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On Circulant Matrices

by

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On Circulant Matrices

by E. KAYA*

SUMMARY

In this article rows, we introduce a more generalized notion of circulant matrix, namely, q-rows *l*-circulant matrices. Suppose the matrix is n x n and q is a divisor of n so that the rows of the matrix can be partitioned into blocks of q–rows each. Let each row block be obtainened from the preceding one by shifting all its entries *l* places to the right. In generalizing the study to those q-row *l*-ciaculant matrices, we have obtained as special cases the reaults on 1-row *l*-circulant and 1-row 1-circulant matrices.

1. INTRODUCTION

Circulant matrices have a long past [1]. B. Friedman has studied the eigenvalues and canonical forms of composite matrices [2]. C. M. Ablow and J. L. Brenner have studied the canonical forms of 1- row and g- circulant matrices [3]

In this paper we shall give new proofs of some therems which have already been proved. We shall also give a new concept about circulant matrices which are not composite and continuant mat rices [1, 2]. By generalizing the notion of circulant matrix (see §2, Definitions 2, 3) we find it possible to locate all the roots and describe all of the vectors of the type of matrix that occurs in the Hurwitz-Routh theory; see the illustration on page 24 of this article.

This paper discusses different kinds of circulant matrices and some of their properties.

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2. Definition 1. Let P_n be the following nxn matrix.

$$\begin{bmatrix} 0 & 1 & 0\ldots0 & 0 \\ 0 & 0 & 1\ldots0 & 0 \\ \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot & \cdot \\ 0 & 0 & 0\ldots0 & 1 \\ 1 & 0 & 0\ldots0 & 0 \end{bmatrix}$$

Lemma 1. $P_n^n = I$

Definition 2. Let R_q be a qxn matrix, where $q < n$.

$$\begin{bmatrix} a_{11} & a_{12}\cdots a_{1n} \\ a_{21} & a_{22}\cdots a_{2n} \\ \cdot & \cdot\ \cdot\cdot\cdot\ \cdot \\ a_{q1} & a_{q2}\cdots a_{qn} \end{bmatrix}$$

Definition 3. Let R_q be a matrix of order qxn, and g, r, q be positive integers such that $1 \leqq g \leqq n$, $n = qr$.

A is the nxn matrix

$$A = \begin{bmatrix} R_q \\ R_q P_n^{qg} \\ R_q P_n^{2qg} \\ \cdot \\ \cdot \\ \cdot \\ R_q P_n^{(r-1)qg} \end{bmatrix}$$

$R_q P_n^{(\nu-1)qg}$ is called the ν-th row block of A.

In the general case, if the ν-th row block of A is $R_q P_n^{(\nu-1)l}$ we shall call A q-rows l- circulant. Thus we have the following definitions.

(i) If the ν-th row block of A is $R_q P_n^{(\nu-1)qg}$, A is called a q-rows qg-circulant matrix. (Called a g-cycle matrix by Friedman [2].)

(ii) If $g = 1$ and the ν-th row block of A is $R_p\ P_n^{(\nu-1)q}$, A is called a q-rows q- circulant matrix (continuant matrix)

(iii) If $q = 1$ and the ν-th row A is $R_1 P_n^{(\nu-1)g}$, A is called a 1- row g-circulant matrix (called a g-circulant matrix by J. L. Brenner [3].)

Theorem 1. A necessary and sufficient condition for a matrix A to be q-rows qg- circulant is

$$(1) \qquad P_n^q A = AP_n^{qg}$$

Proof. The condition is necessary. Let A be a q-rows qg-circulant matrix. If any nxn matrix B, where n is qr, is multiplied on the left by $P_n{}^q$ then the second row block of B is transformed into its first, the third block into the second, the r-th block into the r-1 st block and the first block into the r-th block. A being q-rows gq-circulant we have by definition 3

$$P_n^q A = \begin{bmatrix} R_q P_n^{qg} \\ R_q P_n^{2qg} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_q P_n{}^{(r-1)qg} \\ R_q \end{bmatrix}, AP_n^{qg} = \begin{bmatrix} R_q P_n^{qg} \\ R_q P_n^{2qg} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ R_q P_n{}^{(r-1)qg} \\ R_q P_n^{rqg} \end{bmatrix}$$

By lemma 1 we have $P_n{}^{rqg} = p_n{}^{ng} = (p_n{}^n)^g = I$. Thus every q-row qg-circulant matrix satisfies the relation (1).

