} \; U\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \ldots, \frac{x_n}{r^2}\right).$$

It is obvious that in the case where all k_i are equal to zero, then Lord Kelvin's theorem (16) for the harmonic functions is obtained.

b) Lord Kelvin's theorem for Σ - polyharmonic functions: Lord Kelvin's theorem has been extended for polyharmonic functions of order p [9]. The same method which we used to extend this theorem to the Σ- harmonic functions can be applied here to extend it to Σ- polyharmonic functions of order p. When $U = U(x_1, x_2, \ldots, x_n)$ is a Σ - polyharmonic function,

$$\left\{ \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \right\}^{p} U = 0$$

or

$$\Delta_{\Sigma}^{p} U = 0$$

where p is a positive integer. Appling $\Delta\Sigma$ to both sides of (17) p—1 times successively we can only get a result in the form of (17) if we can prove that $\left\{ m-2 \sum_{i=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \right\} U$ is a Σ - harmonic function. For this purpose the following lemma is given.

LEMMA:

$$\Delta_{\Sigma}^{p} \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right)^m = \left(2p + \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right)^m \Delta_{\Sigma}^{p} .$$

Proof: First let $p = 1$, $m = 1$.

$$(18) \quad \Delta\Sigma \left(x_j \frac{\partial}{\partial x_j} \right) = \left\{ \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \right\} \left(x_j \frac{\partial}{\partial x_j} \right)$$

$$= \left(x_j \frac{\partial}{\partial x_j} \right) \left\{ \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \right\} + \left(\frac{\partial^2}{\partial x_j} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) \left(x_j \frac{\partial}{\partial x_j} \right)$$

We obtain with an easy calculation

$$\frac{\partial}{\partial x_j} \left(x_j \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial x_j} + x_j \frac{\partial^2}{\partial x_j^2} ; \frac{\partial^2}{\partial x_j^2} \left(x_j \frac{\partial}{\partial x_j} \right) = 2 \frac{\partial^2}{\partial x_j^2} + x_j \frac{\partial^3}{\partial x_j^3}$$

and

$$\left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) \left(x_j \frac{\partial}{\partial x_j} \right) = 2 \frac{\partial^2}{\partial x_j^2} + x_j \frac{\partial^3}{\partial x_j^3} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} +$$

$$+ k_j \frac{\partial^2}{\partial x_j^2} = 2 \frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} + x_j \left(\frac{\partial^3}{\partial x_j^3} + \frac{k_j}{x_j} \frac{\partial^2}{\partial x_j^2} \right).$$

On the other hand, we can write

$$\frac{\partial}{\partial x_j} \left(\frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) = \frac{k_j}{x_j} \frac{\partial^2}{\partial x_j^2} - \frac{k_j}{x_j^2} \frac{\partial}{\partial x_j}$$

or

$$\frac{k_j}{x_j} \frac{\partial^2}{\partial x_j^2} = \frac{\partial}{\partial x_j} \left(\frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) + \frac{k_j}{x_j^2} \frac{\partial}{\partial x_j};$$

that is

$$x_j \left(\frac{\partial^3}{\partial x_j^3} + \frac{k_j}{x_j} \frac{\partial^2}{\partial x_j^2} \right) = x_j \frac{\partial}{\partial x_j} \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) + \frac{k_j}{x_j} \frac{\partial}{\partial x_j}$$

or

$$\left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) \left(x_j \frac{\partial}{\partial x_j} \right) = 2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right)$$

$$+ x_j \frac{\partial}{\partial x_j} \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right).$$

Substituting this in (18) we get

$$\Delta \Sigma \left(x_j \frac{\partial}{\partial x_j} \right) = \left(x_j \frac{\partial}{\partial x_j} \right) \left\{ \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \right\}$$

$$+ 2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) = \left(x_j \frac{\partial}{\partial x_j} \right) \Delta \Sigma + 2 \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right).$$

If we take the summation on j we obtain

$$\Delta \Sigma \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right) = 2 \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{k^2}{x_j} \frac{\partial}{\partial x_j} \right) + \left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right) \Delta \Sigma$$

or

$$\Delta\Sigma\left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right) = \left(2+\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right)\Delta\Sigma$$

which proves our lemma for $p = 1$, $m = 1$.

