Self-superposable Fluid Motions (Second Paper)

By

A. N. ERGUN (*)

(Department of Mathematics, Faculty of Science, Ankara University)

Özet : Lüzuci, sıkıştırılamaz ve homogen bir akışkanın hareket diferensiyel denklemleri lineer değildirler. Bu yüzden onların genel çözümlerini bulmak çok güç ve hattâ imkânsızdır. Araştırıcılar ya özel tarzda çözümler bulmakla iktifa ederler veya çözümü sadeleştirici bazı kabuller yaparlar. Bu makalenin yazarının yaptığı sadeleştirici kabul akışkan hareketinin kendi kendisi üzerine bindirilebilme özeliğidir (ss. şartı). Yazar evvelce yayınlamış olduğu bir makalesinde [1] ss. hareketlerden düzlemsel, bir eksene göre simetrik, birinci ve ikinci nevi yarı düzlemsel (pseudo-plane) hareketleri incelemiş, bu yazısında da birinci nevi, bir eksene göre yarı simetrik hareketleri bulmağa çalışmıştır.

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Summary : The equations of motion of a viscous homogeneous incompressible fluid are not linear. Therefore always some assumptions are made to simplify the solution of the equations. The assumption used in this paper is the self-superposability (ss.) property of the motion. Pseudo-axisymmetric ss. motions of the first kind are discussed, both in the steady and non-steady case.

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1. Introduction - As this paper will be a continuation of a previous paper of mine [1], I shall summarise some paragraphs of the first article of that paper in order to explain my aim in this work.

The equations of motion of a viscous incompressible homogeneous fluid can be written as

(*) **Adres** : Doç. Dr. A. N. Ergun. Fen Fakültesi — **Ankara**

$$\frac{\partial U}{\partial t} - U \times (\nabla \times U) + \nu \nabla \times (\nabla \times U) = - \nabla H \qquad (1{\cdot}1)$$

$$\text{and} \qquad \nabla \cdot U = 0 \, , \qquad (1.2)$$

where $U = (u, v, w)$ is the velocity vector,

$$H = \frac{p}{\rho} + \frac{1}{2} U^2 + \Omega \, , \qquad (1.3)$$

ν the kinematic coefficient of viscosity, ρ the density, p the pressure, Ω the force potential $(F = - \nabla \Omega)$, and

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \qquad (1.4)$$

is a differential operator. We assume ρ and ν to be constants.

If we apply the operator $\nabla \times$ to both sides of (1.1) we obtain

$$\nabla \times \frac{\partial U}{\partial t} - \nabla \times [U \times (\nabla \times U)] + \nu \nabla \times [\nabla \times (\nabla \times U] = 0 \, . \quad (1.5)$$

This equation is the kinematic consistency equation. It is the consistency condition of the three scalar equations in (1 1). Hence the determination of a fluid motion will consist of two successive processes. The first is to determine the velocity field by (1.5), and the second is to determine the pressure by (1.1) and (1.3).

The general solution of (1.5) is difficult because of the non-linear terms. For this reason always some assumptions is made to simplify the equation. The principal assumption which we shall make in this memoir is to use the self-superposability (ss.) property of the motion.

Since the equations of motion are not linear their solutions are not, in general, superposable. If U_1 and U_2 are any two solutions of the equations of motion of a viscous incompressible fluid corresponding to given external forces, initial and boundary conditions, not necessarily the same in both cases, they are superposable on each other if and only if

$$U_1 \times (\nabla \times U_2) + U_2 \times (\nabla \times U_1) = \nabla \chi \, , \qquad (1.6)$$

where χ is an arbitrary scalar function of x, y, z and t. This is the superposability condition [2]. If $U_1 = U_2 = U$ we obtain the self-superposability condition (or simply ss. condition)

$$U \times (\nabla \times U) = \nabla \chi \, , \qquad (1.7)$$

where as before χ means any scalar function.

If U is ss. the middle term in (1.5) vanishes, since

$$\nabla \times [U \times (\nabla \times U)] = \nabla \times \nabla \chi = 0, \tag{1.8}$$

and the consistency equation reduces to

$$\nabla \times \frac{\partial U}{\partial t} \times \nu \nabla \times [\nabla \times (\nabla \times U)] = 0. \tag{1.9}$$

Hence the use of ss. condition is to devide the equation into two parts, which is an obvious simplification.

The object of this paper is to find exact solutions of the equations of motion, when they are simplified by the self-superposability condition. In the previous paper plane and pseudo-plane motions and also axisymetric motions were discussed. In this one pseudo-axisymmetric motions of the first kind are discussed. Both the steady and non-steady solutions of the simultaneous equations are obtained.

2. Definitions - An axisymmetric motion is defined as the motion which has the following two properties:

(*a*) Trajectories are situated in planes passing through an axis (say Oz),

(*b*) The motion is identical in every such plane.

In cylindrical coordinates the first property is expressed by writing $v_2 = 0$, and the second one states that v_1 and v_3 do not depend on θ. Hence the component velocities are

$$v_1 = v_1(r, z, t,), \qquad v_2 = 0, \qquad v_3 = v_3(r, z, t,). \tag{2.1}$$

This kind of motion is discussed in the first paper [1].

