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# A General Formulation of an Extremalization Problem and its Application to Analytic Functions with Positive Real Part

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**SUMMARY**

The purpose of this note is to give, first, a general formulation of the extremal function within the class E of analytic functions that can be represented by means of a Stieltjes integral such as considered by Goluzin and then to apply the result in a systematic way, say, to analytic functions with positive real part. We also called attention to some errors in Goluzin's paper (translated by A. W. Goodman).

**INTRODUCTION**

The purpose of this note is to give, first, a general formulation of the extremal function within the Class E of analytic functions that can be represented by means of a Stieltjes integral such as considered by Goluzin and then to apply the result in a systematic way, say, to analytic functions with positive real part. In Goluzin's translated paper Theorem 1 on page 13 is not correct, for equation (9) has at most two roots and not four roots as it is claimed in the paper. Furthermore, the cited example to show that the extremal function (8) does not degenerate into the function s(z, t) is also not correct, for the assertion that the function $z^2/((1-z^2)^2 f(z)) \in T_r$ whenever $f(z) \in T_r$ is not true. Here $T_r$ is the class of typically-real functions analytic in the unit disk. [1].

## 1. *Variations of Analytic functions represented by Stieltjes Integral*

1A. Goluzin has considered a Class E of Analytic functions represented by a Stieltjes integral [1]

$$(1) \qquad f(z) = \int_a^b g(z,t)\, d\mu(t),$$

where a,b are given real numbers, $g(z,t)$ is a given function analytic in the unit disk K: $|z| < 1$ for $a \leqq t \leqq b$, and $\mu(t)$ runs through all possible nondecreasing functions in $a \leqq t \leqq b$, subject to the condition

$$\int_a^b d\mu(t) = \mu(b) - \mu(a) = 1.$$

He has obtained two variation formulas within E, i.e.,

$$(2) \qquad f^*(z) = f(z) + \lambda \int_{t_1}^{t_2} g_t(z,t)\, |\mu(t) - c|\, dt,$$

where $t_1 < t_2$ are arbitrary numbers in the interval $[a,b]$, $\lambda$ is an arbitrary number in the interval $-1 \leqq \lambda \leqq 1$, c is a constant with respect to t and $\lambda$, but depends on the sign of $\lambda$, and $g_t$ is the derivative of g with respect to t. Finally,

$$(3) \qquad f^{**}(z) = f(z) + \lambda\, [g(z,t_1) - g(z,t_2)],$$

where $t_1, t_2 \in [a,b]$ are two jump points for the function $\mu(t)$, and $\lambda$ a sufficiently small number.

1B. Coming back to the representation (1), we observe that if $\mu(t)$ is a step function with jump points at $t_1, t_2, \ldots, t_n$, with

$$a \leqq t_1 < t_2 < \ldots < t_n \leqq b,$$

and if $\lambda_k$ is the corresponding jump in $\mu(t)$, i.e.,

$$\lambda_k = \mu(t_k + 0) - \mu(t_k - 0),\ k = 1, \ldots, n,\ \sum_{k=1}^{n} \lambda_k = 1,$$

then

$$f(z) = \sum_{k=1}^{n} \lambda_k\, g(z,t_k)$$

*Remark.* From here on, we shall assume that for each fixed $z \in K$, $g(z,t)$ is analytic in a region containing the closed interval $[a,b] : a \leqq t \leqq b$. We shall continue to designate by E the class of analytic functions represented by the Stieltjes integral (1) in which $g(z,t)$ has the foregoing specification. Furthermore, for extremalization at a given point z, unless $f(z) = 0$ for all $f(z) \in E$, we may assume without loss of generality that $f(z) \neq 0$.

2. *Extremals within the Class E.*

2A. We consider the Class E, and show that

Teorem 1. For a given entire function $F(w)$ and a given point $z \in K$, either of the functionals

$$\text{(1)} \qquad \operatorname{Re}\{F(\log f(z))\}, \qquad |F(\log f(z))|$$

attains its maximum in E only for functions of the form

$$f(z) = \sum_{k=1}^{n} \lambda_k\, g(z,t_k)$$

with

$$\lambda_k \geqq 0,\ k = 1,\ldots, n,\ \sum_{k=1}^{n} \lambda_k = 1.$$

Here, we exclude from consideration the case in which for the extremal function, $F'(\log f(z)) = 0$.