The condition is sufficient. We shall show that every matrix satisfying relation (1) is a q-rows qg-circulant matrix.

Let the row blocks of the matrix A be U_1, U_2, ..., U_r. By (1) we have

$$U_{i+1} = U_i P_n{}^{qg}, \; i = 1, \ldots, r.$$

which shows that

$$U_1 = U_1, U_2 = U_1 p_n{}^{qg}, U_3 = U_1 p_n{}^{2qg}, \ldots, U_r = U_1 P_n{}^{(r-1)qg} \text{ ; or}$$

$$A = \begin{bmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_r \end{bmatrix} = \begin{bmatrix} U_1 \\ U_1 p_n{}^{qg} \\ \cdot \\ \cdot \\ \cdot \\ U_1 p_n{}^{(r-1)qg} \end{bmatrix}$$

By definition 3, A is a q-rows qg-circulant matrix.

Lemma 2. If A is a q-rows qg-circulant matrix, then

$$p_n^{qg} A = A p_n^{kqg}$$

where k is positive integer.

It can be proved by induction.

Theorem 2. The product of a q-rows qg-circulant matrix and a q-rows qh-circulant matrix is a matrix which is q-rows qgh-circulant.

Proof. We use theorem 1 and lemma 2. The special case for q= 1 is due to Ablow and Brenner [3]. Let A be a q-rows qg-circulant matrix and let B be q-rows qh- circulant. Considering theorem 1 and lemma 2 we have

$$p_n^q A = Ap_n^{qg}, p_n^q B = Bp_n^{qh}, \text{ and}$$
$$p_n^q AB = Ap_n^{qg} \quad B = ABp_n^{qgh}, \text{ or}$$
$$p_n^q AB = ABp_n^{qgh}$$

Corollary. If A and B are q-rows q-circulant matrices then

(i) $p_n^q A = AP_n^q$

(ii) $p_n^q AB = ABP_n^q$

(iii) $p_n^q A^k = A^k P_n^q$, where k is a positive integer.

Theorem 3. The inverses of q-rows q-circulant matrices are also q-rows q-circulant matrices.

Proof. Let A be a non-singular q-rows q-circulant matrix of order nxn. Multiplying both sides of the relation $AA^{-1} = I$ by p_n^q form the left, we have

$$P_n^q AA^{-1} = p_n^q, \ AP_n^q A^{-1} = P_n^q$$
$$p_n^q A^{-1} = A^{-1} p_n^q$$

Corollary. The adjoints of q-rows q-circulant matrices are q-rows q- circulant matrices.

Theorem 4. If A is a q-rows q-circulant matrix, then

$$p_n^q f(A) = f(A) p_n^q$$

where f(x) is a polymonial in the scalar variable x.

Proof. It can be proved by the corollary of theorem 2.

Theorem 5. (The commutative property of the multiplication of the 1-row 1-circulant matrices). If A and B are 1-row 1-circulant matrices of order nxn, then we have

$$AB = BA$$

Proof. This is well-known [1], [4, p. 95]. Let A and B be given as follows.

$$A = \begin{bmatrix} R \\ RP_n \\ \cdot \\ \cdot \\ \cdot \\ Rp_n^{n-1} \end{bmatrix} \quad B = \begin{bmatrix} R_1 \\ R_1P_n \\ \cdot \\ \cdot \\ \cdot \\ R_1p_n^{n-1} \end{bmatrix}$$

where $R = [a_{11}\ a_{12}\cdots a_{1n}]$, $R_1 = [b_{11}\ b_{12}\cdots b_{1n}]$.