But these two properties may not simultaneously be satisfied. Then we obtain pseudo axisymmetric motions (*). If for example (*a*) is satisfied but (*b*) is not satisfied, we have pseudo axisymmetric motions of the first kind. The velocity components are

$$v_1 = v_1(r, \theta, z, t), \qquad v_2 = 0, \qquad v_3 = v_3(r, \theta, z, t), \tag{2.2}$$

which contains (2.1) as a particular case.

Again if (*b*) is satisfied, but (*a*) is not satisfied, we obtain pseudo axisymmetric motions of the second kind. In this case the velocity components are of the form

$$v_1 = v_1(r, z, t), \qquad v_2 = v_2(r, z, t), \qquad v_3 = v_3(r, z, t), \tag{2.3}$$

which also contains (2.1) as a particular case.

Hence pseudo-axisymmetric motions are more general than

(*) The nomenclature is due to R. Berker [3].

real axisymmetric motions. In the next three articles we shall discuss the pseudo-axisymmetric ss. motions of the first kind, leaving motions of the second kind to a next paper.

3. Pseudo-axisymmetric ss. motions of the first kind. These are the motions in planes passing through an axis (say, Oz, in cylindrical coordinate system), but the motion is not identical in every plane, i.e. it depends on θ, which varies as we pass from one plane to the other. Hence the velocity components are

$$v_1 = v_1(r, \theta, z, t), \qquad v_2 = 0, \qquad v_3 = v_3(r, \theta, z, t). \tag{3.1}$$

The equation of continuity is

$$\frac{\partial (r v_1)}{\partial r} + \frac{\partial (r v_3)}{\partial z} = 0 ,$$

which suggests the existence of a stream function $\psi = \psi(r,\theta,z,t)$, such that

$$v_1 = \frac{1}{r}\psi_z , \qquad v_3 = -\frac{1}{r}\psi_r , \tag{3.2}$$

where ψ_z denotes the partial derivative of ψ with respect to z.

Now ss. condition (1.8) gives three equations

$$\left.\begin{aligned} &\left[\frac{1}{2r^2}(\psi_r^2+\psi_z^2)\right]_{\theta z} - \left[\psi_z \cdot \frac{1}{r^2} D_2\psi\right]_\theta = 0 \\ &\left[\frac{1}{2r^2}(\psi_r^2+\psi_z^2)\right]_{\theta r} - \left[\psi_r \cdot \frac{1}{r^2} D_2\psi\right]_\theta = 0 \end{aligned}\right\} , \tag{3.3}$$

$$\frac{D\left(\psi , \dfrac{1}{r^2} D_2 \psi\right)}{D(r , z)} = 0 , \tag{3.4}$$

where the last equation is a Wronski determinant.

The consistancy equation (1.9) also gives three equations

$$\left.\begin{aligned} &\nu\left[(D_2'\psi_r) + \frac{2}{r^3}\psi_{\theta\theta} + \frac{2}{r}\psi_{zz}\right]_\theta - \psi_{r\theta t} = 0 \\ &\nu\left[(D_2'\psi)_z - \frac{2}{r}\psi_{rz} + \frac{4}{r^2}\psi_z\right]_\theta - \psi_{z\theta t} = 0 \end{aligned}\right\} , \tag{3.5}$$

$$\nu\left[D_2(D_2'\psi) + \frac{2}{r^3}\psi_{r\theta\theta} - \frac{8}{r^4}\psi_{\theta\theta}\right] - (D_2\psi)_t = 0 , \tag{3.6}$$

where

$$D_2\psi = \psi_{rr} + \frac{1}{r}\psi_r + \psi_{zz} \tag{3.7}$$

$$D'_2\psi = \psi_{rr} + \frac{1}{r}\psi_r + \psi_{zz} + \frac{1}{r^2}\psi_{\theta\theta} \,. \tag{3.8}$$

The vorticity components are

$$\xi_1 = -\frac{1}{r^2}\psi_{r\theta}\,, \qquad \xi_2 = \frac{1}{r}D_2\psi\,, \qquad \xi_3 = -\frac{1}{r^2}\psi_{z\theta}\,. \tag{3·9}$$

The system (3.3) is equivalent to the system

$$\left[\frac{1}{2r^2}(\psi_r^2+\psi_z^2)\right]_z - \psi_z \cdot \frac{1}{r^2}D_2\psi = P_z\,, \tag{3.10}$$

$$\left[\frac{1}{2r^2}(\psi_r^2+\psi_z^2)\right]_r - \psi_r \cdot \frac{1}{r^2}D_2\psi = P_r\,, \tag{3.11}$$

and (3.4) is a consequence of these two, where P is an arbitrary function of r, θ z and t, such that $P_{z\theta} = P_{r\theta} = 0$, i. e. neither P_r nor P_z depends on θ.

The equations (3.5) cen be integrated once with respect to θ, which give

$$\nu\left[(D_2'\psi)_r + \frac{2}{r^3}\psi_{\theta\theta} + \frac{2}{r}\psi_{zz}\right] - \psi_{rt} = A(r, z, t,)\,,$$

$$\nu\left[(D_2'\psi)_z - \frac{2}{r}\psi_{rz} + \frac{4}{r^2}\psi_z\right] - \psi_{zt} = B(r, z, t)\,,$$

where A and B are two arbitrary functions depending only on r, z and t, but not on θ.