Proof. Since E is compact and that the functionals are continuous on E, it follows that the extremal function exists. Indeed, the continuity is evident, as to the compactness of E, it is a consequence of the uniform boundedness of E inside the unit disk K, i.e., on every closed subset of K. In fact,

$$|f(z)| \leqq \max_{\substack{|z|=r \\ a \leqq t \leqq b}} |g(z,t)|, \quad |z| \leqq r < 1.$$

Moreover if $f(z)$ extremalizes the second of the functionals in (1) then for a suitable $\gamma$, it will extremalize

$$\operatorname{Re}\{e^{i\gamma}\, F(\log f(z))\}$$

which is nothing else but the first functional in (1), where $F(w)$ is replaced by the entire function $e^{i\gamma} F(w)$. Consequently it will suffice to prove the theorem only for the first functional in (1).

Denoting for simplicity by $G(z)$ the coefficient of $\lambda$ in formula (2) of paragraph 1, we have in view of this formula,

$$F(\log f^*(z)) = F(\log(f(z) + \lambda G(z))).$$

But, in view of the Remark, we may write,

$$\begin{aligned} \log(f(z) + \lambda G(z)) &= \log f(z)\,(1+\lambda G(z)/f(z)) \\ &= \log f(z) + \log(1+\lambda G(z)/f(z)) \\ &= \log f(z) + \lambda G(z)/f(z) + 0(\lambda^2). \end{aligned}$$

Hence,

$$F(\log f^*(z)) = F(\log f(z) + \lambda G(z)/f(z) + 0(\lambda^2)).$$

Or,

$$F(\log f^*(z)) = F(\log f(z)) + \lambda F'(\log f(z))G(z)/f(z) + 0(\lambda^2).$$

Finally, going back to the old notation, and setting

$$I_f = \mathrm{Re}\, F(\log f(z)),$$

we have,

$$I_{f^*} = I_f + \lambda \int_{t_1}^{t_2} \mathrm{Re}(F'(\log f(z))\, g_t(z,t)/f(z))\, |\mu(t) - c|\, dt + 0(\lambda^2).$$

For fixed z, in order that $I_f$ be extremal it is necessary that the coefficient of $\lambda$ vanishes. Namely,

$$\text{(2)} \qquad \int_{t_1}^{t_2} \mathrm{Re}(F'(\log f(z))\, g_t(z,t)/f(z))\, |\mu(t) - c|\, dt = 0,$$

where $t_1$, $t_2$ are arbitrary numbers in the interval $[a,b]$. From (2) it follows that $\mu(t) = c$ constant for any segment $(t_1, t_2)$ in which there are no roots of the equation

$$\text{(3)} \qquad \mathrm{Re}(F'(\log f(z))\, g_t(z,t)/f(z)) = 0.$$

But because of the fact that $g_t(z,t)$ is analytic in a region containing $[a,b]$, and assuming that $F'(\log f(z)) \neq 0$, it follows that for a fixed z, (3) has only a finite number of roots on $[a,b]$. If s is the number of roots, then $\mu(t)$ is a step function with no more than s points of discontinuity, say,

$$a \leqq t_1 < t_2 < \ldots < t_s \leqq b.$$

We now construct the second variation according to (3) of paragraph 1. We have as before

$$F(\log f^{**}(z)) = F(\log f(z)) + \lambda F'(\log f(z))(g(z,t_k) - g(z,t_l))/f(z) + 0(\lambda^2)$$

Taking real part of both sides, we obtain

$$I_{f^{**}} = I_f + \lambda \operatorname{Re}(F'(\log f(z))(g(z,t_k) - g(z,t_l))/f(z)) + 0(\lambda^2).$$

Here $(t_k,t_l)$ is either one of the intervals $(t_1,t_2),\ldots,(t_{s-1}, t_s)$.

For fixed z, in order that $I_f$ be extremal it is necessary that the coefficient of $\lambda$ vanishes. Namely,

$$\operatorname{Re}(F'(\log f(z))(g(z,t_k) - g(z,t_l))/f(z)) = 0.$$

This condition shows that

$$\operatorname{Re}(F'(\log f(z))\, g(z,t)/f(z)) \tag{4}$$

has the same value for $t_k$ and $t_l$. Hence the derivative of (4) with respect to t, i.e., the left hand side of (3) must vanish at points inside the intervals $(t_1, t_2),\ldots, (t_{s-1}, t_s)$. But then the equation (3) will have more than s roots which is impossible. This proves that $\mu(t)$ has at most $(s+1)/2$ jumps if s is odd and $s/2$ if s is even. In all cases if n denotes the maximum number of jumps of $\mu(t)$, then the extremal function will be of the form such as stated in Theorem 1.