By $P_nA=AP_n$ and $p_n^k\ A=Ap_n^k$ ν-th rows of AB and BA are $RP_n^{\nu-1}B=RBP_n^{\nu-1}$ and $R_1P_n^{\nu-1}A=R_1AP_n^{\nu-1}$. Now we show RB and R_1A are the same.

$$RB=[a_{11}a_{12}\cdots a_{1n}]\begin{bmatrix} R_1 \\ R_1P_n \\ \cdot \\ \cdot \\ \cdot \\ R_1p_n^{n-1} \end{bmatrix}=[a_{11}R_1+a_{12}R_1P_n+\cdots+a_{1n}R_1P_n^{n-1}] \text{ or}$$

$$RB = R_1[a_{11}I+a_{12}P_n+\cdots+a_{1n}P_n^{n-1}] = R_1A$$

3. In order to obtain the eigenvalues and canonical forms of a q-rows l- circulant matrix, we shall be concerned with certain subsets of the set of residue classes modulo r defined as follows.
Definition 4. Let $(g, r) = 1$

$$g^kh_i \equiv h_{i+1} \pmod r$$

where k is some positive integer.

If h_i is any residue modulo r, then

$$h_i,\ h_i\ g,\ h_i\ g^2,\ldots,\ h_ig^{t-1} \pmod r$$

is a subset of the residues, where t is the least positve integer such that $g^t \equiv 1 \pmod{r}$. Note that t divides $\varphi(r)$. This kind of subset modulo r is called by Friedman a minimal invariant set under multiplication by g (mod r). This form of the definition is given by J. L. Brenner [3,5].

We note that if $(g, r) = 1$, then Euler's theorem shows that t exists and satisfies the conditions $g^t = 1 \pmod{r}$, $t \mid \varphi(r)$. If $(g,r) > 1$, then no such integer t exists [8, p. 50].

As an illustration of this definition, consider the case where $r = 21$, $g = 4$. Then $t = 3$ and the minimal invariant sets are the following.

[0], [1, 4, 16], [2, 8, 11] [3, 12, 6], [5, 20, 17], [7],

[9, 15, 18] [10, 19, 13], [14].

We define the direct product of matrices [6]. Suppose that A is an nxn and B an mxm matrix. Then the direct product $A \otimes B$ is the nmxnm matrix defined by

$$A \otimes B = \begin{bmatrix} b_{11}A & b_{12}A & \dots & b_{1m}A \\ . & . & \dots & . \\ b_{m1}A & b_{m2}A & \dots & b_{mm}A \end{bmatrix}$$

By the definition, we have

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = A_1A_2 \otimes B_1B_2$$

Lemma 3. Let $n = qr$; $(q,r) = 1$ and let s be the n-th root of unity $s = \exp[2\pi i/n]$, and $X_l(s^h)$ be the column vector $(1, s^h, s^{2h}, \dots, s^{(l-1h)})^T$. Then

(i) $P_nX_n(s^h) = s^hX_n(s^h)$; $h = 1, \dots, n$.

(ii) $P_n^kX_n(s^h) = s^{hk}X_n(s^h)$, k is any positive integr.

(iii) The r-th roots of unity are

$$s^{qh}; \ h = 1, \dots, r$$

(iv) $X_q(s^h) \otimes X_r(s^{qh}) = X_n(s^h)$

(v) $I_q \otimes P_r = p_n^q$

(vi) $I_q \otimes P_r^k = P_n^{kq}$, k is any positive integer.