By taking account of the equation (3.6), it can easily be shown that A and B satisfy the relation

$$\frac{\partial}{\partial r}\left(\frac{A}{r}\right) + \frac{\partial}{\partial z}\left(\frac{B}{r}\right) = 0\,.$$

Hence there exists an arbitrary function Q of r, θ, z and t, such that

$$A/r = -Q_z\,, \qquad B/r = Q_r$$

and the equations above can be written as follows

$$\nu\left[(D_2'\psi)_r + \frac{2}{r^3}\psi_{\theta\theta} + \frac{2}{r}\psi_{zz}\right] - \psi_{rt} = -rQ_z\,, \tag{3.12}$$

$$\nu\left[(D_2'\psi)_z - \frac{2}{r}\psi_{rz} + \frac{4}{r^2}\psi_z\right] - \psi_{zt} = rQ_r\,, \tag{3.13}$$

where $Q_{r\theta} = Q_{z\theta} = 0$.

Since the equation (3.6) is a consequence of these two last equations the system of equations (3.10) — (3.13) form the kinematic conditions of the motions in question.

As for the dynamical condition, (1.1) gives the following result after some calculations :

$$H = Q - P + \frac{2\nu}{r^2}\psi_z + \frac{1}{2r^2}(\psi_r^2 + \psi_z^2) + C(t)\,,$$

where $C(t)$ is an arbitrary function of time. Hence from (1.3) we find

$$\frac{p}{\rho} + \Omega = Q(r, \theta, z, t) - P(r, \theta, z, t) + \frac{2\nu}{r^2}\psi_z + C(t), \tag{3.14}$$

where P and Q, as they are defined above, are two arbitrary functions such that P_r, P_z, Q_r, Q_z, do not depend on θ, and $C(t)$ is arbitrary. H and therefore p is determined only except for an arbitrary additive function of time.

4. Solution of the equations in the steady case. In cylindrical coordinates the stream function ψ must be a solution of the system of equations

$$\left[\frac{1}{2r^2}(\psi_r^2 + \psi_z^2)\right]_z - \psi_z \cdot \frac{1}{r^2} D_2\psi = P_z\,, \tag{3.10}$$

$$\left[\frac{1}{2r^2}(\psi_r^2 + \psi_z^2)\right]_r - \psi_r \cdot \frac{1}{r^2} D_2\psi = P_r\,, \tag{3.11}$$

$$\nu\left[(D_2'\psi)_r + \frac{2}{r^3}\psi_{\theta\theta} + \frac{2}{r}\psi_{zz}\right] = -rQ_z\,, \tag{3.12'}$$

$$\nu\left[(D_2'\psi)_z - \frac{2}{r}\psi_{rz} + \frac{4}{r^2}\psi_z\right] = rQ_r\,, \tag{3.13'}$$

where P, Q are arbitrary functions of r, θ, z, such that P_r, P_z, Q_r and Q_z are independent of θ.

$$D_2\psi = \psi_{rr} + \frac{1}{r}\psi_r + \psi_{zz} \quad ,$$

$$D_2'\psi = \psi_{rr} + \frac{1}{r}\psi_r + \psi_{zz} + \frac{1}{r^2}\psi_{\theta\theta} \, .$$

In each case we shall first try to satisfy the equation

$$\frac{D\left(\psi, \frac{1}{r^2} D^2\psi\right)}{D(r,z)} = 0 \, , \tag{3.4}$$

which is a direct consequence of (3.10) and (3.11) expressing $P_{zr} - P_{rz} = 0$.

A. Let ψ be linear in z.

$$\psi = F(r, \theta)z + G(r, \theta) \, , \tag{4.1}$$

where F and G are two unknown functions of r and θ alone.

Now (3.4) gives

$$(zF_r + G_r)\left(F_{rr} - \frac{1}{r}F_r\right)$$
$$- F\left[zF_{rrr} + G_{rrr} - \frac{3}{r}(zF_{rr} + G_{rr}) + \frac{3}{r^2}(zF_r + G_r)\right] = 0,$$

which must be true for all values of z, hence

$$F_r\left(F_{rr} - \frac{1}{r}F_r\right) - F\left(F_{rrr} - \frac{3}{r}F_{rr} + \frac{3}{r^2}F_r\right) = 0 \, , \tag{4.2}$$

$$G_r\left(F_{rr} - \frac{1}{r}F_r\right) - F\left(G_{rrr} - \frac{3}{r}G_{rr} + \frac{3}{r^2}G_r\right) = 0 \, . \tag{4.3}$$

Both the equations are satisfied if $F = 0$, i.e.

$$\psi = G(r, \theta) \, .$$

Then we easily obtain

$$P_z = P_r = 0 \, ,$$

i. e. P is a function of θ only.

$$Q_r = 0 \, ,$$

$$-Q_z = \frac{\nu}{r}\left(G_{rrr} - \frac{1}{r}G_{rr} + \frac{1}{r^2}G_r + \frac{1}{r^2}G_{r\theta\theta}\right) \qquad (4\ 4)$$

and $Q_{rz} - Q_{zr} = 0$ requires that

$$G_{rrr} - \frac{1}{r}G_{rr} + \frac{1}{r^2}G_r + \frac{1}{r^2}G_{r\theta\theta} = C(\theta)\, r\,, \qquad (4.5)$$

where C (θ) is arbitrary. But $Q_{z\theta}^{'} = 0$ also requires $C' = 0$, hence C is an absolute constant.