2B. If we choose $F(w) = e^w$, then theorem 1 takes the form.

Theorem 2. Suppose that for each $z \in K$, $g(z, t)$ is analytic in a region containing the closed interval $a \leqq t \leqq b$. Then for a given $z \in K$, and $f(z) \in E$,

$$|f(z)| \leqq \left| \sum_{k=1}^{n} \lambda_k\, g(z,t_k) \right|.$$

3. *Application to Analytic Functions with Positive Real Part.*

3A. *Compactness and Distortion Theorem.* We consider analytic functions $h(z)$ defined in the unit disk $K$: $|z| < 1$, with the normalization $h(0) = 1$ and positive real part

$$\operatorname{Re}(h(z)) > 0.$$

We shall establish a lemma similar to the one given by G. Labelle and Q. I. Rahman. [2].

Lemma. Let $w=h(z)$ be analytic in the unit disk K, normalized and with positive real part. Then the image of the circle $|z| = r<1$ under $w = h(z)$ lies in the disk with centre $\omega = (1+r^2)/(1-r^2)$ and radius $\rho = 2r/(1-r^2)$.

In fact, we have

$$(1) \qquad \left| w - \frac{1+r^2}{1-r^2} \right| \leqq \frac{2r}{1-r^2}, \quad |z| \leqq r.$$

Proof. For analytic functions $w = h(z)$ defined in the unit disk K, normalized and with positive real part, we have the Herglotz-Stieltjes integral representation

$$(2) \qquad h(z) = \int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z} \, d\mu(t),$$

where $\mu(t)$ is non decreasing in $[0,2\pi]$ and $\mu(2\pi) - \mu(0) = 1$.

For each $t \in [0,2\pi]$, the linear substitution

$$T_t : z \longrightarrow \frac{1+e^{it}z}{1-e^{it}z}$$

transforms the circle $|z| = r < 1$ into the circle with centre $\omega = \dfrac{1+r^2}{1-r^2}$ and radius $\rho = \dfrac{2r}{1-r^2}$. Hence it follows from (2), that for $|z| \leqq r$

$$|w - \omega| \leqq \rho.$$

Theorem 1. If $w = h(z)$ is analytic in the unit disk K, normalized and with positive real part, then

$$(3) \qquad \frac{1-r}{1+r} \leqq |h(z)| \leqq \frac{1+r}{1-r}, \quad |z| \leqq r < 1.$$

Proof. The theorem follows at once from the triangle inequalities. The triangle in question has for vertices: 0, $\omega$, w, where 0 is the origin of the complex w-plane. Hence (1) reads

$$\left| \frac{1+r^2}{1-r^2} - |w-\omega| \right| \leqq |w| \leqq \frac{1+r^2}{1-r^2} + |w-\omega|.$$

Here $|w-\omega| \leqq \rho$. And (3) follows.

Note that the right hand side inequality in (3) follows at once from the representation (2).

From here on P will denote the class of functions analytic in the unit disk K, normalized and with positive real part.

Theorem 2. The class P is compact (hence normal) inside K.

Proof. Indeed, the inequality on the right-hand side of (3) shows that P is uniformly bounded inside K, i.e., on every closed subset of K.

Theorem 3. (Distortion theorem). If $h(z) \in P$, then

$$|h'(z)| \leqq \frac{2}{(1-r)^2}, \quad |z| \leqq r < 1. \tag{4}$$

Proof. We have for h'(z) the representation

$$h'(z) = \int_0^{2\pi} \frac{2e^{it}}{(1-e^{it}z)^2} d\mu(t). \tag{5}$$

From (5), the theorem follows.

