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by

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On the Librational Orbits in the Elliptic Restricted Three Body Problem

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Motion near the Lagrangean triangular points has been investigated by Rabe 1961, and Long periodic orbits have been established numerically in the circular case. i.e. when the two primaries revolve around the common center of mass in circular orbits. Following the same procedure as Rabe, Goodrich found short periodic orbits in the same case 1965. We showed, 1967, that one of the Rabes initial Conditions also gives periodic orbit in the elliptic case. On the other hand Szebehely treated the elliptic case analytically. There are agreements between our numerical and Szebehely's analytical results.

The restricted problem of three bodies studies the motion of an infinitesimal particle moving under the influence of two finite point masses. It is assumed that the two bodies move about their center of mass in concentric circles and their orbits are undisturbed by the infinitesimal third body.

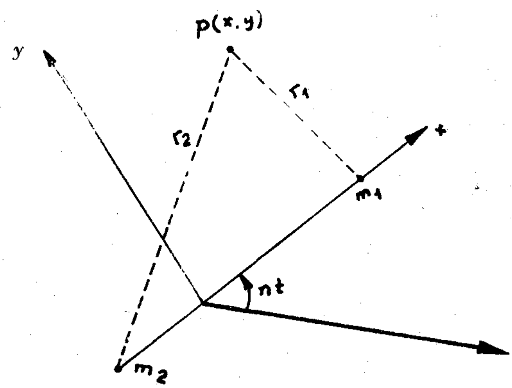

Fig. 1

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Let x, y be the coordinates of the third body; r_1, r_2 its distances from the primaries; m_1, m_2 the masses of the primaries; l the distance between them; n the mean motion of the primaries and t the time. We put

$$\frac{x}{l} = \xi, \quad \frac{y}{l} = \eta; \quad \frac{m_2}{m_1+m_2} = \mu, \quad \frac{m_1}{m_1+m_2} = 1-\mu;$$

$$\frac{r_1}{l} = \varrho_1, \quad \frac{r_2}{l} = \varrho_2 \quad \text{and } nt = \tau$$

We now have ξ, η dimensionless coordinates of the third body; ϱ_1, ϱ_2 its dimensionles distances from the primaries. μ, $1-\mu$ dimensionless masses of the primaries and τ the dimensionless time.

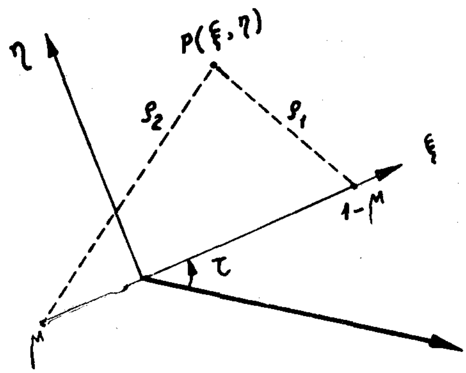

Fig. 2

The equations of motion of the third body in this uniformly rotating synodical dimensionless $\xi-\eta$ Cartesian rectangular coordinates system are

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \Omega_\xi \\ \ddot{\eta} + 2\dot{\xi} &= \Omega_\eta \end{aligned} \tag{1}$$

where

$$\Omega = \frac{1-\mu}{\varrho_1} + \frac{\mu}{\varrho_2} + \frac{1}{2}(\xi^2+\eta^2)$$

and the. shows derivatives with respect to τ.

We know that equations (1) have the well-known Jacobi integral

$$\dot{\xi}^2 + \dot{\eta}^2 = 2\Omega - C \tag{2}$$

where C is the Jacobi constant.

Now let us suppose that the primaries revolve around their center of mass in elliptic orbits. Taking the true anomaly θ of the primaries as the independent variable it can be shown, ref. 2, that the equations of motion of the third body are of the following form (now nonuniformly rotating coordinate system)

$$\xi'' - 2\eta' = \frac{1}{1+e\cos\theta}\,\Omega_\xi \tag{3}$$

$$\eta'' + 2\xi' = \frac{1}{1+e\cos\theta}\,\Omega_\eta$$

where Ω is the same function as before. The $'$ shows derivative with respect to θ and e is the common eccentricity of the elliptic orbits. The integral of equations (3) that corresponds to the Jacobi integral in circular case has now the following form.

$$\xi'^2 + \eta'^2 = \frac{2\Omega}{1+e\cos\theta} - C - 2e\int\frac{\Omega\sin\theta}{(1+e\cos\theta)^2}\,d\theta \tag{4}$$

Note that if we put $e = 0$ in the the equations (3) and (4) we get equations (1) and (2). So θ corresponds to τ in circular case. Note also that the distance between the primaries in circular case is fixed, but in elliptic case it is not. In fact if we show it now with r we have

$$r = \frac{a\,(1-e^2)}{1+e\cos\theta}$$

where a is the semi major axis of the elliptic orbit of the smaller primary about the other.