To solve the equation (4.5) we assume the relation

$$G_{r\theta\theta} = \mu\, G_r\,, \qquad (4.6)$$

where μ is an arbitrary constant. We distinguish three cases.

(*i*) If μ is positive, let $\mu = m^2$. Then (4 6) gives

$$G_r = A\, e^{m\theta} + B\, e^{-m\theta},$$

where A and B are arbitrary functions of *r*, and (4.5) requires

$$r^2A'' - rA' + (1 + m^2)A = 0,$$
$$r^2B'' - rB' + (1 + m^2)B = 0,$$
$$\text{and} \qquad C = 0.$$

These equations have the solutions

$$A(r) = a_1 r^{1+mi} + a_2 r^{1-mi}$$
$$B(r) = b_1 r^{1+mi} + b_2 r^{1-mi}\,,$$

where a_i, b_i are arbitrary constants of integration. Hence

$$\psi = G(r,\theta) = \left(\frac{a_1}{2+mi}\, r^{2+mi} + \frac{a_2}{2-mi}\, r^{2-mi}\right)e^{m\theta} +$$
$$\left(\frac{b_1}{2+mi}\, r^{2+mi} + \frac{b_2}{2-mi}\, r^{2-mi}\right)e^{-m\theta},$$

which can be written as

$$\psi = \sum_m r^n\left(\alpha_m\, e^{m\theta} + \beta_m\, e^{-m\theta}\right), \qquad (4.7)$$

where α_m, β_m are arbitrary constants, and

$$n = 2 \mp mi\,,$$

since the equation (4.5) is linear.

(*ii*) If μ is negative, let $\mu = -m^2$. Then (4.6) gives

$$G_r = A\cos m\theta + B\sin m\theta, \qquad (4.8)$$

where A, B are arbitrary fonctions of r. (4.5) requires

$$r^2A'' - rA' + (1 - m^2)A = 0 ,$$

$$r^2B'' - rB' + (1 - m^2)B = 0 ,$$

and $C = 0$.

After solving and replacing their values in (4.8), we can show that the expression for $\psi = G(r, \theta)$ can be written

$$\psi = \sum_m r^n (\alpha_m \cos m\theta + \beta_m \sin m\theta) , \quad (4.9)$$

where $n = 2 \mp m$.

(*iii*) When $\mu = 0$,

$$G_r = A\theta + B ,$$

A, B being arbitrary functions of r. (4.5) furnishes the equations

$$r^2A'' - rA' + A = 0 ,$$

$$r^2B'' - rB' + B = Cr^3 ,$$

and C is an arbitrary constant.

After solving we obtain

$$A(r) = (a_1 \log r + a_2) r,$$

$$B(r) = (b_1 \log r + b_2) r + \frac{1}{4} Cr^3.$$

Hence, after integrating once with respect to r, we find

$$\psi = G(r, \theta) = r^2 (\alpha \log r + \beta) \theta + (\gamma \log r + \delta) r^2 + \frac{1}{16} Cr^4 , \quad (4.10)$$

where α, β, γ, δ are arbitrary constants.

The solutions (4.7) and (4.9) are just R. Berker's solutions (16.23) and (16.24) with $k = 0$. [3]. This shows that those solutions are self-superposable when $k = 0$. But (4.10) looks to be more general than his (16.25).

For the solutions (4.7), (4.9) and (4.10) the pressure is given by

$$\frac{p}{\rho} + \Omega = - \nu Cz + K(\theta) ,$$

where $K(\theta)$ is an arbitrary function of θ. The solutions (4.7), (4.9) and (4.10) represent flows parallel to the axis of z.

If $F \neq 0$, the equation (4.2) can be written in the form

$$\frac{F_r}{F} = \frac{F_{rrr} - \frac{1}{r} F_{rr} + \frac{1}{r} F_r}{F_{rr} - \frac{1}{r} F_r} - \frac{2}{r},$$

which after integrating with respect to r gives

$$F_{rr} - \frac{1}{r} F_r - h(\theta).r^2 F = 0, \qquad (4.11)$$

where $h(\theta)$ is arbitrary.

Inserting $F_{rr} - \frac{1}{r} F_r$ from (4.11), (4.3) becomes

$$G_{rrr} - \frac{3}{r} G_{rr} + \left(\frac{3}{r^2} - hr^2 \right) G_r = 0 . \qquad (4.12)$$

If we change the independent variable by $r^2 = \rho$, the equation (4.11) transforms into

$$F_{\rho\rho} - \frac{1}{4} h(\theta) F = 0 . \qquad (4.13)$$

Now there are three cases as $h(\theta)$ is zero, positive or negative.

(*i*) $h(\theta) = 0$. Then

$$F(r, \theta) = A(\theta) r^2 + B(\theta), \qquad (4.14)$$

where $A(\theta)$ and $B(\theta)$ are arbitrary functions of θ to be determined by the remaining conditions (3.10) — (3.13′).

