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On The Motion of The Frenet Trihedron of a Space Curve

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· In 1963 Hans Vogler in his paper called "Die auf einer Torse verlaufenden Linien konstanten Gratabstandes als duale Seitenstücke zu den pseudorektifizieren den Torsen einer Raumkurve" has studied the geometrical varieties formed by the elements of the moving FRENET trihedron along the curve traced on a developable at a constant distance from its edge of regression. The developable itself is the dual corresponding of the pseudo-rectifying developable of a space curve.

In this paper we have studied the geometrical varieties of the elements of the moving Frenet trihedron along the parametric curves of the skew surface generated by a straight line fastened in the rectifying plane, to the moving Frenet trihedron of a space curve.

The parametric curves which are taken into consideration have given more general results than the curves at a constant distance from the edge of regression. Thus we were able to deduce the same results of Hans Vogler as special cases of the problem.

I. INTRODUCTION

Let k be any twisted curve, we denote by D the Frenet trihedron at a point X of the curve, and its motion along the curve by T. And let S denote the instant helicoidal motion which represent T at the time t. Let a straight line tightly fastened to D at the origin be d, and the ruled surface generated by this straight line during the motion T be (d). We denot by P_v a point on d at a distance v from the point X. The loci of the points P_v during the motion T are the parametric curves C_v of the surface (d). ·

The unit vector $\vec{d}$ on d referred to the Frenet trihedron is:

(1) $$\vec{d} = d_1 \vec{t} + d_2 \vec{n} + d_3 \vec{b}$$

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If $\vec{X}$ is the position vector of the point X and s is the arc length of k, then the ruled surface (d) can be given as:

(2) $$\vec{X}\ (s,v)\ =\ \vec{x}\ (s)\ +\ v\ \vec{d}\ (s)$$

The parameter of distribution of (d) wille be:

(3) $$\lambda_d\ =\ \frac{(d_2{}^2\ +\ d_3{}^2)\ \tau\ -\ d_1\ d_3\ \varkappa}{d_3{}^2\ (\varkappa^2\ +\ \tau^2)\ +\ (d_1\ \varkappa\ -\ d_3\ \tau)^2}$$

A necessary and sufficient condition for (d) to be devolopable is that:

$$(d_2{}^2\ +\ d_3{}^2)\ \tau\ -\ d_1\ d_3\ \varkappa = 0$$

or

(4) $$\frac{\varkappa}{\tau}\ =\ \frac{d_2{}^2\ +\ d_3{}^2}{d_1\ d_3}\ =\ tg\ \theta \qquad \text{(where } \theta\ =\ \text{cte)}$$

As an immediate result of this we have:

Theorem Ia: The ruled surface (d) generated by the straight line d fastened at the origin to the Frenet trihedron D of a twisted curve k, during its motion T, is a developable if and only if k is a general helix satisfying the relation:

$$\frac{\varkappa}{\tau}\ =\ \frac{d_2{}^2\ +\ d_3{}^2}{d_1\ d_3}\ =\ tg\theta \text{ (where } \theta\ =\ \text{cte.)}$$

The curve traced by P_v of the line d during the motion T on the ruled surface (d) for v=cte will be the parametric curve C_v with the equation:

(5) $$\vec{X}\ (s)\ =\ \vec{x}\ (s)\ +\ v\vec{d}\ (s) \qquad\qquad \text{(v = cte)}$$

The tangent $\vec{t}_v$ of the curve C_v at the point P, is:

(6) $$\dot{\vec{X}} = \vec{t}_v = (1\ -\ vd_2\ \varkappa)\ \vec{t}\ +\ v\ (d_1\ \varkappa\ -\ d_3\ \tau)\ \vec{n}\ +\ vd_2\ \tau\ \vec{b}$$

It can immediately be seen from (6) that the tangent $\vec{t_v}$ has components on the three axes of D. Without imposing any restriction on the twisted curve k, only the third component of $\vec{t_v}$ can be zero. Which implies:

(7) $$d_2 = 0$$

meaning that the straight line d lie in the rectifying plane Thus during the motion T the tangents $\vec{t_v}$ of the orbits of the points which are in the rectifying plane are parallel to the osculating plane of k at that instant. By substitution:

(8) $$\begin{cases} \vec{d} \cdot \vec{t} = \text{Cos } \alpha = d_1 \\[2mm] \vec{d} \cdot \vec{b} = \text{Sin } \alpha = d_3 \end{cases}$$

from (4) we have:

(8') $$\frac{\varkappa}{\tau} = tg\,\theta = \frac{d_3}{d_1} = tg\,\alpha$$

which gives $\theta = \alpha$. For this special case theorem Ia becames:

> Theorem Ib : The ruled surface (d), generated by a straight line d fastened to the rectifying plane of a twisted curve k satisfying the relation $tg\,\alpha = \dfrac{d_3}{d_1}$, during the motion T of its Frenet trihedron D is a developable if and only if k is a general helix with an angle of inclination equal to α.