We will now see Rabe's application of the circular case to the Sun-Jupiter system. He takes the Sun and Jupiter as the two primaries and the Trojan moving about the Lagrangean triangular point L_5 as the third infinitesimal body. Jupiter describes a circular orbit relative to the Sun. The radius of this orbit is adopted as the unit of distance, the Sun's mass as the unit of mass. The unit of time is chosen such that the gravitational constant becomes unity. After a transformation of the ξ—η system to x—y system centered at Jupiter, rotating with uniform angular velocity with

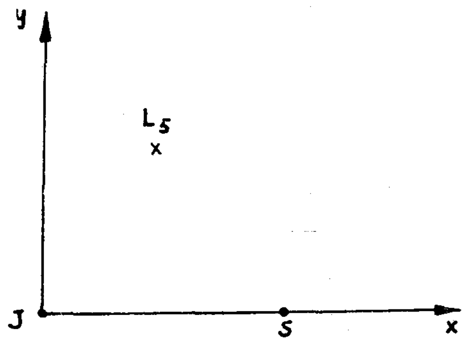

Fig. 3

the Sun fixed on the positive x axis, equations (1) of the third small body become

$$\ddot{x} - 2N y = (x-1)\left(1 - \frac{1}{\rho_1^3}\right) + Mx\left(1 - \frac{1}{\rho_2^3}\right)$$

$$\ddot{y} + 2N\dot{x} = y\left(1 - \frac{1}{\rho_1^3}\right) + My\left(1 - \frac{1}{\rho_2^3}\right) \qquad (5)$$

where N is the mean motion of Jupiter and M is the mass of Jupiter. The . shows derivative with respect to the time t. For numerical integration of equations (5) Rabe uses a method due to Steffenson 1956. He assumes that the solution can be represented as power series in the following manner

$$x = \sum_{n=0}^{\infty} a_n (t-t_0)^n$$
$$y = \sum_{n=0}^{\infty} b_n (t-t_0)^n \tag{6}$$

where t_0 is the starting time and a_n, b_n are constants. a_0, b_0 are the position components at time t_0. Derivating equations (6) once we see that a_1, b_1 are the velocity components at time t_0. These four constants are identified as the four constants of integration. If we know the initial position and velocity we put (6) into (5) and get the recursion formulas necessary to find a_2, b_2; a_3, b_3;... To find the right initial conditions to get a periodic orbit around L_5 Rabe supposes that the Trojan gets its maximum velocity relative to Jupiter at the time of the closest approach to the libration point L_5 and this point is on the line connecting L_5 to the Sun. This condition gives quadratic equations the small root of which is a good approximation for initial conditions to get a periodic orbit of long period $\sim$ 147.4 years around L_5. Rabe suggests that the other root can approximate the short periodic orbits $\sim$ 11.9 years around L_5.

On the other hand, following Rabe's way Goodrich 1965 found short period orbits around L_5. In fig. 4 wee see a long and short period orbit for the same initial position but different initial

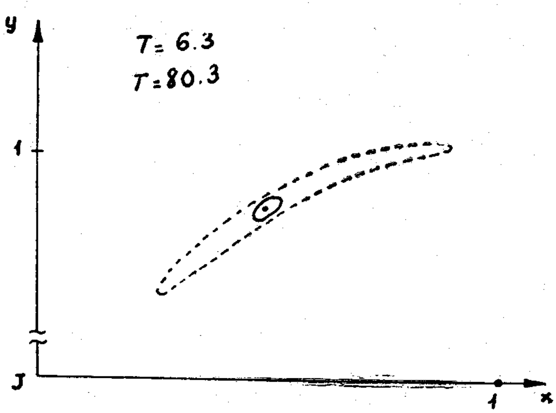

Fig. 4

velocity. The time unit being $\sim$ 1.89 years. These results are already expected from the linearized theory of the circular case see for instance ref. 6.