As for the equation (4.12) for G, let

$$G_r = r f(r, \theta) ,$$

then (4.12) becomes

$$f_{rr} - \frac{1}{r} f_r - h(\theta) r^2 f = 0 ,$$

i. e. just the same equation as (4.11). Hence we can take

$$f(r, \theta) = k(\theta). F(r, \theta) ,$$

where $k(\theta)$ is now arbitrary.

Now from $G_r = rf$, we have

$$G_r = rkF$$
$$= k(Ar^3 + Br) \qquad \text{by (4.14)},$$
$$\therefore \qquad G(r, \theta) = k\left(\frac{1}{4}Ar^4 + \frac{1}{2}Br^2\right),$$

neglecting an arbitrary function of θ, since ψ is determined only except for an arbitrary additive function of θ. But as A, B and k are arbitrary functions of θ, we can assume

$$kA = C(\theta), \qquad kB = D(\theta),$$

C and D being the new arbitrary functions of θ, and write

$$G(r, \theta) = \frac{1}{4}Cr^4 + \frac{1}{4}Dr^2.$$

Hence (4.1) becomes

$$\psi = (Ar^2 + B)z + \frac{1}{4}Cr^4 + \frac{1}{2}Dr^2, \tag{4.16}$$

where all the coefficients are functions of θ, to be determined.

Now let us satisfy the equations (3.10) — (3.13′), and determine the coefficients as functions of θ. After some calculations we obtain

$$P_z = 4A^2z + 2(AD - BC),$$
$$P_r = A^2r - \frac{B^2}{r^3},$$
$$-Q_z = \nu\left(C'' + 4C + \frac{1}{r^2}D''\right),$$
$$Q_r = \frac{4B\nu}{r^3},$$

where dashes refer to diferentiation with respect to θ.

$P_{zr} = P_{rz}$ is always satisfied, since we have first satisfied (3.4).

$Q_{zr} = Q_{rz}$ requires $D'' = 0$, (4.17)

$P_{z\theta} = 0$,, $AA' = 0$, (4.18)

and $(AD - BC)' = 0$ (4.19)

$P_{r\theta} = 0$ requires $AA' = BB' = 0$, (4.20)

$Q_{z\theta} = 0$,, $D''' = C''' + 4C' = 0$, (4.21)

$Q_{r\theta} = 0$,, $B' = 0$. (4.22)

The conditions (4.18), (4.20) and (4.22) are satisfied if A, B are numerical constants, say

$$A = a\,, \qquad B = b\,.$$

(4.17) gives $\quad D = c\theta + c_1\,,$

(4.21) gives $\quad C = \alpha \cos 2\theta + \beta \sin 2\theta + d\,,$

and the last one (4.19) requires

$$ac - 2b\,(\beta \cos 2\theta - \alpha \sin 2\theta) = 0\,, \qquad (4.23)$$

where all the coefficients a, b, c, d, α, β are absolute constants.

The relation (4.23) may be satisfied in one of the following four ways

(*i*) $a = b = 0$,

(2) $a = \alpha = \beta = 0$,

(3) $c = b = 0$,

(4) $c = \alpha = \beta = 0$.

To these correspond the following solutions

(1) $a = b = 0$,

$$\therefore\ \psi = \frac{1}{4} r^4 (\alpha \cos 2\theta + \beta \sin 2\theta + d) + \frac{1}{2} r^2 (c\theta + c_1). \qquad (4.24)$$

This represents a one dimensional streaming parallel to z-axis. If the region of motion contains some part of the axis, the uniformity of the velocity requires $c = 0$. Similar restrictions exist in the above solutions (4.7) and (4.10).

(2) $a = \alpha = \beta = 0$.

$$\therefore\ \psi = bz + \frac{1}{4} dr^4 + \frac{1}{2} r^2(c\theta + c_1). \qquad (4.25)$$

This is a two dimensional motion in planes passing through Oz. The axis is a line source or sink as $b > 0$ or $b < 0$. It is general than R. Berker's solution (16.25) when $k = 0$, [3].

(3) $c = b = 0$.

$$\therefore\ \psi = ar^2 z + \frac{1}{4} r^4 (\alpha \cos 2\theta + \beta \sin 2\theta + d) + \frac{1}{2} c_1 r^2. \qquad (4.26)$$

A two dimensional motion in planes passing through Oz.

The motion on the axis is entirely axial.

(4) $c = \alpha = \beta = 0.$

$$\therefore \psi = (ar^2 + b)z + \frac{1}{4} dr^4 + \frac{1}{2} c_1 r^2. \tag{4.27}$$

This is a two dimensional rotational motion which does not depend on θ at all. Hence it is an axisymmetric ss. motion.

In all cases above the vorticity vector lies in planes perpendicular to the axis, since $\psi_{z\theta}$ and therefore $\xi_3 = 0$. The pressure is given by

$$\frac{p}{\rho} + \Omega = 2\nu(a - 2dz) - \frac{1}{2} a^2(r^2 + 4z^2) - \frac{b^2}{r^2}$$
$$- 2z[a(c\theta + c_1) - b(\alpha \cos 2\theta + \beta \sin 2\theta + d)] + K(\theta), \tag{4.28}$$

where $K(\theta)$ is arbitrary. For example in the first case (1)

$$\frac{p}{\rho} + \Omega = -4d\nu z + K(\theta).$$

(*ii*) $h(\theta) > 0.$

Let $\frac{1}{4} h(\theta) = \lambda^2$, then the solution of (4.13) is

$$F(\rho, \theta) = A e^{\lambda\rho} + B e^{-\lambda\rho},$$

or

$$F(r, \theta) = A e^{\lambda r^2} + B e^{-\lambda r^2},$$

where A, B are arbitrary functions of θ, and λ is either a constant or an arbitrary function of θ.