We denote by $(d)_0$ the skew surface generated by the straight line d which is fastened to D at the origin and which lies in the rectifying plane of the twisted curve k during the motin T and the parametric curve v = cte of $(d)_0$ by C_v. The skew surface

(d)$_0$ and its parametric curves will be the main topic of our investigations.

If we include the condition $d_3 = O$ to (8) then the straight line d will coincide with the tangent $\overrightarrow{t}$ of D and (d)$_0$ becomes the torse generated by the tangents of the rtwisted curve k. For this special case the parametric curves C$_v$ are the curves at a constant distance from the edge of regression of the surface generated by the tangents. Hans VOGLER in his paper [1] has studied the curves traced on a torse at a constant distance from its edge of regression, the torse being the dual of the pseudo-rectifying torse of a space curve [3].

In this paper by investigating the parametric curves C$_v$ of the skew surface (d)$_0$ we will see that the results Hans Vogler obtained will also apply to the parametric curves.

II. The Tangents $\overrightarrow{t_v}$ of the Curves C$_v$.

The straight line d satisfying the condition (7), during the motion T generates the skew surface (d)$_0$ and a point P$_v$ on d traces a parametric curve C$_v$ on (d)$_0$. The tangent $\overrightarrow{t_v}$ of C$_v$ at the point P$_v$ by (6) and (7) is:

(6') $\overrightarrow{t_v} = \overrightarrow{t} + v \ (d_1 \varkappa - d_3 \tau) \ \overrightarrow{n}$

The angle β between $\overrightarrow{t_v}$ and $\overrightarrow{d}$ will be:

(9) $tg\beta = \dfrac{\sqrt{1 + v^2 \ (d_1 \varkappa - d_3 \tau)^2 - d_1^2}}{d_1}$

The curves C$_v$ being the loci of the fixed points of D during the motion T, the tangents of these curves at the time t will belong to the quadratic ray complex Q$_0$ of the tangents of the orbits during the motion T [2]. Thus:

Theorem II. At an instant t the tangents of the parametric curves of the ruled survace (d)$_0$ are fastened to

D and belong to the quadratic ray complex for-
med by the torsal lines of the ruled surfaces
generated by the tangents during the motion T.

Let $B = d_1 \varkappa - d_3 \tau$, as a special case if k is a Bertrand curve
which means $B = cte$, by (6') two following theorems can be given

Theorem III. If k is a Bertrand curve then the tangents
$\vec{t_v}$ of the parametric curves C_v of the skew
surface $(d)_0$ are fastened to D.

Theorem IV. If the tangents $\vec{t_v}$ of the parametric curves
C_v of the skew surface $(d)_0$ are fastened to D
then the space curve k is a Bertrand curve.

As an immediate result of (3) and (7) we have:

Theorem V. If k is a Bertrand curve then the parameter
of distribution of the skew survace $(d)_0$ is
constant and has the value

$$(9') \qquad \lambda_d = - \frac{d_3}{B}$$

for every generator d.

To find *the envelope of the projections of the tangents*
$\vec{t_v}$ on the osculating plane, at an instant t:

Let (x, y, z) be any point on the line directed by
$\vec{t_v}$ passing through the point $P_v (vd_1, 0, vd_3)$ then its equation
becomes:

$$(10) \qquad \begin{cases} z - vd_3 = 0 \\ y = vB (x - vd_1) \end{cases}$$

The envelope of their projection on the osculating plane as
v varies will be the parabola

$$(11) \qquad x^2 = \frac{4 d_1}{B} y$$

On the basis of this result we can formulate the following theorems:

Theorem VI.   The envelope of the tangents $\overrightarrow{t_v}$ of the parametric curves C_v of the rulate surface $(d)_0$ on the osculating plane at an instant t is the parabola

$$x^2 = \frac{4\,d_1}{B}\,v$$

referred to the system D. This parabola is tangent to the curve k and its focal distance is

$$\frac{1}{B}.$$

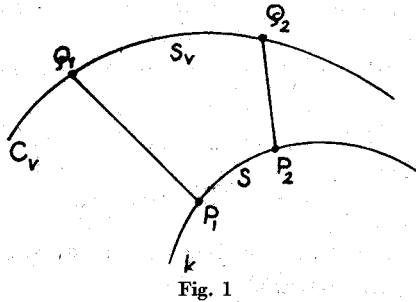Theorem VII.   As a particular case, if the curve k is a Bertrand curve then the parabola is fastened to D.