Now the question is do we still have periodic orbits around L_5 when we take elliptic orbit for Jüpiter around the Sun. We will first give Szebehely's analytical treatment. The equations of motion of the third body in elliptic case i.e. equations (3) are given in ref. 2 and are fully derived in ref. 7. Suppose x_0, y_0 are the particular solution at L_5. Introducing $x = x_0 + \xi$, $y = y_0 + \eta$ the variational equations of (3) becomes, neglecting the second and higher order terms in Taylor expansion, of the following form.

$$\xi'' - 2\eta' = \frac{1}{1+e\cos\theta}\left[\Omega_{xx}(x_0, y_0)\,\xi + \Omega_{xy}(x_0, y_0)\eta\right]$$

$$\eta'' + 2\xi' = \frac{1}{1+e\cos\theta}\left[\Omega_{yx}(x_0, y_0)\,\xi + \Omega_{yy}(x_0, y_0)\eta\right] \qquad (7)$$

where

$$\Omega_{xx} = \frac{3}{4}, \quad \Omega_{yy} = \frac{9}{4}$$

and

$$\Omega_{xy} = \Omega_{yx} = 3\sqrt{3}\,(1-2\mu)/4$$

Equations (7) describe the small amplitude motion of the infinisimal body around L_5, and are a system of linear differential equations with variable coefficients. The solution proposed by Szebehely is a power series in the eccentricity i.e.

$$\xi = \sum_{n=0}^{\infty} \xi_n(\theta)\, e^n$$

$$\eta = \sum_{n=0}^{\infty} \eta_n(\theta)\, e^n \qquad (8)$$

The zeroth order solution in the eccentricity becomes identical with the one known for the circular case where we have two types of periodic orbits long period (librational) orbits of period 147.4 years and short period orbits of period 11.9 years as mentioned before. Introducing the first and higher order solutions Szebehely gets additional frequences. He proves that for special μ values it is possible to find periodic orbits also in elliptic case. The short period terms in the solution show up in form of loops superposed on the smooth librational orbits. These loops diminish with decreasing eccentricity of Jupiters orbit.

Now we turn our attention to the numerical solutions of equation (3). We first make a transformation of coordinates from the center of mass to Jupiter as origin. Equations (3) will be now of the following form.

$$\ddot{x} - 2N\dot{y} = \frac{1}{1+e\cos Nt}\left[(x-1)\left(1-\frac{1}{\varrho_1 3}\right)+Mx\left(1-\frac{1}{\varrho_2 3}\right)\right]$$

$$\ddot{y} + 2N\dot{x} = \frac{1}{1+e\cos Nt}\left[y\left(1-\frac{1}{\varrho_1 3}\right)+My\left(1-\frac{1}{\varrho_2 3}\right)\right] \qquad (9)$$

where $N = \sqrt{1+M}$, $Nt = \theta$ and the . shows derivatives with respect to the time t.

To integrate equations (9) we used the Runge-Kutta fourth order numerical integration method taking $e = 0.04833499$ and $M = 0.00095478$. We took Rabe's initial values to start the integration. After making differential corrections the following initial condition gave us a periodic orbit of period 78.1913713 in elliptic case:

$$x_0 = 0.495, \quad y_0 = 0.874685658, \quad \dot{x}_0 = 0.013114103,$$
$$\dot{y}_0 = 0.007433322$$

for detail see ref. 7. This orbit is shown in fig. 5. The short period terms show up as loops superposed on the orbit as Szebehely remarked. Then Changing e from 0 to 1 we found that these loops get more and bigger for e's getting close to 1, and diminish when e goes to 0 which also confirms with Szebehely's results.

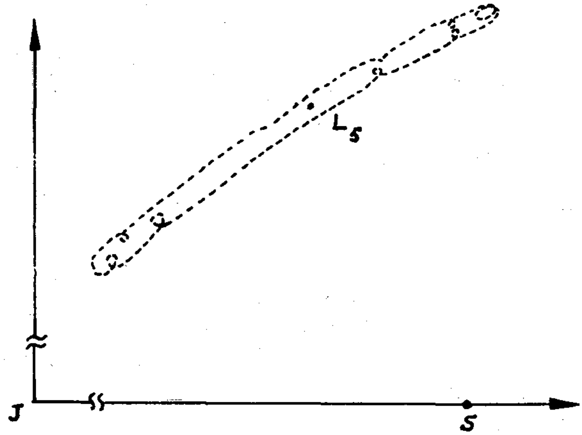

Fig. 5

As a result of both analytical and numerical treatments mentioned above we conclude that there are periodic orbits around the Lagrangean triangular points even when we take elliptic orbit for Jupiter.

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ÖZET

Lagrange üçgensel noktaları etrafındaki hareket 1961 de Rabe tarafından incelenmiş ve dairesel halde, yani iki esas cisim kütle merkezleri etrafında çemberler çizdiği zaman, uzun periyodlu yörüngelerin varlığı nümerik olarak ıspatlanmıştır. Aynı metodla Goodrich 1965'te kısa periyodlu yörüngeler elde etmiştir.

1967 de, biz Rabe'nin ilk şartlarından birinin eliptik halde de periyodik yörünge verdiğini gösterdik. Szebehely ise eliptik hali analitik olarak inceledi. Bu yazıda bizim nümerik neticelerimizle Szebehely'in analitik neticeleri arasındaki uygunlukları göstereceğiz.

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