In order to complete the proof it is sufficient to apply the operators $\Delta\Sigma$ p–1 times and $\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$ m—1 times; indeed

$$\Delta\Sigma^{p}\left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right) = \left(2p+\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right)\Delta\Sigma^{p}$$

and

$$\Delta\Sigma^{p}\left(\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right)^{m} = \left(2p+\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}\right)^{m}\Delta\Sigma^{p} .$$

When U is a Σ – harmonic function with necessary continuity conditions

$$\left\{ m - 2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \right\} U$$

is also Σ – harmonic since

$$\Delta\Sigma \left\{m—2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j}\right\} U = m\,\Delta\Sigma\, U — 2\,\Delta\Sigma\left(\sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j}\right) U$$

$$= m\,\Delta\Sigma\, U — 2\left(2+\sum_{j=1}^{n} \xi_i \frac{\partial}{\partial \xi_j}\right)\Delta\Sigma\; U$$

is an identity

Now the operator $\Delta\Sigma$ can be applied successively to (17) in order to obtain a Σ – polyharmonic function of order p.

$$\Delta\Sigma^{2}\;[r^{m}\, U] = \left(m—2+n+\sum_{i=1}^{n} k_j\right)\Delta\Sigma\;\left[r^{m-2}\left\{m-2\sum_{j=1}^{n}\xi_j\frac{\partial}{\partial \xi_j}\right\}U\right]$$

$$= (m-2+n+\sum_{i=1}^{n} k_i)\,(m-4+n+\sum_{i=1}^{n} k_i)\, r^{m-4} \{ (m-2)$$

$$-2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \} \{ m-2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \} U .$$

It is evident by the lemma that

$$\{ (m-2) -2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \} \{ m-2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \} U$$

is Σ - harmonic. Proceeding in this manner, one obtains

$$(19) \qquad \Delta_{\Sigma}^{p} [r^{m} U] = r^{m-2p} \quad \{ \prod_{i=1}^{p} (m-2\,i+n+ \sum_{i=1}^{n}) \} \; \vartheta\,(U)$$

where

$$\vartheta\,(U) = [\prod_{i=1}^{p} \{ m-2\,(p-i) -2 \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial \xi_j} \}] \quad U \quad .$$

If $V = r^{m} U$ is required to be a Σ - harmonic function we must have

$$\prod_{i=1}^{p} (m-2\;i+n+ \sum_{i=1}^{n} k_i) = 0$$

or

$$m-2\,i+n+ \sum_{i=1}^{n} k_i = 0 \; .$$

So if the power m is chosen as

$$m = -(n-2i+ \sum_{i=1}^{n} k_i\,),\; i \leqq \text{p}$$

then V is a Σ - polyharmonic function of order p. Here the most important term is

$$m = -(n-2\,p+ \sum_{i=1}^{n} k_i) \; .$$

Hence it is proved that Lord Kelvin's theorem for Σ – polyharmonic functions with n independent variables has the form

$$V = r^{-(n-2p+\sum_{i=1}^{n} k_i)} \, U\left(\frac{x_1}{r^2}, \ldots, \frac{x_n}{r^2}\right) .$$

c) Almansi's expansion for Σ – polyharmonic functions: It is well-known that E. Almansi had given an expansion of polyharmonic functions in terms of harmonic functions [11]. In this paragraph a similar expansion for Σ – polyharmonic functions will be given.

Let $U = U(x_1, x_2, \ldots, x_n)$ be a Σ–harmonic function, that is

$$\Delta_\Sigma \, U = 0 ;$$

and let

(20) $$V(x_1, x_2, \ldots, x_n) = r^m \, U(x_1, x_2, \ldots, x_n)$$

be a function which satisfies the equation

(21) $$\overset{p}{\Delta}_\Sigma \, V = \overset{p}{\Delta}_\Sigma \, (r^m U) = 0 .$$

Thus the problem is reduced to find a number m which fits the condition (21). Using derivatives of (20) we have from (21) for $p = 1$ and $p = 2$ respectively

$$\Delta_\Sigma \, [r^m U] = m r^{m-2} \left\{ \left(m - 2 + n + \sum_{i=1}^{n} k_i\right) + 2 \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right\} U$$

and

$$\overset{2}{\Delta}_\Sigma \, [r^m U] = m(m-2) \, r^{m-4} \left\{ \left(m - 4 + n + \sum_{i=1}^{n} k_i\right) \right.$$

$$\left. + 2 \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right\} \left\{ \left(m - 2 + n + \sum_{i=1}^{n} k_i\right) + 2 \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right\} U.$$

Proceeding in this manner and after necessary simplifications one obtains

$$(22) \qquad \Delta \overset{2}{\Sigma} [r^m U] = r^{m-2p} \left\{ \prod_{i=0}^{p-1} (m-2i) \right\} \left\{ \prod_{i=0}^{p-1} \left[m-2(p-i) + n + \sum_{i=1}^{n} k_i + 2 \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right] \right\} U = 0 .$$