Now by (4.15) we have

$$G_r = k(\theta) . rF$$
$$= kr(Ae^{\lambda r^2} + Be^{-\lambda r^2}).$$

$$\therefore \quad G(r, \theta) = \frac{1}{2\lambda} k(\theta).(A e^{\lambda r^2} - Be^{-\lambda r^2}),$$

or, since $k(\theta)$ is arbitrary, we can take

$$G(r, \theta) = \frac{1}{2\lambda}(C e^{\lambda r^2} - De^{-\lambda r^2}).$$

Hence the stream function (4.1) becomes

$$\psi = (A e^{\lambda r^2} + B e^{-\lambda r^2}) z + \frac{1}{2\lambda} (C e^{\lambda r^2} - D e^{-\lambda r^2}) , \qquad (4.29)$$

where all the coefficients are arbitrary functions of θ, to be determined by the equations (3.10) – (3.13'). After some lengthy calculations we obtain P_z, P_r, Q_z and Q_r, but unfortunately no values of A, B, C and D different from zero, can be found satisfying all the conditions, even in case when $h(\theta) = a$ constant. Hence my conclusion is that no ss. motions of this type exist, at least in the case when $h(\theta)$ is a constant.

B. Let ψ be linear in r.

$$\psi = F(z, \theta) r + G(z, \theta) . \qquad (4.30)$$

The equation (3.4) requires

$$F\left(G_{zzz} + rF_{zzz} - \frac{1}{r} F_z\right) - (rF_z + G_z)\left(\frac{3}{r^2} F - F_{zz} - \frac{2}{r} G_{zz}\right) = 0,$$

which furnishes

$$\left.\begin{aligned} F\, F_{zzz} + F_z F_{zz} &= 0 , \\ F\, G_{zzz} + F_{zz} G_z + 2F_z G_{zz} &= 0 , \\ 2F\, F_z - G_z G_{zz} &= 0 , \\ F\, G_z &= 0. \end{aligned}\right\} \qquad (4.31)$$

The last shows that either $F = 0$ or $G_z = 0$.

(*i*) If $F = 0$, the first two equations in (4.31) are satisfied, and the third requires $G_{zz} = 0$. Hence

$$\psi = G(z, \theta) = A(\theta) z + B(\theta) ,$$

i. e. ψ is linear in z, and this is included in (4.1).

(*ii*) If $G_z = 0$, the third equation in (4.31) shows that F_z is also zero. Hence $\psi = \psi(r, \theta)$, and we obtain the case §4A, (4.3).

C. Let ψ be of the form

$$\psi = F(r,z) . h(\theta) .$$

Then we have

$$D_2\psi = h\left(F_{rr} - \frac{1}{r} F_r + F_{zz}\right) = h\, D_2 F ,$$

and the equation (3.4) gives

$$F_r(D_2F)_z - F_z(D_2F)_r + \frac{2}{r} F_z \cdot D_2F = 0 .$$

This is satisfied particularly by (*i*) $F_z = 0$, (*ii*) $D_2F = Cr^2$, where C is an arbitrary constant.

In case (*i*) $F_z = 0$, and ψ becomes *a* function of r and θ alone, *i.e*

$$\psi = F(r).h(\theta) = \psi(r, \theta) ,$$

and this form is included in (4.3).

(*ii*) If $D_2F = Cr^2$, then $D_2\psi = h\, D_2F = hCr^2$, and we find

$$P_z = \frac{h^2}{2r^2}(F_r^2 + F_z^2)_z - hCF_z ,$$

$$P_r = \frac{1}{2} h^2 \left[\frac{1}{r^2}(F_r^2 + F_z^2)\right]_r - hCF_r .$$

Now $P_{z\theta} = P_{r\theta} = 0$ requires $h' = 0$, *i.e.* $h(\theta) = a$ constant. We also find out that all the other conditions $Q_{r\theta} = Q_{z\theta} = 0$, $Q_{rz} - Q_{zr} = 0$ are satisfied if $h' = 0$. In this case ψ does not depend on θ, and we obtain the well known solutions of

$$D_2\psi = Cr^2 ,$$

first given by U. Crudeli [4].

To resume our results, we can say that (4.7), (4.9) and (4.10) are the only solutions of the steady case in the form $\psi = G(r, \theta)$, and (4.24) (4.25) and (4.26) in the form $\psi = F(r, \theta)z + G(r, \theta)$. As far as I know, the solutions (4.10), (4.24), (4.25) and (4,26) are new.

5. Solution of the equations in the non-steady case.

In cylindrical coordinates the stream function ψ is a solution of the simultaneous equations (3.10) — (3.13).