The element of arc ds_v of the vurves C_v from (6) is:

(12) $ds_v = \sqrt{1 + v^2\,B^2}\;ds$

If k is a Bertrand curve then from (12)

(12') $S_v = \sqrt{1 + v^2\,B^2}\;s$

is deduced. This property can be given by:

Fig. 1

Theorem VIII. If the curve k is a Bertrand curve then the corresponding arc lengths $P_1\,P_2$ of k

and Q_1 Q_2 of C_v are proportional. Their ratio is:

$$\gamma_v = \sqrt{1 + v^2\, B^2}$$

III. The normal planes and the polar axes of the curves C_v.

The equation of the normal plane of a curve C_v referred to D is:

(14) $$(\vec{X} - \vec{x} - \vec{vd}) \cdot \vec{t_v} = 0$$

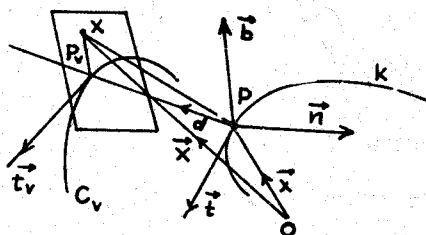

Fig. 2

Where $\vec{X}$ is any point on the normal plane. In virtue of (7) and (6') the equation above cane be written as:

(14') $$X + vBy = vd_1$$

It follows from (14') that for every value of v there corresponds a normal plane parallel to the Z axis and all these planes pass through the intersection line m of the planes,

(15) $$\begin{cases} X = 0 \\ y = \dfrac{d_1}{B} \end{cases}$$

This line m passes through the focus of the parabola (11) and is perpendicular to the osculating plane. Thus we can give the theorems:

Theorem IX. The normal planes of the parametric curves C_v from a pencil, its axis is the straight

line m which passes through the focus of the

parabola $x^2 = \dfrac{4\,d_1}{B}\, y$ and is parallel to

the polar axis of the curve k.

Theorem X. If k is a Bertrand curve then the normal pla-
nes v_v of the curves C_v from a pencil of planes
fastened to D. The axis which is fastened to
D passes through the focus of the parabola,

$x^2 = \dfrac{4\,d_1}{B}\, y$, and is parallel to the polar

axis of the curve k.

Let us investigate the polar axis of the curves C_v:

By differentiating the equation (14) of the normal planes
with respect to the arc length s of k and from (6′) we have:

(16) $\qquad -v\varkappa Bx + (\varkappa + vB')\, y + v\tau Bz = 1$

The polar axis of C_v has to satisfy both the equations (14)
and (16), therefore it has the direction of

(17) $\qquad \vec{b}_v = v^2\,\tau\,B^2\,\vec{t} - v\tau B\,\vec{n} + (\varkappa + vB' + v^2\varkappa B^2)\vec{b}$

Here $\vec{b}_v$ is the binormal of the curve C_v. The intersection
point of the line m and the consecutive normal plane (16) is a point
on m_v. The coordinates of this point from (15) and (16) is:

(18) $\qquad \begin{cases} x = 0 \\[2mm] y = \dfrac{d_1}{B} \\[2mm] z = p = -\,\dfrac{d_3\,\tau + d_1\,vB'}{v\tau B^2} \end{cases}$

E being a point on the ruled surface Ψm_v generated by the
polar axis m_v (fig. 3) we have:

$$\vec{E} = \vec{E}\,(u,v) = \frac{d_1}{B}\,\vec{n} + p\,\vec{b} + u\vec{b}_v$$

or by replacing

$$C = \varkappa + vB' + v^2\,\varkappa B^2$$

$$(19)\quad \vec{E} = uv^2\,\tau B^2\vec{t} + \left(\frac{d_1}{B} - uv\tau B\right)\vec{n} + \left[-\frac{d_3\tau + d_1 vB'}{v\tau B^2} + uC\right]\vec{b}$$

is deduced, the equation (19)
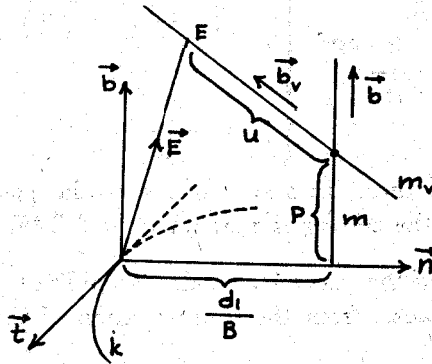
referred to the system D has the parametric form:

Fig. 3

$$(19')\quad \begin{cases} x = uv^2\,\tau B^2 \\ y = \dfrac{d_1}{B} - uv\tau B \\ z = -\dfrac{d_3\,\tau + d_1\,v\,B'}{v\tau B^2} + uC \end{cases}$$