This relation gives

$$\prod_{i=0}^{p-1} (m-2\,i) = 0$$

or

$$m = 2\,i \; ; \qquad i = 0, 1, \ldots, p-1 .$$

Thus, the equation (21) is satisfied for $m = 0, 2, \ldots, 2p - 2$; that is a Σ– polyharmonic function V of order p can be expressed in the form

$$(23) \qquad V = U_0 + r^2 U_1 + \ldots + r^{2p-2} U_{p-1}$$

or briefly

$$V = \sum_{i=1}^{p-1} r^{2i} \; U_i$$

where $U_0, U_1, \ldots, U_{p-1}$ are Σ – harmonic functions.

The expansion (23) which was established for Σ– polyharmonic functions is Almansi's expansion for the polyharmonic functions when U_i $(i = 0,1,\ldots, p-1)$ are harmonic.

III. THE EXTENSION OF BATEMAN'S AND BRILL'S THEOREMS TO Σ – HARMONIC FUNCTIONS

a) **Bateman's theorem:** H. Bateman had given the following transformation for the harmonic functions in n independent variables in his book [4] which will be referred here as Bateman's theorem.

$$(24) \qquad (x_1 + ix_2)^{1-n/2} \; F \left[\frac{r^2 - a^2}{2\,(x_1 + ix_2)}, \frac{r^2 + a^2}{2\,i\,(x_1 + ix_2)}, \frac{ax_3}{x_1 + ix_2}, \ldots, \frac{ax_n}{x_1 + ix_2} \right]$$

is a harmonic function if $F\,(x_1, x_2, \ldots, x_n)$ *is harmonic*

$\left(r^2 = \sum_{j=1}^{n} x_j^2,\; i = \sqrt{-1},\; a = \text{real constant}\right).$

In this paragraph we want to determine m such that

$$(25) \qquad V = (x_1 + ix_2)^m \; U \left[\frac{r^2 - a^2}{2\,(x_1 + ix_2)}, \frac{r^2 + a^2}{2\,i\,(x_1 + ix_2)}, \frac{ax_3}{x_1 + ix_2}, \ldots, \frac{ax_n}{x_1 + ix_2} \right]$$

be a solution of generalized Tricomi's equation *with* $k_1 = k_2 = 0$, when U is a solution of the same equation.

Let $\xi_1, \xi_2, \ldots, \xi_n$ denote the variables in the brackets in (25):

$$(26) \qquad V = (x_1 + ix_2)^m \; U\,(\xi_1, \xi_2, \ldots, \xi_n) \;.$$

Replacing V in the equation $\Delta_\Sigma\, U = 0$ and dividing into two parts as follows

$$\Lambda_1\,(V) = \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2}$$

$$\Lambda_2\,(V) = \sum_{j=3}^{n} \left(\frac{\partial^2 V}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial V}{\partial x_j} \right),$$

then we can write

$$\Delta_\Sigma\, V = \Lambda_1\,(V) + \Lambda_2\,(V)$$

Applying the Bateman's transformation first to the part $\Lambda_1\ (V)$ and calculating the derivatives

$$\frac{\partial V}{\partial x_1}, \quad \frac{\partial V}{\partial x_2}, \quad \frac{\partial^2 V}{\partial x_1^2}, \quad \frac{\partial^2 V}{\partial x_2^2}$$

from (26) with respect to $\xi_1, \xi_2, \ldots, \xi_n$ and substituting these in $\Lambda_1\ (V)$, we obtain

$$\Lambda_1 (V) = (x_1 + ix_2)^m \left\{ \sum_{j=1}^{n} \frac{\partial_2 U}{\partial \xi_j^2} \left[\left(\frac{\partial \xi_j}{\partial x_1}\right)^2 + \left(\frac{\partial \xi_j}{\partial x_2}\right)^2 \right] \right.$$

$$+ \sum_{\substack{j,l=1 \\ j \neq l}}^{n} \frac{\partial^2 U}{\partial \xi_j \partial \xi_l} \left[\frac{\partial \xi_j}{\partial x_1} \frac{\partial \xi_l}{\partial x_1} + \frac{\partial \xi_j}{\partial x_2} \frac{\partial \xi_l}{\partial x_2} \right] + \sum_{j=1}^{n} \frac{\partial U}{\partial \xi_j}$$

$$\left. \left[\frac{\partial^2 \xi_j}{\partial x_1^2} + \frac{\partial^2 \xi_j}{\partial x_2^2} \right] \right\} + 2m\,(x_1 + ix_2)^{m-1} \sum_{j=1}^{n} \left(\frac{\partial \xi_j}{\partial x_1} + i \frac{\partial \xi_j}{\partial x_2} \right) \frac{\partial U}{\partial \xi_j}$$