First suppose that ψ is of the form

$$\psi = T \cdot \varphi(r, \theta, z) ,$$

where T is a function of t alone. The equations (3.10) and (3.11) become

$$\left[\frac{1}{2r^2}(\varphi_r^2+\varphi_z^2)\right]_z - \frac{1}{r^2}\varphi_z\cdot D_z\varphi = P_z/T^2\,, \tag{5.1}$$

$$\left[\frac{1}{2r^2}(\varphi_r^2+\varphi_z^2)\right]_r - \frac{1}{r^2}\varphi_r\cdot D_z\varphi = P_r/T^2\,, \tag{5.2}$$

and the equations (3.12) and (3.13) take the form

$$\nu\left[(D_2'\varphi)_r + \frac{2}{r^3}\varphi_{\theta\theta} + \frac{2}{r}\varphi_{zz}\right]T - \varphi_r\cdot T' = -\,rQ_z\,, \tag{5.3}$$

$$\nu\left[(D_2'\varphi)_z - \frac{2}{r}\varphi_{rz} + \frac{4}{r^2}\varphi_z\right]T - \varphi_z\cdot T' = rQ_r\,, \tag{5.4}$$

$Q_{z\theta}=0$ requires

$$\frac{\nu}{r}\left[(D_2'\varphi)_r + \frac{2}{r^3}\varphi_{\theta\theta} + \frac{2}{r}\varphi_{zz}\right]_\theta\cdot T - \frac{1}{r}\varphi_{r\theta}\cdot T' = 0\,,$$

$$\therefore\ \frac{T'}{T} = \frac{\nu\left[(D_2'\varphi)_r + \frac{2}{r^3}\varphi_{\theta\theta} + \frac{2}{r}\varphi_{zz}\right]_\theta}{\varphi_{r\theta}} = \text{constant} = -\nu k^2 \text{ say,}$$

where k is *a* constant. Hence

$$T \sim e^{-\nu k^2 t}\,,$$

and
$$(D_2'\varphi)_r + \frac{2}{r^3}\varphi_{\theta\theta} + \frac{2}{r}\varphi_{zz} + k^2\varphi_r = A(r,\,z).$$

Similarly $Q_{r\theta}=0$ requires

$$(D_2'\varphi)_z - \frac{2}{r}\varphi_{rz} + \frac{4}{r^2}\varphi_z + k^2\varphi_z = B(r,\,z)\,,$$

where A and B are arbitrary functions of r and z. But (5.3) and (5.4) show that

$$\nu AT = -\,rQ_z \quad \text{and} \quad \nu BT = rQ_r\,.$$

That is

$$(D_2'\varphi)_r + \frac{2}{r^3}\varphi_{\theta\theta} + \frac{2}{r}\varphi_{zz} + k^2\varphi_r = -\,rQ_z/\nu T\,, \tag{5.5}$$

$$(D_2'\varphi)_z - \frac{2}{r}\varphi_{rz} + \frac{4}{r^2}\varphi_z + k^2\varphi_z = rQ_r/\nu T\,. \tag{5.6}$$

The equations (5.1), (5.2), (5.5) and (5.6) are four equations to determine φ, P and Q corresponding to (3 10) — (3.13′) in the steady case. They are not independent since $P_{rz} = P_{zr}$ requires

$$\frac{1}{r^2} D_2\varphi = f'(\varphi) ,$$

and then (5.1) and (5.2) give

$$\frac{1}{2r^2}(\varphi_r^2 + \varphi_z^2) - f(\varphi) = P/T^2 .$$

Thus we have four equations to determine four unknown functions φ, P, Q, and f.

As in the steady case we shall consider the different forms of φ

A. Let φ be linear in z.

$$\varphi = F(r, \theta)z + G(r, \theta) .$$

F and G satisfy the ebuations (4,2) and (4,3), and both are satisfied if $F = 0$, i.e.

$$\varphi = G(r, \theta) .$$

Then we can easily obtain

$$P_z = P_r = 0 ,$$

i.e. P is *a* function of θ and t only.

$$\left.\begin{aligned} &Q_r = 0 , \\ &-Q_z = \frac{\nu T}{r}\left(G_{rrr} - \frac{1}{r} G_{rr} + \frac{1}{r^2} G_r + \frac{1}{r^2} G_{r\theta\theta} + k^2 G_r\right) \end{aligned}\right\} \quad (5.7)$$

Now $Q_{rz} - Q_{zr} = 0$ requires

$$G_{rrr} - \frac{1}{r} G_{rr} + \frac{1}{r^2} G_r + \frac{1}{r^2} G_{r\theta\theta} + k^2 G_r = C(\theta) r , \quad (5.8)$$

where $C(\theta)$ is arbitrary. But $Q_{z\theta} = 0$ also requires $C' = 0$, hence C is an absolute constant.

To solve the equation (5.8) we assume

$$G_{r\theta\theta} = \mu\, G_r , \quad (5.9)$$

where μ is an arbitrary constant.