In particular,

$$|h'(0)| \leqq 2. \tag{6}$$

Note that all our inequalities are sharp as the function $(1+z)/(1-z)$ in P shows. Furthermore, for $h(z) \in P$ with real coefficients,

$$\mathrm{Re}\left\{ \frac{1-z^2}{z} f(z) \right\} > 0, \quad |z| < 1$$

being necessary and sufficient in order that f(z) belongs to $T_r$, it follows from (6) that if $f(z) = z + a_2 z^2 + \dots \in T_r$, then

$$-2 \leqq a_2 \leqq 2.$$

3B. *Variation within the Class P.* Applying formally the formulas (2)and(3) of paragraph 1 to a function of Class P, we have

$$h^*(z) = h(z) + \lambda \int_{t_1}^{t_2} \frac{2ie^{it}z}{(1-e^{it}z)^2} |\mu(t) - c| \, dt \tag{7}$$

and

$$h^{**}(z) = h(z) + \lambda \left( \frac{1 + e^{it_1}z}{1 - e^{it_1}z} - \frac{1 + e^{it_2}z}{1 - e^{it_2}z} \right). \tag{8}$$

If $\Delta$ is any fixed closed subset of K, then for all $z \in \Delta$, $h^*(z)$ and $h^{**}(z)$ will have positive real part as soon as $\lambda$ is chosen sufficiently small.

3C. *Extremals within the Class P.* For each z, $|z| < 1$,

$$g(z,t) = \frac{1 + e^{it}z}{1 - e^{it}z}$$

is analytic with respect to t. On the other hand, if $F'(\log h(z)) \neq 0$ and $z \neq 0$, then

$$\operatorname{Re} \left( \frac{F'(\log h(z))}{h(z)} \; \frac{2ie^{it}z\,(1 - e^{-it}\bar{z})^2}{|1 - e^{it}z|^4} \right)$$

yields a cubic equation in $e^{-it}$ with at most three roots. Hence, $n = 2$ and we have.

Teorem 1. For a given entire function F(w) and a given point $z \in K$, either of the functionals

$$\operatorname{Re}\,(F(\log h(z)), \quad |F(\log h(z))|$$

attains its maximum in P only for functions of the form

$$h(z) = \frac{1 + e^{it_1}z}{1 - e^{it_1}z} \lambda_1 + \frac{1 + e^{it_2}z}{1 - e^{it_2}z} \lambda_2 \tag{9}$$

with $\lambda_1, \lambda_2 \geqq 0$, $\lambda_1 + \lambda_2 = 1$. Here, we exclude from consideration the case in which for the extremal function $F'(\log h(z)) = 0$.

In particular, if we choose $F(w) = e^w$, then

Theorem 2. For a given $z \in K$, and $h(z) \in P$, $|h(z)|$ attains its maximum in P for a function of the form (9).

4. *Application to Typically-Real Functions*

4A. *Variations within the Class $T_r$*. The function

$$f(z) = z + a_2z^2 + \ldots \tag{1}$$

analytic in the unit disk $K: |z| < 1$ is called typically-real in $|z| < 1$, if it is real on the diameter $-1 < z < 1$, and if at other points of K, $\operatorname{Im} f(z) > 0$ for $\operatorname{Im} z > 0$, $\operatorname{Im} f(z) < 0$ for $\operatorname{Im} z < 0$. We let $T_r$ denote the Class of functions (1) that are analytic and typically-real in K.

Each $f(z) \in T_r$ can be represented in K by the formula [3]

$$f(z) = \frac{1}{\pi} \int_0^\pi \frac{z}{1 - 2z\cos\theta + z^2} d\mu(\theta) \tag{2}$$

where $\mu(\theta)$ is a real non decreasing function in $0 \leqq \theta \leqq \pi$ with

$$\mu(\pi) - \mu(0) = \pi.$$

The formula (2) can be written in the equivalent form

$$f(z) = \int_{-1}^{1} \frac{z}{1 - 2tz + z^2} d\mu(t) \tag{3}$$

where $\mu(t)$ is a real non decreasing function in $-1 \leqq t \leqq 1$ with

$$\mu(1) - \mu(-1) = 1.$$

Note that for any fixed t, $-1 \leqq t \leqq 1$,

$$g(z,t) = \frac{z}{1 - 2tz + z^2} \tag{4}$$

is schlicht and belongs to $T_r$. Furthermore, we maintain that for any fixed z, $|z| < 1$, the denominator in (4) does not vanish in the interval $-1 \leqq t \leqq 1$. In fact if we write

$$-t = \frac{1+z^2}{2z}$$

we see that t is real only if z is on the unit circle or on the real axis. But for $-1 < z < 1$, one has $|t| > 1$. From this the assertion follows. Accordingly, g(z,t) is analytic in a region containing $-1 \leqq t \leqq 1$. Finally, both varied functions

$$f^*(z) = f(z) + \lambda \int_{t_1}^{t_2} g_t(z,t)\,|\,\mu(t) - c\,|\,dt$$

and

$$f^{**}(z) = f(z) + \lambda(g(z,t_1) - g(z,t_2)),$$

belong to $T_r$.