Taking into notice that $\xi_\nu = \dfrac{a\, x_\nu}{x_1 + ix_2}$; $v \geqq 3$ and deriving it with respect to x_1 and x_2 we get

$$\frac{\partial \xi_\nu}{\partial x_1} = - \frac{a\, x_\nu}{(x_1 + ix^2)^2} ; \quad \frac{\partial \xi_\nu}{\partial x^2} = - \frac{iax_\nu}{(x_1 + ix_2)^2}, \quad (\nu \geqq 3)$$

and

$$\frac{\partial^2 \xi_\nu}{\partial x_1^2} = \frac{2\, a\, x_\nu}{(x_1 + ix_2)_2} ; \quad \frac{\partial^2 \xi_\nu}{\partial x_2^2} = - \frac{2\, a\, x_\nu}{(x_1 + ix_2)^2}, \quad (\nu \geqq 3)$$

Using these derivatives we have

$$(27) \begin{cases} \left(\dfrac{\partial \xi_j}{\partial x_1}\right)^2 + \left(\dfrac{\partial \xi_j}{\partial x_2}\right)^2 = 0, (j \geqq 3) ; \dfrac{\partial \xi_j}{\partial x_1} \dfrac{\partial \xi_l}{\partial x_1} + \dfrac{\partial \xi_j}{\partial x_2} \dfrac{\partial \xi_l}{\partial x_2} = 0, (j,l \geqq 3) \\ \dfrac{\partial^2 \xi_j}{\partial x_1^2} + \dfrac{\partial^2 \xi_j}{\partial x_2^2} = 0,\ (j \geqq 3) ; \dfrac{\partial \xi_j}{\partial x_1} + i \dfrac{\partial \xi_j}{\partial x_2} = 0, (j \geqq 3). \end{cases}$$

These equations simplify the form of $\Delta_1 (V)$.

ξ_1 and ξ_2 are the form of

$$\xi_1 = \frac{r^2 - a^2}{2(x_1 + ix_2)}, \quad \xi_2 = \frac{r^2 + a^2}{2i(x_1 + ix_2)} = -\frac{i(r^2 + a^2)}{2(x_1 + ix_2)};$$

we get

$$\frac{\partial \xi_1}{\partial x_1} = -\frac{1}{2} - \frac{x_3^2 + \ldots + x_n^2 - a^2}{2(x_1 + ix_2)^2} \quad \frac{\partial^2 \xi_1}{\partial x_1^2} = -\frac{x_3^2 + \ldots + x_n^2 - a^2}{(x_1 + ix_2)^3}$$

$$\frac{\partial \xi_1}{\partial x_2} = -\frac{i}{2} - \frac{i(x_3^2 + \ldots + x_n^2 - a^2)}{2(x_1 + ix_2)^2}; \quad \frac{\partial^2 \xi_1}{\partial x_2^2} = -\frac{x_3^2 + \ldots + x_n^2 - a^2}{(x_1 + ix_2)^3}$$

$$\frac{\partial \xi_2}{\partial x_1} = -\frac{i}{2} + \frac{i(x_3^2 + \ldots + x_n^2 - a^2)}{2(x_1 + ix_2)^2}; \quad \frac{\partial^2 \xi_2}{\partial x_1^2} = -\frac{i(x_3^2 + \ldots + x_n^2 - a^2}{(x_1 + ix_2)^3}$$

$$\frac{\partial \xi_2}{\partial x_2} = -\frac{1}{2} - \frac{x_3^2 + \ldots + x_n^2 - a^2}{2(x_1 + ix_2)^2}; \quad \frac{\partial^2 \xi_2}{\partial x_2^2} = -\frac{i(x_3^2 + \ldots + x_n^2 - a^2)}{(x_1 + ix_2)^3}$$