(vii) $R_q\,(X_q \otimes X_r\,(s^{qh})\,) = M(s^{qh})\,X_q$ where X_q is an arbitrary column vector with q components and $M\,(s^{qh})$ is a matrix of order qxq defined by

$$M\,(s^{qh}) = \sum_{\nu=1}^{r} A_\nu s^{(\nu-1)qh}, \text{ where}$$

$$A\nu = \begin{bmatrix} a_{1\,,\,(\nu-1)q+1} & a_{1\,,\,(\nu-1)q+2} & \cdots\cdots & a_{1\,,\,(\nu-1)q+q} \\ a_{2\,,\,(\nu-1)q+1} & a_{2\,,\,(\nu-1)q+2} & \cdots\cdots & a_{2\,,\,(\nu-1)q+q} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ a_{q\,,\,(\nu-1)q+1} & a_{q\,,\,(\nu-1)q+2} & \cdots\cdots & a_{q\,,\,(\nu-1)q+q} \end{bmatrix}$$

Theorem 6. Let A be q-rows qg-circulant matrix of order nxn. Let $h_i\,,\, h_t g,\, g_i g^2, \ldots,\, h_i g^{t-1};\ i = 1\,,\, \ldots,\, j$ be the distinct minimal invariant subsets of g(mod r); $(g, r) = 1$. Then the matrix A has the following representation,

$$W_1 \oplus W_2 \oplus \cdots \oplus W_j\,,$$

where $B \oplus C$ denotes $\begin{bmatrix} B & O \\ O & C \end{bmatrix}$ and

$$W_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & M(s^{hq^{t-1}h_i}) \\ M(s^{qh_i}) & 0 & \cdots & 0 & 0 \\ 0 & M(s^{qgh_i}) & \cdots & 0 & 0 \\ \cdot & & \cdots & & \cdot \\ 0 & 0 & \cdots & M(s^{qg^{t-2}h_i}) & 0 \end{bmatrix}$$

Proof. Let X_q be an arbitrary column vector with q components. We multiply both sides of the following relation on the right by $X_q \otimes X_r\,(s^{qh})$

$$A = \begin{bmatrix} R_q \\ R_q p_n^{qg} \\ \cdot \\ \cdot \\ \cdot \\ R_q p_n^{(r-1)qg} \end{bmatrix}$$

Using lemma 3 and the direct product of matrices , we reduce the ν-th row block of the product to the following form:

$$R_q P_n^{(\nu-1)qg} (X_q \otimes X_r (s^{qh})) = R_q (I_q \otimes P_r^{(\nu-1)g}) (X_q \otimes X_r s^{qg}) \text{ or}$$

$$R_q P_n^{(\nu-1)qg} (X_q \otimes X_r (s^{qh})) = R_q (X_q \otimes X_r (s^{qh})) \, s^{(\nu-1)qgh}.$$

Therefore we have

$$A(X_q \otimes X_r (s^{qh_i})) = (M(s^{qh_i}) X_q) \otimes X_q (s^{qgh_i})$$

$$A(X_q \otimes X_r (s^{qgh_i})) = (M(s^{gqh_i}) X_q) \otimes X_r (s^{gq2h_i})$$

$$\cdots \qquad \cdots$$

$$A(X_q \otimes X_r (s^{qg^{t-1}h_i})) = M(s^{qg^{t-1}h_i}) X_q) \otimes X_r (s^{qh_i}) \text{ or}$$

$$W_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & M(s^{qg^{t-1}h_i}) \\ M(s^{qh_i}) & 0 & \cdots & 0 & 0 \\ & & & & \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & M(s^{qg^{t-2}h_i}) & 0 \end{bmatrix}, i = 1, \ldots, j$$

where W_i is a broken diagonal matrix of order tqxtq. This result is due to Friedman [2].

Theorem 7. If A is a g-rows qg-circulant matrix, then

$$AX_n (s^h) = (M(s^{qh}) X_q(s^h)) \otimes X_r (s^{qgh}).$$

Prof. By lemma 3 (i), (ii) and (iv), and (vii)

$$AX_n (s^h) = R_q X_n (s^h) \otimes X_r (s^{qgh}) \text{ or}$$

$$AX_n (s^h) = M(s^{qh}) X_q (s^h) \otimes X_r (s^{qgh}).$$

As a special case for $q = 1$, $g = 1$, we have

$$AX_n(s^h) = (a_{11} + a_{12}s^h + \ldots + a_{1n}s^{(n-1)h}) X_n (s^h).$$

This is Ablow and Brenner's Theorem [3].