By eliminating u and v from (19') we have as the equation of the surface Ψ_{m_v}:

$$(19'')\quad \left(y - \frac{d_1}{B}\right)^2 + x^2 = \frac{\tau}{\varkappa}\,x \cdot \left[z - \frac{d_1}{\tau}\left(\frac{1}{B}\right)'\right]$$

$$+ \frac{\tau}{\varkappa}\,x\left(\frac{B'}{B\tau} - \frac{d_3}{B\varkappa}\right)\left(y - \frac{d_1}{B}\right)$$

The intersection of this surface with the plane   which is parallel to (x, z) plane and which passes through the focus of the parabola (11), are the lines given by

$$\varkappa\, x^2 = \tau z x - d_1 .(\frac{1}{B})' .x$$

or

$$(20) \quad \begin{cases} x = 0 \\[2mm] y = \dfrac{d_1}{B} \end{cases} \text{and} \quad \begin{cases} x = \dfrac{\tau}{\varkappa}\, z - \dfrac{d_1}{\varkappa}\, (\dfrac{1}{B})' \\[3mm] y = \dfrac{d_1}{B} \end{cases}$$

The first of these lines is the axis of the pencil of normal planes v_v , and the secong is parallel to the DARBOUX vector $\vec{\omega} = \tau \vec{t} + \varkappa \vec{b}$ of the curve k and also is the limit position of m_v for $v \to \infty$. Because, from the first equation of the system (19′) we have:

$$u = \frac{x}{v^2\, \tau\, B^2}$$

If this is replaced in the second and third equations we have:

$$y = \frac{d_1}{B} - \frac{x}{vB}$$

$$z = - \frac{d_3\, \tau + d_1\, B'v}{v\, \tau\, B^2} + \frac{C}{v^2\, \tau\, B^2}\, x$$

and for $v \to \infty$

$$y = \frac{d_1}{B}$$

$$x = \frac{\tau}{\varkappa}\, z - \frac{d_1}{\varkappa}\, (\frac{1}{B})'$$

are deduced.

For this property this line is called the seperating generator.

If k is a Bertrand curve then (19'') becomes:

$$(21) \qquad (y - \frac{d_1}{B})^2 + x^2 = \frac{\tau}{\varkappa} zx - \frac{\tau d_3}{\varkappa B}(y - \frac{d_1}{B})$$

Again the intersection of this surface with the plane

$$y = \frac{d_1}{B}$$

we have the lines given by:

$$x^2 = \frac{\tau}{\varkappa} zx$$

or

$$(22) \qquad \begin{cases} x = 0 \\ \\ y = \frac{d_1}{B} \end{cases} \quad \text{and} \quad \begin{cases} \varkappa x = \tau z \\ \\ y = \frac{d_1}{B} \end{cases}$$

the first being the axis m and the other a generator parallel to

the vector $\vec{\delta} = \tau \vec{t} + \varkappa \vec{b}$ (the seperating generator).

The parameter of distribution of the ruled surface Ψ_{m_v} along the generator m_v is:

$$(23) \quad \lambda_{m_v} = - \frac{d_3 \cdot \{ v^2\tau^2 B^2 (1+v^2 B^2) + [\varkappa(1+v^2 B^2)+ vB']^2\}}{B \cdot \{v^4 \tau^2 B^4 + \varkappa^2 (1+v^2 B^2)^2 + v^3 B^2 B'(4\varkappa+vB')\}}$$

This shows that Ψ_{m_v} is a skew surface. If k is a Bertrand curve, as (23) remains unchanged, the surface is still a scroll, For the surface Ψ_{m_v} to be a developable $d_3 = 0$. The special case for $d_3 = 0$ has been investigated by Hans Vogler [1]. These properties can be gives as a theorem.

Theorem XI. The polar axis m_v of the parametric curve C_v of the surface $(d)_0$ passing through the

point P_v of the generator d generates a quadric skew surface at an instant t. The intersection of this surface with the plane are the straight lines: (i) $x = 0$, $y = \dfrac{d_1}{B}$ parallel to the polar axis of k, and the other: (ii)

$$\varkappa x = z\tau - d_1 \left(\dfrac{1}{B}\right)', y = \dfrac{d_1}{B} \quad \text{parallel} \quad \text{to}$$

the DARBOUX vector $\overrightarrow{W} = \tau \overrightarrow{t} + \varkappa \overrightarrow{b}$.