The above derivatives give the following results:

$$(28) \quad \left\{ \begin{array}{l} \left(\frac{\partial \xi_1}{\partial x_1}\right)^2 + \left(\frac{\partial \xi_1}{\partial x_2}\right) = -\frac{x_3^2 + \ldots + x_n^2 - a^2}{(x_1 + ix_2)^2}; \\ \left(\frac{\partial \xi_2}{\partial x_1}\right)^2 + \left(\frac{\partial \xi_2}{\partial x_2}\right)^2 = \frac{x_2^2 + \ldots + x_n^2 + a^2}{(x_1 + ix_2)^2}; \\ \frac{\partial x_1}{\partial x_1} \frac{\partial x_2}{\partial x_1} + \frac{\partial x_1}{\partial x_2} \frac{\partial x_2}{\partial x_2} = i \frac{x_2. + \ldots + x_n.}{(x_1 + ix_2)^2}; \\ \frac{\partial^2 \xi_1}{\partial x_1^2} + \frac{\partial^2 \xi_2}{\partial x_2^2} = 0; \quad \frac{\partial^2 \xi_2}{\partial x_1^2} + \frac{\partial^2 \xi_2}{\partial x_2^2} = 0; \\ \frac{\partial \xi_1}{\partial x_1} + i \frac{\partial \xi_1}{\partial x_2} = 1; \quad \frac{\partial \xi_2}{\partial x_1} + i \frac{\partial \xi_2}{\partial x_2} = -i. \end{array} \right.$$

By (27) and (28) $\Lambda_1(V)$ takes the form

$$\Lambda_1(V) = (x_1 + ix_2)^m \{ - \frac{x_3^2 + \ldots + x_n^2 - a^2}{(x_1+ix_2)^2} \frac{\partial^2 U}{\partial \xi_1^2} + \frac{x_3^2 + \ldots + x_n^2 + a^2}{(x_1+ix_2)^2} \frac{\partial^2 U}{\partial \xi_2^2} + 2i \frac{x_3^2 + \ldots + x_n^2}{(x_1+ix_2)^2} \frac{\partial^2 U}{\partial \xi_1 \partial \xi_2} \} + 2m(x_1+ix_2)^{m-1} \left(\frac{\partial U}{\partial \xi_1} - i \frac{\partial U}{\partial \xi_2} \right).$$

Applying the transformation on $\Lambda_2(V)$ and substituting the derivatives of the function V we obtain

$$(29) \quad \Lambda_2(V) = (x_1 + ix_2)^m \{ \frac{\partial^2 U}{\partial \xi_1^2} [\sum_{j=3}^{n} \left(\frac{\partial^2 U}{\partial x_j} \right)^2] + \frac{\partial^2 U}{\partial \xi_2^2} [\sum_{j=3}^{n} \left(\frac{\partial \xi_2}{\partial x_j} \right)^2] + \sum_{l=3}^{n} \frac{\partial^2 U}{\partial \xi_l^2} [\sum_{j=3}^{n} \left(\frac{\partial \xi_l}{\partial x_j} \right)^2]$$

$$+ 2 \frac{\partial^2 U}{\partial \xi_1 \partial \xi_2} [\sum_{j=3}^{n} \frac{\partial \xi_1}{\partial x_j} \frac{\partial \xi_2}{\partial x_j}]$$

$$+ \sum_{\substack{l,h=3 \\ l \neq h}}^{n} \frac{\partial^2 U}{\partial \xi_l \partial \xi_h} [\sum_{j=3}^{n} \frac{\partial \xi_l}{\partial x_j} \frac{\partial \xi_h}{\partial x_j}]$$

$$+ \frac{\partial U}{\partial \xi_1} [\sum_{j=3}^{n} \left(\frac{\partial^2 \xi_1}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial \xi_1}{\partial x_j} \right)]$$

$$+ \frac{\partial U}{\partial \xi_2} [\sum_{j=3}^{n} \left(\frac{\partial^2 \xi_2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial \xi_2}{\partial x_j} \right)]$$

$$+ \sum_{l=3}^{n} \frac{\partial U}{\partial \xi_l} [\sum_{l=3}^{n} \left(\frac{\partial^2 \xi_l}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial \xi_l}{\partial x_j} \right)] \}.$$

On the other hand we get the following results with the derivatives of ξ_j with respect to x_j, keeping in view that