4. In the present section we shall investigate the q-rows g-circulant matrices. The type of matrices that occur in the Hurwitz-Routh theory [7; v. 2, Chapter XV].

Theorem 8. If A is a q-rows g-circulant matrix, then

$$AX_n(s^h) = M (s^{qh}) X_q(s^h) \otimes X_r(s^{gh})$$

Proof. We multiply both sides of the following relation on the right by $X_n (s^h)$.

$$A = \begin{bmatrix} R_q \\ R_q p_n^{\,g} \\ \cdot \\ \cdot \\ \cdot \\ R_q p_n^{(r-1)g} \end{bmatrix}$$

Using lemma 3 and the direct product of matrices, we reduce the ν-th row of the product to the following form:

$$R_q P_n^{(\nu-1)g} X_n(s^h) = R_q X_n(s^h)\, s^{(\nu-1)gh} = M(s^{qh}) X_q(s^h)\, s^{(\nu-1)gh}$$

Therefore we have

$$AX_n(s^h) = M(s^{qh}) X_q(s^h) \otimes X_r(s^{gh})$$

We shall discuss an application of theorem 8 and shall investigate the Hurwitz matrices [7; v. 2, p. 190].

A Hurwitx matrix H may be defined as follows. Let.

$$R_2 = \begin{bmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}$$

be a matrix of order 2xn, where $n = 2r$ and

$$a_k = 0 \text{ for } k > r$$
$$b_k = 0 \text{ for } k > r\text{-}1$$

We then define H by

$$H = \begin{bmatrix} R_2 \\ R_2 P_n \\ \cdot \\ \cdot \\ \cdot \\ R_2 p_n^{\,r-1} \end{bmatrix}$$

This is a 2-rows 1- circulant matrix.

Theorem 9. If H is a Hurwitz matrix of order nxn where $n = 2r$. Then.

$$HX_n(s^h) = M(s^{2h}) X_2(s^h) \otimes X_r(s^h)$$

Proof. This is a special case of theorem 8 for $q = 2$, $g = 1$. We have

$$HX_n(s^h) = M(s^{2h}) X_2(s^h) \otimes X_r(s^h)$$

As an illustration of this theorem consider the Hurwitz matrix of order 6x6, $q = 2$, $r = 3$.

$$
\begin{bmatrix}
b_0 & b_1 & b_2 & 0 & 0 & 0 \\
a_0 & a_1 & a_2 & a_3 & 0 & 0 \\
0 & b_0 & b_1 & b_2 & 0 & 0 \\
0 & a_0 & a_1 & a_2 & a_3 & 0 \\
0 & 0 & b_0 & b_1 & b_2 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & a_3
\end{bmatrix}
\cdot
\begin{bmatrix} 1 \\ s^h \\ s^{2h} \\ s^{3h} \\ s^{4h} \\ s^{5h} \end{bmatrix}
$$

$$
= \begin{bmatrix} b_0 + b_2 s^{2h} & b_1 \\ a_0 + a_2 s^{2h} & a_1 + a_3 s^{2h} \end{bmatrix}
\begin{bmatrix} 1 \\ s^h \end{bmatrix}
\otimes
\begin{bmatrix} 1 \\ s^h \\ s^{2h} \end{bmatrix}
$$

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ÖZET

Bu çalışmada. sirkülant matrisler daha genel anlamda yani q-satırlı l- sirkülant matrisler olarak ifade edilmiştir. q, n nin bir böleni olmak üzere n x n mertebeden bir matris q satırlı bloklara bölünmüş olduğunu kabul edelim. Her satır blok, bir önceki satır blokun sağdan l tane elemanını sol tarafa yer değişmsile elde edilmiş olsun,

Bu genelleştirme ile. ı- satırlı l- sirkülant ve ı- satırlı ı- sirkülant matrislere ait teoremler. q- satırlı l- sirkülant matrislere ait teoremlerin özel halleri olduğu görülür.

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