IV. The Osculating Planes of the Curves C_v.

If φ is the angle between the tangent plane of the surface $(d)_0$ at the point P_v, and the osculating plane of the parametric curve C_v passing through P_v, taking $A = d_1 \tau + d_3 \varkappa$ we have:

$$(24) \qquad tg\varphi = \dfrac{d_3 (\varkappa + vB') + v^2 B^2 A}{vB (B + v^2 B^3 + d_1 vB')} \sqrt{1 + v^2 B^2}$$

If k is a Bertrand curve then $B' = 0$ and the above formula is reduced to:

$$(24') \qquad tg\varphi = \dfrac{d_3 \varkappa + v^2 B^2 A}{vB^2 \sqrt{1 + v^2 B^2}}$$

As it is given by theorem II, the tangents $\overrightarrow{t_v}$ of the curves C_v at instant t are the torsal lines of the surfaces generated by the straight lines fastened to D. Therefore the tangents $\overrightarrow{t_v}$ belong to the quadratic ray complex Q_0 formed by the tangents of the orbit. The tangent $\overrightarrow{t_v}$ at an instant t is the generator of the torse (T). The angle φ^* between the tangent planes of the developable (T) and the surface $(d)_0$ at the point P_v is:

$$(25) \qquad tg\,\varphi^* = \dfrac{d_3 \varkappa + v^2 B^2 A}{vB \sqrt{1 + v^2 B^2}}$$

at the instant t both $\varkappa$ and τ can be taken as constants so $B' = 0$. (25) is the same of (24'). The relation between the angles φ and φ^* found in (24) and (25) is:

$$(26) \quad \text{Cotg } \varphi - \text{Cotg } \varphi^* = \frac{d_1 \, v^2 \, BB'}{[d_3 \, (\varkappa + vB') + v^2 B^2 A \,]\sqrt{1 + v^2 B^2}}$$

Thus we can give the following theorems:

Theorem XII. The osculating plane of the parametric curve C_v traced on the surfaces $(d)_0$ coincides with the torsal plane which corresponds to the torsal line $\vec{t_v}$ of C_v if and only if $\varkappa$ and τ are stationary.

Theorem XIII. If k is a Bertrand curve then the osculating plane of the parametric curve C_v of the surface $(d)_0$ coincides with the torsal plane corresponding to the torsal lines $\vec{t_v}$ of C_v.

Theorem XIV. The osculating plane of the parametric curves C_v, traced on the surface $(d)_0$ generated by the binormals $\vec{b}$ of a space curve k, coincides with the torsal plane corresponding to the torsal line $\vec{t_v}$ of C_v.

If the space curve k, is taken such that the osculating plane of the parametric curve C_v. at the point P_v is tightly fastened to the Frenet trihedron D of the curve k than as the vectors $\vec{t_v}$ and $\vec{b_v}$ will be tightly fastened to D, from (6') and (17) the first and second curvatures $\varkappa$ and τ of the curve k shall be constant. We have the following theorem:

Theorem XV. If the space curve k is a circular helix then the osculating planes of the parametric curve C_v of the survface $(d)_0$, is tightly

fastened to the Frenet trihedron D of the
curve k.

Now consider the set of the osculating planes of the curves C_v
traced by the points of a generator d of the surface $(d)_0$. If $\underset{v}{\infty}$
denotes the osculating plane of the curve C_v, its equation referred
to the system D will be

(27) $v^2 B^2 \tau x - vB \tau y + cz = d_1 v^3 B^2 \tau' + vd_3 c$

u_0, u_1, u_2, u_3 being the homogeneous coordinates of $\underset{v}{\infty}$ we have:

(28) $u_0 : u_1 : u_2 : u_3 = - \{ d_1 v^3 B^2 \tau + vd_3 c \} : v^2 B^2 \tau : - vB\tau : c$

In the particular case when k is a Bertrand curve then $B' = 0$
and from (28) we have:

(29) $u_0 : u_1 : u_2 : u_3 = - \{d_1 v^3 B^2\tau + vd_3\varkappa(1+v^2B^2)\} : v^2B^2\tau : vB\tau : \varkappa.(1+$
                                                                $v^2 \ B^2)$

Thus:

Theorem XVI. The osculating developable at the points
P_v of the curves C_v passing through the po-
ints P_v of a generator d of the surface $(d)_0$
is a surface generated by the tangents of a
cubic parabola.

In view of theorem XII, the tangent planes $\underset{v}{\infty}$ of (T), which
correspond to the generators $\overrightarrow{t_v}$ will coincide with the osculating
planes $\underset{v}{\infty}$ at the points where $\varkappa$ and τ of k are stationary, so the
envelope of $\underset{v}{\overline{\infty}}$ will also be the surface generated by the tangents
of cubic parabola.