$$\xi_j = \frac{a\, x_j}{x_1 + i x_2}, \; j \geqq 3 .$$

$$(30) \left\{ \begin{aligned}
& \sum_{j=3}^{n} \left(\frac{\partial \xi_1}{\partial x_j}\right)^2 = \frac{x_3^2 + \dots x_n^2}{(x_1 + i x_2)^2}, \; \sum_{j=3}^{n} \left(\frac{\partial \xi_1}{\partial x_j}\right)^2 = - \frac{x_3^2 + \dots + x_n^2}{(x_1 + i\, x_2)^2}, \\
& \sum_{j=3}^{n} \left(\frac{\partial \xi_l}{\partial x_j}\right)^2 = \frac{a^2}{(x_1 + i x_2)^2}; \; (l \geqq 3) \\
& \sum_{j=3}^{n} \frac{\partial \xi_1}{\partial x_j} \frac{\partial \xi_2}{\partial x_j} = -i \frac{x_3^2 + \dots + x_n^2}{(x_1 + i x_2)^2}, \; \sum_{j=3}^{n} \frac{\partial \xi_l}{\partial x_j} \frac{\partial \xi_h}{\partial x_j} = 0, \; (l,h \geqq 3); \\
& \sum_{j=3}^{n} \left(\frac{\partial^2 \xi_1}{\partial x_j^2} + \frac{k_j}{x_j} \cdot \frac{\partial \xi_1}{\partial x_j}\right) = \sum_{j=3}^{n} \frac{1 + k_j}{x_1 + i x_2} = \frac{n - 2 + \sum_{j=3}^{n} k_j}{x_1 + i x_2}, \\
& \sum_{j=3}^{n} \left(\frac{\partial^2 \xi_2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial \xi_2}{\partial x_j}\right) = -i \sum_{j=3}^{n} \frac{1 + k_j}{x_1 + i x_2} = -i \frac{n - 2 + \sum_{j=3}^{n} k_j}{x_1 + i x_2}, \\
& \sum_{j=3}^{n} \left(\frac{\partial^2 \xi_l}{\partial x_j} + \frac{k_j}{x_j} \frac{\partial \xi_l}{\partial x_j}\right) = \frac{k_l}{x_l} \frac{a}{x_1 + i x_2} .
\end{aligned} \right.$$

Using (30), $\Lambda_2\,(V)$ takes the form

$$\Lambda_2\,(V) = (x_1 + i x_2)^m \Big\{ \frac{a^2}{(x_1 + i x_2)^2} \sum_{j=3}^{n} \left(\frac{\partial^2 U}{\partial \xi_j^2} + \frac{k_j}{x_j} \frac{\partial U}{\partial \xi_j} \right)$$

$$+ \frac{x_3^2 + \dots + x_n^2}{(x_1 + i x_2)^2} \frac{\partial^2 U}{\partial \xi_1^2}$$

$$- \frac{x_3^2 + \dots + x_n^2}{(x_1 + i x_2)^2} \frac{\partial^2 U}{\partial \xi_2^2} - 2\, i \frac{x_3^2 + \dots + x_n^2}{(x_1 + i x_2)^2} \frac{\partial^2 U}{\partial \xi_1 \partial \xi_2} +$$

$$+ \frac{n - 2 + \sum_{j=3}^{n} k_j}{x_1 + ix_2} \left(\frac{\partial U}{\partial \xi_1} - i \frac{\partial U}{\partial \xi_2} \right) \}$$

and after necessary simplifications we find for the sum

$$\Delta_\Sigma V = \Lambda_1 (V) + \Lambda_2 (V)$$

(31) $$\Delta_\Sigma V = (x_1 + ix_2)^m \left\{ \frac{a^2}{(x_1 + ix_2)^2} \Delta_\Sigma U \right.$$

$$\left. + \frac{2m - 2 + n + \sum_{j=3}^{n} k_j}{x_1 + ix_2} \left(\frac{\partial U}{\partial \xi_1} - i \frac{\partial U}{\partial \xi_2} \right) \right\}.$$

It is obvious that, if U is a solution of generalized Tricomi's equation (with $k_1 = k_2 = 0$) and if V is required to be a solution of the same equation we have

(32) $$\Delta_\Sigma V = \Delta_\Sigma \left[(x_1 + ix_2)^m U \right]$$

$$= (2m - 2 + n + \sum_{j=3}^{n} k_j) (x_1 + ix_2)^{m-1} \left(\frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \right) U = 0$$

which gives us the condition

$$2m - 2 + n + \sum_{j=3}^{n} k_j = 0$$

or

$$m = 1 - \frac{n + \sum_{j=3}^{n} k_j}{2}.$$

So, Bateman's theorem for Σ- harmonic functions (with the condition $k_1 = k_2 = 0$) takes the form

$$V = (x_1 + ix_2)^{1 - \frac{n + \sum_{j=3}^{n} k_j}{2}} \; U \left[\frac{r^2 - a^2}{2\,(x_1 + ix_2)}, \right.$$

$$\left. \frac{r^2 + a^2}{2\,i\,(x_1 + ix_2)}, \frac{a\,x_3}{x_1 + ix_2}, \ldots, \frac{ax_n}{x_1 + ix_2} \right].$$

b) Bateman's theorem for Σ– polyharmonic functions: The function

$$U_1 = \left(\frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \right) U$$

in (32) is also a Σ– harmonic function; for it is evident that

$$\Delta\Sigma \left(\frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \right) = \left(\frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \right) \Delta\Sigma .$$