4B. *Extremals within the Class $T_r$.* We see that all the conditions of the Remark are satisfied, and that for $F'(\log f(z)) \neq 0$ and $z \neq 0$,

$$\operatorname{Re}\left(\frac{F'(\log f(z))}{f(z)}\, 2z^2\,(1-2t\bar{z}+\bar{z}^2)^2\right) = 0$$

has at most two roots in the interval $-1 \leqq t \leqq 1$. Hence

Theorem 1. For a given entire function F(w) and a given point $z \in K$, either of the functionals

$$\operatorname{Re}(F(\log f(z)), \quad |F(\log f(z))|$$

attains its maximum in $T_r$ only for a function of the form (4), i.e.,

$$f(z) = g(z,t).$$

Here, we exclude from consideration the case in which for the extremal function $F'(\log f(z)) = 0$.

In particular if we choose $F(z) = e^w$, then

Theorem 2. For a given $z \in K$, and $f(z) \in T_r$, $|f(z)|$ attains its maximum in $T_r$ for a function of the form (4).

4C. *Connection between the classes P and $T_r$.* In paragraph 3A we recalled that for functions in P with real coefficients [4]

$$\operatorname{Re}\left\{\frac{1-z^2}{z} f(z)\right\} > 0, \quad |z| < 1$$

if and only if $f(z) \in T_r$[1]. This means that if h(z) is a function of positive real part with real coefficients, then

$$h(z) = \frac{1-z^2}{z} f(z), \qquad f(z) \in T_r.$$

Conversely, if $f(z) \in T_r$, then $h(z) \in T_r$ with real coefficients. Hence, for the same z, using Theorem 1 of paragraph 4B, theorem 1 of paragraph 3B takes the form

Theorem 1. For a given entire function F(w) and a given point $z \in K$, either of the functionals

$$\operatorname{Re}(F(\log h(z))), \quad |F(\log h(z))|$$

attains its maximum in the Class of functions of positive real part with real coefficients only for a function of the form

$$h(z) = \frac{1}{2}\left(\frac{1+e^{i\theta}z}{1-e^{i\theta}z} + \frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\right) \tag{5}$$

Here, we exclude from consideration the case in which for the extremal function $F'(\log h(z)) = 0$.

Proof. For the extremal function in $T_r$, instead of (4), we use the expression under the sign integral in (2). Then the extremal in the class of functions of positive real part with real coefficients takes the form (5), i.e.,

$$\frac{1-z^2}{z} \frac{z}{1-2z\cos\theta+z^2} = \frac{1}{2}\frac{1+e^{i\theta}z}{1-e^{i\theta}z} + \frac{1}{2}\frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}.$$

In particular, if we choose $F(w) = e^w$, then

1 This again is not stated correctly in Goluzin's paper (translated by A.W. Goodman).

Theorem 2. For a given $z \in K$, and $h(z) \in P$ with real coefficients, $|h(z)|$ attains its maximum for a function of the form (5).

Conversely, Theorem 1 and Theorem 2 of 4C yield Theorem 1 and Theorem 2 of 4B, respectively.

## LITERATURE

[1] G. M. Goluzin, On a variational method in the theory of analytic functions, Leningrad. gosud. Univ. 144, 85-101, (1952). (Russian) Translated by A. W. Goodman, Amer. Math. Soc. Trans., II. Ser. 18, 1–14 (1961).

[2] G. Labelle et Q. I. Rahman, Remarque sur la moyenne arithmetique de fonctions univalentes convexes, Canadian J. Math., 977–981 (969).

[3] W. Rogosinski, Über positive harmonische Entwicklungen und typishrelle potenzreihen, Math. Z. 35, 93–121 (1932).

[4] A. C. Schaeffer and D. C. Spencer, Coefficient Regions for Schlicht functions, Amer. Math. Soc. 2–3, (1950).

## ÖZET

Bu makalede, Goluzin tarafından tetkik edilmiş Stieltjes integrali yardımıyla gösterilen analitik fonksiyonların E sınıfı içinde ekstremal fonksiyonun genel bir formülasyonu verilmektedir. Bu formül özel olarak reel kısmı pozitif olan analitik fonksiyonlara tatbik edilmiştir. Ayrıca, Goluzin'in makalesinde (A. W. Goodman tarafından tercümesinde) bazı önemli hatalara işaret edilmiştir.

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