(i) If μ is positive, let $\mu = m^2$. Then (5.9) gives

$$G_r = A e^{m\theta} + B e^{-m\theta},$$

where A , B are arbitrary functions of r , and (5.8) requires

$$r^2A'' - rA' + (1 + m^2 + k^2r^2)A = 0,$$

$$r^2B'' - rB' + (1 + m^2 + k^2r^2)B = 0,$$

$$\text{and} \quad C = 0 .$$

If we write $A = r\alpha$, where α is a function of r, we see that both A and B satisfy the equation

$$r^2\alpha'' + r\alpha' + (k^2r^2 + m^2)\,\alpha = 0,$$

whose solution is expressed in terms of Bessel's functions of imaginary order :

$$\alpha = \frac{1}{r} A = c_1 J_{mi}(kr) + c_2 Y_{mi}(kr) ,$$

$$\frac{1}{r} B = d_1 J_{mi}(kr) + d_2 Y_{mi}(kr) .$$

Hence

$$\psi = TG_r = e^{-\nu k^2 t}\{r[c_1 J_{mi}(kr) + c_2 Y_{mi}(kr)]e^{m\theta} + r[d_1 J_{mi}(kr) + d_2 Y_{mi}(kr)]e^{-m\theta}\} \quad (5.10)$$

$$\therefore \quad v_1 = v_2 = 0 , \quad v_3 = -\frac{1}{r}\psi_r .$$

(ii) If μ is negative, let $\mu = -m^2$. Then (5.9) gives

$$G_r = A\cos m\theta + B\sin m\theta ,$$

where A, B are arbitrary functions of r, and (5.8) requires

$$r^2A'' - rA' + (1 - m^2 + k^2r^2)\,A = 0 ,$$

$$r^2B'' - rB' + (1 - m^2 + k^2r^2)\,B = 0 ,$$

$$\text{and} \quad C = 0.$$

Again by writing $A = r\alpha$ we prove that A and B satisfy the equation

$$r^2\alpha'' + r\alpha' + (k^2r^2 - m^2)\,\alpha = 0,$$

$$\therefore \quad \alpha = \frac{1}{r} A = c_1 J_m(kr) + c_2 Y_m(kr),$$

$$\frac{1}{r} B = d_1 J_m(kr) + d_2 Y_m(kr).$$

Hence

$$\psi_r = TG_r = e^{-\nu k^2 t}\{ r(c_1 J_m + c_2 Y_m)\cos m\theta + r(d_1 J_m + d_2 Y_m)\sin m\theta \}, \tag{5.11}$$

$$v_1 = v_2 = 0, \qquad v_3 = -\frac{1}{r}\psi_r .$$

(iii) If $\mu = 0$, (5.9) gives

$$G_r = A\theta + B,$$

where A, B are arbitrary functions of r, and (5.8) requires

$$r^2A'' - rA' + (1 + k^2r^2)A = 0,$$
$$r^2B'' - rB' + (1 + k^2r^2)B = Cr^3 .$$

The substitution $A = r\alpha$ transforms the first equation into

$$r^2\alpha'' + r\alpha' + k^2r^2\alpha = 0.$$

Hence

$$A = r\alpha = r\left[c_1 J_0(kr) + c_2 Y_0(kr)\right],$$
$$B = r\left[d_1 J_0(kr) + d_2 Y_0(kr) + \beta(r)\right],$$

where $\beta(r)$ is a particular solution of

$$r^2\alpha'' + r\alpha' + k^2r^2\alpha = Cr^2,$$
$$\therefore \quad \beta(r) = C/k^2 = a \text{ constant.}$$

Hence

$$\psi_r = TG_r = e^{-\nu k^2 t}\{r\left[c_1 J_0(kr) + c_2 Y_0(kr)\right]\theta + r\left[d_1 J_0(kr) + d_2 Y_0(kr) + C/k^2\right]\}. \tag{5.12}$$

In all three cases above the pressure is given by

$$\frac{p}{\rho} + \Omega = -\nu Cz\cdot e^{-\nu k^2 t} + K(\theta, t). \tag{5.13}$$

If $F \neq 0$, we obtain as in § 4.A (i)

$$\varphi = (Ar^2 + B)z + \frac{1}{4}Cr^4 + \frac{1}{2}Dr^2, \tag{5.14}$$

where A, B, C, D are arbitrary functions of θ.
Now the equations (5.1) and (5.2) give

$$4A^2z + 2(AD - BC) = P_z/T^2,$$

$$A^2 r - \frac{B^2}{r^3} = P_r/T^2 .$$

$P_{r\theta} = P_{z\theta} = 0$ require $A' = B' = 0$, and

$$(AD - BC)' = 0.$$

Thus $A = a$, $B = b$, and $aD' - bC' = 0$.

The equations (5.5) and (5.6) show that $C = 0$, $D' = 0$. Hence we obtain finally

$$\psi = \left[(ar^2 + b)\, z + \frac{1}{2}\, dr^2\right] e^{-\nu k^2 t}, \qquad (5.15)$$

where a, b, d are constants, and the solution does not depend on θ.

The pressure is given by

$$\frac{p}{\rho} + \Omega = \left[2\nu a + \nu k^2 \left(\frac{1}{2}\, ar^2 + b \log r - az^2 - dz\right)\right] \cdot e^{-\nu k\ t}$$
$$- \left(\frac{1}{2}\, a^2 r^2 + \frac{b^2}{2r^2} + 2a^2 z^2 + 2adz\right) e^{-2\nu k\ t} + K(\theta,\ t).$$

The solutions (5.10), (5.11), (5.12) and (5.15) are some cases of the solutions of the equations of motion in the non-steady case, in the form $\psi = T \cdot \varphi\, (r,\ \theta,\ z)$. They are not the only solutions, since one may replace the condition (5.9) by any other, and also the expression for φ may not be linear in z.

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