Then applying the operator $\Delta\Sigma$ $p - 1$ times successively to the left hand side of (32) we get

$$\overset{p}{\Delta\Sigma} \left[(x_1 + ix_2)^m \, U \right] = \left\{ \prod_{\nu=1}^{p} \left[2\,(m - \nu) \right.\right.$$

$$\left.\left. + n + \sum_{j=3}^{n} k_j \right] \right\} (x_1 + i\,x_2)^{m-p} \left(\frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \right)^{(p)} U .$$

This expression must be zero if the function $(x_1 + i\,x_2)^m\, U$ is required to be a Σ–polyharmonic function of order p, that is

$$\left\{ \prod_{\nu=1}^{p} \left[2\,(m - \nu) + n \right.\right.$$

$$\left.\left. + \sum_{j=3}^{n} k_j \right] \right\} (x_1 + ix_2)^{m-p} \left(\frac{\partial}{\partial \xi_1} - i \frac{\partial}{\partial \xi_2} \right)^{(p)} U = 0 .$$

So we get the result

$$\prod_{\nu=1}^{p} \left[2(m-\nu) + n + \sum_{j=3}^{n} k_j \right] = 0$$

or

$$m = \nu - \frac{n + \sum_{j=3}^{n} k_j}{2}, \qquad \nu = 1, 2, \ldots, p.$$

A Σ–polyharmonic function of order q is always a Σ – polyharmonic function of order $q + 1$; so the power in the Σ– polyharmonic function $(x_1 + x)^m U$ which plays the most important part is

$$m = p - \frac{n + \sum_{j=3}^{n} k_j}{2}.$$

That is, Bateman's theorem for Σ– polyharmonic functions of order p has the following form:

$$(33)\quad V = (x_1 + ix_2)^{p - \frac{n + \sum_{j=3}^{n} k_j}{2}} U \left[\frac{r^2 - a^2}{2(x_1 + ix_2)}, \frac{r^2 + a^2}{2i(x_1 + ix_2)}, \frac{a\,x_3}{x_1 + ix_2}, \ldots, \frac{a\,x_n}{x_1 + ix_2} \right].$$

Remark: It is clear that in the case of $p = 1$ and $k_1 = k_2 = \ldots = kn = 0$ (33) is Bateman's theorem for harmonic functions; and in the case of $p = 1$ we have the same theorem for Σ – harmonic functions. If all the indices vanish, then we get the Bateman's theorem for polyharmonic functions.

c) Brill's theorem for Σ – harmonic functions: H. Bateman has used the theorem which we dealt with in the last paragraph to prove the following property, called Brill's theorem.

Replacing $x_1 + ix_2 = t$, $x_1 - ix_2 = s$ then the Laplace's equation which is satisfied by $F = F(x_1, x_2, \ldots, x_n)$ takes the the form

$$(34) \qquad 4 \frac{\partial^2 F}{\partial s \, \partial t} + \frac{\partial^2 F}{\partial x_3^2} + \ldots + \frac{\partial^2 F}{\partial x_n^2} = 0 \, .$$

If $R^2 = x_3^2 + x_4^2 + \ldots + x_n^2$ *and if* $F(s, t, x_3, \ldots, x_n)$ *is a solution of (34) then by Bateman's theorem*

$$(35) \qquad t^{1-n/2} \, F \left(\frac{st + R^2}{t}, -\frac{a^2}{t}, \frac{ax_3}{t}, \ldots, \frac{ax_n}{t} \right)$$

is a solution of the equation (34). Now we will extend this property to Σ - harmonic functions with a different point of view.

Let $U(x_1, x_2, \ldots, x_n)$ be a solution of generalized Tricomi's equation with any two indices (for example k_1 and k_2) are zerc:

$$(36) \quad \Delta_\Sigma U = \sum_{j=1}^{n} \frac{\partial^2 U}{\partial x_j^2} + \sum_{j=1}^{n-2} \frac{k_{j+2}}{x_{j+2}} \frac{\partial U}{\partial x_{j+2}} = 0 \, .$$