Consider the transformation which will transform the oscu-
lating planes $\underset{v}{\infty}$ to the torsal planes $\underset{v}{\overline{\infty}}$: The equation of $\underset{v}{\infty}$ is
(27). The equation of $\underset{v}{\overline{\infty}}$ is:

(30) $v^2 B^2 \tau \rho x - vB \tau \rho y + (1 + v^2 B^2) z = d_1 v^3 B^2 \tau \rho + d_3 v(1 + v^2 B^2)$

$\bar{u}_0$, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, being the plane homogeneous coordinates, the connection between u_i and $\bar{u}_i$ will be

(31)
$$
\begin{cases}
\bar{u}_0 = \rho \left(u_0 - d_3 \dfrac{B'}{B^2 \tau} u_1 \right) \\[2mm]
\bar{u}_1 = \ldots\ldots\ldots\ldots\ldots \rho\, u_1 \\[2mm]
\bar{u}_2 = \ldots\ldots\ldots\ldots\ldots \rho\, u_2 \\[2mm]
\bar{u}_3 = \ldots\ldots\ldots\ldots\ldots \rho\left(-\dfrac{B'}{B\tau} u_2 + u_3 \right)
\end{cases}
$$

x_i and $\bar{x}_i$ being the homogeneous coordinates of the points of $\underset{v}{\infty}$ and $\overline{\underset{v}{\infty}}$ respectively, from (31) we have:

(31')
$$
\begin{cases}
x_0 = \rho\, \overline{X}_0 \\[2mm]
x_1 = -\rho\, \dfrac{d_3 B'}{\tau B^2}\, \overline{X}_0 + \rho\, \overline{X}_1 \\[2mm]
x_2 = \ldots\ldots\ldots\ldots\ldots \rho\, \overline{X}_2 - \rho\, \dfrac{B'}{\tau B}\, \overline{X}_3 \\[2mm]
x_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \rho\, \overline{X}_3
\end{cases}
$$

The caracteristic equation which gives the double points of this Transfromation is $(\rho - 1)^4 = 0$, which has four equal real roots. This shows that we have four different cases, either one double point or all the points of a line or all the points of a plane or all the points of the space will corrrespond for each root [4]. For the planes $\underset{v}{\infty}$ and $\overline{\underset{v}{\infty}}$ the second case is valid. The center of this transformation is the point

$$(x_i) = (0,\ 0,\ 1,0)$$

This infinitely distant point is at the same time on the constant plane

$$(u_i) = (0,\ 0,\ 0,1),$$

Thus the planes $\underset{v}{\infty}$ and $\underset{v}{\overline{\infty}}$ correspond by an affine perspective.

V. The Principal Normals $\vec{n}_v$ and The Binormals $\vec{b}_v$ of The Curves C_v.

In this paragraph we will try to investigate closely the principal

normals $\vec{n}_v$ and the binormals $\vec{b}_v$ of the curves C_v at the points P_v of a generator d of the surface $(d)_0$.

The explicit expressions for the vectors $\vec{t}_v$, $\vec{n}_v$, $\vec{b}_v$ are:

$$\vec{t}_v = \vec{t} + B v \vec{n}$$

(32) $$\vec{n}_v = - v B C \vec{t} + C \vec{n} + B \tau v (1 + v^2 B^2) \vec{b}$$

$$\vec{b}_v = B^2 \tau v^2 \vec{t} - B\tau v \vec{n} + C \vec{b}$$

By the following stage by stage operations the Frenet trihedron D of the curve k and D_v of the curve C_v can be put on each other:

a) The vertex P of D will be moved along the generator d of the surface $(d)_0$ to a distance v,

b) The tangent vector $\vec{t}$ of D will be rotated around the new vertex P_v at an angle Ψ, where tg $\Psi = B v$.

c) The osculating plane of D will be rotated about $\vec{t}_v$ at angle δ where

$$tg \ \delta = \frac{B \tau v \sqrt{1 + v^2 B^2}}{\varkappa + B' v + B^2 v^2 \varkappa}$$

The generators $\vec{n}_v$ of the ruled Ψ_n and the axis m of the theorem IX are in the same normal plane v_v of C_v as they are not parallel, they intersect at a point R_v (fig. 4) The lines $\vec{n}_v$ also intersect the line d at the points P_v. The generator d of the surface $(d)_0$ and the line m parallel to the polar axis of the

curve k are the directrix lines (directrix curves) of the ruled surface Ψ_n. Thus Ψ_n is a net of rays of the hyperbolic type [2].
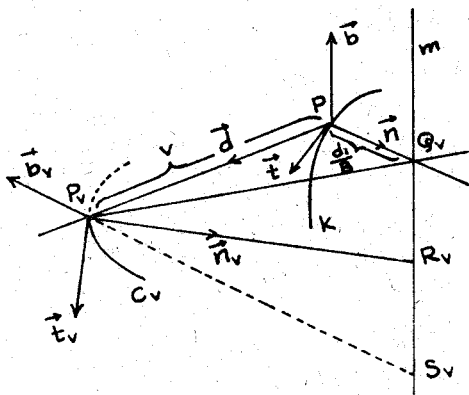

Fig. 4

The generator d of the surface $(d)_0$ in terms of the parameter v is:

(33) $(x_i) = (d_1\ v,\ 0,\ d_3\ v)$

And the line m in terms of parameter w is:

(34) $(x_i) = (0,\ \dfrac{d_1}{B},\ w)$

so the relation between the directrix lines d and m because of

$$\overrightarrow{P_v R_v} = \lambda\ \vec{n}_v$$

is given by:

(35) $d_1\ \tau\ v\ (1 + v^2\ B^2) = (w - d_3\ v)\ C$

Here to every value of v there corresponds a value of w and to every value of w there are three different corresponding values of v. Thus the generator d of the surface $(d)_0$ is a simple directrix of Ψ_n and the line m is a multiple directrix of order three (fig. 5). The equation of the surface Ψ_n referred to the system D is:

(36) $z(d_1 - By) \ [(d_1 - By)^2 \ \varkappa + (d_1 - By) \ B'x + B^2 \ x^2 \ \varkappa] =$
$= d_3 x \ [(d_1 - By)^2 \varkappa + (d_1 - By) B'x + B^2 x^2 \varkappa] + xy B\tau \ [(d_1 - By)^2 + B^2 x^2]$
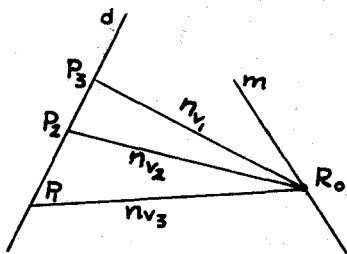
thus we have the theorem:

Fig. 5

Theorem XVII. The principal normals $\overrightarrow{n}_v$ of the curves C_v passing through the points P_v of the generator d of the surface $(d)_0$ are on a surface of rays Ψ_n of the degree four. The multiple tangent planes of order three of the surface Ψ_n pass through the generator d of the surface $(d)_0$ and the line m is the multiple directrix line of order three.

If k is a Bertrand curve then the equation (36) can be factorized as:

(36') $[(d_1 - By)^2 + B^2 x^2] . [\varkappa(d_1 z - d_3 x) - By(\varkappa z + \tau x)] = 0$

From the first factor we have:

(37) $\dfrac{d_1}{B} - y = \pm i \ x$

which means that the surface Ψ_n is degenerated into two pencil of rays by isotropic planes. The vertex of the pencil being

(38) $(\pm i \ \dfrac{d_1}{B}, \ 0)$

corresponding to the value of the parameter:

(39) $v = \pm\, i\, \dfrac{d_1}{B}$

From the second factor

(40) $(d_1 z - d_3 x)\, \varkappa - B\, (\varkappa z + \tau\, x)\, y = 0$

it can be deduced that the surface Ψ_n is transfomed into a hyperbolic paraboloid

In this particular case $B' = 0$ therefore (35) becomas:

(35') $A\, v = \varkappa\, w$

Thus for the particular case of $B' = 0$ the points of the directrix

lines d and m are mapped by the generators $\overrightarrow{n}_v$ bi-univocally.

Let Ψ_b denote the ray surface generated by the binormals

$\overrightarrow{b}_v$ of the curves C_v. From (fig. 4) it is easily seen that this surface too has the generator d of the surface $(d)_0$ and the line m parallel to the polar axis of the curve k as directrix lines. These two directrix lines are expressed in terms of the parameters v and w by (33) and (34), this shows that the points of these lines correspond

through the generators $\overrightarrow{b}_v$ of the surface Ψ_b by:

(41) $B^3 v^2 + (d_1\, B' + B^2\, \tau\, w)\, v + d_1\, \varkappa = 0$

so that the generator d of the surface $(d)_0$ is a simple directrix line of Ψ_b and the line is a multiple directrix line of order two. (41) gives for every value of w two values of v. Generally these two values of v are different. For some values of w the values of v be equal; the points corresponding to these value are the cuspidal points For the cuspidal points from (41) we have

(42) $w_{1,2} = \dfrac{d_1\, B' \pm \sqrt{4\, d_1\, \varkappa\, B^3}}{B^2\, \tau}$

and the values of v corresponding to these values of w are:

(43) $v_{1,2} = \pm\, \sqrt{\dfrac{d_1\, \varkappa}{B^3}}$

The values of w given by (42) are the distances between the surface Ψ_b and the cuspidal points of the tangent planes ∞ of the surface (d)₀ (43) shows that the cuspidal points are real.

The equation of the surface Ψ_b refrred to the system D

(44) $(d_1-By)[\varkappa y(d_1-By) + B'xy + B\tau xz]+B^2\varkappa x^2y = 0$

is a surface of degree three. Thus we have the theorem:

> Theorem XVIII. The binormals $\vec{b}_v$ of the curves C_v passing through the points P_v of a generator d of the surface (d)₀ are on a surface of rays Ψ_b of the degree three. The line d is a simple directrix and the line m is a multiple directrix of order two.

In the particular case k being a Bertrand curve the distance between Ψ_b and the cuspidal points of the tangent planes ∞ of the surface (d)₀ from (42) is:

(42') $w_{1,2} = \pm \dfrac{2}{\tau} \sqrt{\dfrac{d_1 \varkappa}{B}}$

In this particular case the intersection of the surface of rays with the infinitely distant plane (infinitely distant curve) is interesting to study. In order to do it in (44) substituting B'=0 and using the homogeneous coordinates for the surface of rays Ψ_b we have:

(44') $(d_1x_0-Bx_2)(d_1\varkappa x_2x_0-\varkappa Bx^2_2+B\tau x_1x_3) + B\varkappa x^2_1x_2 = 0$

The infinitely distant curve (the intersection by the plane $x_0 = 0)$ is given by:

(44'') $Bx_2[\varkappa(x^2_1+x^2_2)-\tau x_1x_3] = 0$

From which we have:

(45) $\begin{cases} x_0 = 0 \\ x_2 = 0 \end{cases}$

and

(46)
$$\begin{cases} x_0 = 0 \\ \\ x^2{}_1 + x^2{}_2 = \dfrac{\tau}{\varkappa} \ x_1 \ x_3 \end{cases}$$

The first one is the infinitely distant line of the rectifying plane of the curve k; and the second is the infinitely distant conic of the right cone given by the equation

$$x_1{}^2 + x_2{}^2 = \frac{\tau}{\varkappa} \ x_1 \ x_3$$

VI. The rectifying planes of the curves C_v.

In this paragraph we shall investigate the rectifying planes ρ_v of the curves C_v at points P_v of a generator d of the surface $(d)_0$.

The equation of one of the rectifying planes ρ_v referred to the system D is:

(47) $-Bvcx + cy + B\tau v(1 + B^2 v^2)z = d_3 B\tau v^2(1 + B^2 v^2) - d_1 \quad Bv^2 \quad C$

The rectifying planes ρ_v being family of one parameter v their envelope is a torse. Because of the degree of the parameter v is four the envelope of the planes ρ_v is a torse of degree four.

If the curve k is a Bertrand curve (47) becomes:

(47') $(1 + v^2 B^2) \ (-B\varkappa vx + \varkappa y + B\tau vz + B^2 v^2) = 0$

From the first factor we have

(48) $v = \pm i \ \dfrac{1}{B}$

which shows that the rectifying planes of the two curves C_v corresponding to these values of the parameter are indefinite. One the other hand the points P_v corresponding to the values

$$v = \pm i \ \frac{1}{B}$$

are P_v $(\pm\ i\ \dfrac{d_1}{B},\ 0,\ \pm\ i\ \dfrac{d_3}{B})$; the projections of these points on the plane $z = O$ being the same as (38), the rectifying planes $\rho_{\cdot\cdot}$ corresponding to the parameter values $v = \pm\ i\ \dfrac{1}{B}$ under the condition $z = o$ verify the isotropic planes of the surface Ψ under the particular condition $B' = 0$.

The envelopes of the rectifying planes corresponding to the values of v other than the values $v = \pm\ i\ \dfrac{1}{B}$ are:

$$(49) \qquad y = \frac{\rho}{4}\ (-\ \varkappa x\ +\ \tau z)^2$$

which is obtained by eliminating v in between the second factor of (47') and its derivative with respect to v. It is a parabolic cylinder whose geneators arer parallel to the vector

$$\vec{\delta} = \vec{\tau t}\ +\ \vec{\varkappa b}$$ the DARBOUX vector of the surface $(d)_0$.

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ÖZET

Hans Vogler'in 1963 yılında yayınladığı "Die auf einer Torse Verlaufenden Linien konstanten Gratabstandes als duale Seitenstücke zu den pseudorektifizieren den Torsen einer Raumkurve" isimli makalesinde, bir uzay eğrisinin pseudo rektifiyan torslarına dual karşılık olan, bir torzs üzerine çizilmiş sabit sırt uzaklık eğrilerinin Frenet üçüzlülerinin elementleri tarafından teşkil edilen geometrik varyeteler tetkik edilmiştir.

Bu çalışmada bir uzay eğrisinin, Frenet üçyüzlüsünün eğri boyunca hareketi esnasında, rektifiyan düzlemine sıkı surette bağlı bir doğrunun tevlid ettiği açılamıyan bir yüzey üzerine çizilmiş parametre eğrilerinin (sabit sırt uzaklık eğrilerinden daha genel) FRENET üçyüzlülerinin elementleri tarafından tevlid edilen varyeteler incelenmiştir. Bu suretle Hans Vogler'in bulduğu sonuçların daha genel hallerde de mümkün olabileceği gösterilmiştir.