Putting $x_1 + ix_2 = t$, $x_1 - ix_2 = s$ this equation becomes

$$(37) \; \Delta_\Sigma U = 4 \frac{\partial^2 U}{\partial s \partial t} + \sum_{j=1}^{n-2} \left(\frac{\partial^2 U}{\partial x_{j+2}^2} + \frac{k_{j+2}}{x_{j+2}} \frac{\partial U}{\partial x_{j+2}} \right) = 0 .$$

It is required to find a convinient value for m in

$$V = t^m \, U \left(\frac{st + R^2}{t}, -\frac{a^2}{t}, \frac{ax_3}{t}, \ldots, \frac{ax_n}{t} \right)$$

or

$$V = t^m \, U(\xi_1, \xi_2, \ldots, \xi_n)$$

such that if $U(x_1, x_2, \ldots, x_n)$ is a solution of (37) then V be a solution of the same equation, where $R^2 = x_3^2 + x_4^2 + \ldots + x_n^2$. Substituting the derivatives in (37), we obtain

$$\Delta_\Sigma V = t^{m-2} \{ a^2 [4 \frac{\partial^2 U}{\partial \xi_1 \partial \xi_2} + \sum_{j=1}^{n-2} \left(\frac{\partial^2 U}{\partial \xi^2_{j+2}} + \frac{k_{j+2}}{\xi_{j+2}} \frac{\partial U}{\partial \xi_{j+2}} \right)]$$

$$+ 2 (n-2) t \frac{\partial U}{\partial \xi_1} + 4 mt \frac{\partial U}{\partial \xi_1} + 2 (\sum_{j=3}^{n} k_j) t \frac{\partial U}{\partial \xi_1} \} = 0$$

or

$$\Delta_\Sigma V = t^{m-1} \, 2 (2 m - 2 + n + \sum_{j=3}^{n} k_j) \frac{\partial U}{\partial \xi_1} = 0 .$$

So we find

$$m = 1 - \frac{n + \sum_{j=3}^{n} k_j}{2} .$$

Thus the extension of Brill's theorem to Σ - harmonic functions can be expressed as follows:

If $U(s, t, x_3, \ldots, x_n)$ is a solution of (37) *then*

$$V = t^{1 - \frac{n + \sum_{j=1}^{n} k_j}{2}} U \left(\frac{st + R^2}{t}, - \frac{a^2}{t}, \frac{ax_3}{t}, \ldots, \frac{ax_n}{t} \right)$$

is also a solution.

The extension of this property to the Σ- polyharmonic functions is quite clear by using the preceding method. We will only give the result:

If $U(s, t, x_3, \ldots, x_n)$ is a solution of (37) *then*

$$V = t^{p - \frac{n + \sum_{j=3}^{n} k_j}{2}} U \left(\frac{st + R^2}{t}, - \frac{a^2}{t}, \frac{ax_3}{t}, \ldots, \frac{ax_n}{t} \right)$$

is a solution of

$$\Delta \overset{p}{\Sigma} V = 0 \ .$$

Application : Let us look for a solution of (37) in the form of

$$V = e^{-s/4\varkappa} U \ (t, \ x_3 \ , \ldots, \ x_n). \tag{38}$$

If the derivatives of (38) are substituted in the differential equation, we have

$$\sum_{j=3}^{n} \left(\frac{\partial^2 U}{\partial x_j^2} + \frac{k_j}{x_j} \ \frac{\partial U}{\partial x_j} \right) = \frac{1}{\varkappa} \ \frac{\partial U}{\partial t} \ . \tag{39}$$

That is, if $U \ (t, x_3, \ldots, x_n)$ is a solution of the heat equation (39), then (38) is a solution of (37).

In conclusion I wish to express my thanks to Prof. S. Süray for his great help in preparing this paper.

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ÖZET

Σ – monojenik fonksiyonlar ve

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{k}{y} \frac{\partial U}{\partial y} = 0$$

eliptik diferensiyel denkleminin çözümleri arasındaki bağıntı aşikardır. Bu makalede yaptığımız şey

$$(*) \qquad \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j{}^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) U = 0$$

genelleştirilmiş Tricomi denkleminin ve onun Σ – harmonik fonksiyonlar diyeceğimiz çözümlerinin özelliklerinin araştırılmasından ibarettir. Çalışma üç kısımdan meydana gelmiştir.

I inci kısımda (*) denkleminin özellikleri incelenmiş, II inci kısımda Σ – harmonic ve Σ – poliharmonik fonksiyonlar için tanınmış Lord Kelvin teoremi ve Almansi açılımı tesis edilmiş ve III üncü kısımda evvelce yine harmonik fonksiyonlar için elde edilmiş olan Brill teoremi Σ – poliharmonik fonksiyonlar için ispatlanmıştır.

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