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by

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Bresse and Inflection Congruences*

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ABSTRACT

The well known Bresse circle and inflection circle of planar kinematics are calculated on the unit dual sphere. Spatial equivalents of these circles are derived using the Study mapping. Hence two line congruences are obtained. At any time t there are three common lines of these congruences. These lines are the instantaneous acceleration axes of one parameter spatial motion which corresponds to the congruences. In general case and in some special cases, the properties of these congruences are discussed.

I. INTRODUCTION

The oriented lines in Euclidean space R^3 are in one-to-one correspondence with the points of the dual unit sphere in dual space D^3 [1]. Using this correspondence, one can derive the properties of the spatial motion of a line. Because this correspondence allows the geometry of congruences to be represented by the geometry of the two-parameter motion of a point on the unit dual sphere.

II. BASIC CONCEPTS

a) Plückerian Coordinates of an Oriented Line.

An orinted line in R^3 may be given by two points on it, $\vec{x}$ and $\vec{y}$. If p is any nonzero constant, the parametric equation of the line can be given in the form

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$$\vec{y} = \vec{x} + p\vec{a} , \qquad (2.1)$$

where $\vec{a}$ is the direction vector of the line. If $\vec{a^*}$ denotes the moment of the vector $\vec{a}$ with respect to the origin O we have

$$\vec{a^*} = \vec{x} \Lambda \vec{a} = \vec{y} \Lambda \vec{a} , \qquad (2.2)$$

where Λ denotes the exterior product of the vectors. This means that the direction vector $\vec{a}$ of the line and its moment vector $\vec{a^*}$ are independent of the choice of the points of the line. The two vectors $\vec{a}$ and $\vec{a^*}$ are not independent of one another; they satisfy the following equations:

$$< \vec{a}, \vec{a} > = 1, < \vec{a}, \vec{a^*} > = 0. \qquad (2.3)$$

The six components a_i , a^*_i ($i = 1, 2, 3$) of $\vec{a}$ and $\vec{a^*}$ are *Plückerian homogeneous line coordinates.* Hence the two vectors $\vec{a}$, $\vec{a^*}$ determine the oriented line.

The set of oriented lines in R^3 is one-to-one correspondence with pairs of vectors in R^3 subject to the conditions (2.3), and so we may expect to represent it as a certain four dimentional set in R^6 of six tuples of real numbers; we may take the space D^3 of triples of dual numbers with coordinates

$$X_1 = x_1 + \varepsilon x^*_1 , X_2 = x_2 + \varepsilon x^*_2 , X_3 = x_3 + \varepsilon x^*_3 , \varepsilon^2 = 0. \quad (2.4)$$

Each line in R^3 is represented by the dual vector

$$A = \vec{a} + \varepsilon \vec{a^*}, \vec{A}^2 = <\vec{a} , \vec{a} > + 2\varepsilon < \vec{a} , \vec{a^*}> = 1 \quad (2.5)$$

in D^3. Thus the following theorem of E. Study can be given [1].

Theorem(2.1): The oriented lines in R^3 are in one-to-one correspondence with the points of the dual unit sphere $< \vec{A} , \vec{A} > = 1$ in D^3.

Inner product: If we carry over the formal definition of the products of vectors to dual space D^3 we may write

$$<\vec{A}\,,\,\vec{B}> = <\vec{a}\,,\,\vec{b}> + \varepsilon\,[\,<\vec{a}^*\,,\,\vec{b}> + <\vec{a}\,,\vec{b}^*>\,] \quad (2.6)$$

which denotes the inner products of the dual vectors $\vec{A} = \vec{a} + \varepsilon\,\vec{a}^*$ and $\vec{B} = \vec{b} + \varepsilon\,\vec{b}^*$. If $\vec{A}$ and $\vec{B}$ are unit dual vectors then we have

$$<\vec{a}\,,\,\vec{b}> = \cos\varphi\,,\; <\vec{a}^*\,,\,\vec{b}> + <\vec{a}\,,\,\vec{b}^*> = -\varphi^*\sin\varphi,$$

where φ and φ^* denote the angle, and the shortest distance, respectively, of the two lines $\vec{A}$ and $\vec{B}$ [1]. Hence we have the the following theorem:

Theorem (2. 2): Let φ and φ^* denote the angle and the shortest distance between the lines $\vec{A}$ and $\vec{B}$. Then

$$<\vec{A}\,,\,\vec{B}> = \cos\Phi$$

where $\Phi = \varphi + \varepsilon\varphi^*$ denotes the dual angle between the two lines.

The taylor polynomial of an analytic dual variable function has just two terms:

$$f(t + \varepsilon h) = f(t) + \varepsilon f'(h),$$

for example

$$\sin\Phi = \sin\,(\varphi + \varepsilon\varphi^*) = \sin\varphi + \varepsilon\varphi^*\cos\varphi\,,$$

$$\cos\Phi = \cos\,(\varphi + \varepsilon\varphi^*) = \cos\varphi - \varepsilon\varphi^*\sin\varphi\,.$$

The following special cases of inner product are important:

(i) If

$$<\vec{A}\,,\,\vec{B}> = 0 \quad (2.7)$$

then $\varphi = \frac{\pi}{2}$ and $\varphi^* = 0$; this means that two lines $\vec{A}$ and $\vec{B}$ meet at a right angle.

(ii) If

$$<\vec{A}\,,\ \vec{B}> \ = \text{pure dual}, \tag{2.8}$$

then $\varphi = \dfrac{\pi}{2}$ and $\varphi^* \neq 0$; the lines $\vec{A}$ and $\vec{B}$ are orthogonal skew lines.

(iii) If

$$<\vec{A}\,,\ \vec{B}> \ = \text{pure real}, \tag{2.9}$$

then $\varphi \neq \dfrac{\pi}{2}$ and $\varphi^* = 0$; the lines $\vec{A}$ and $\vec{B}$ intersect.

(iv) If

$$<\vec{A}\,,\vec{B}> \ = \mp 1, \tag{2.10}$$

then $\varphi = 0$ and $\varphi^* = 0$; the lines $\vec{A}$ and $\vec{B}$ are coincident (their senses are the same or opposite).

b) *Spatial Motions*:

Since an euclidean motion in R^3 leaves unchanged the angle and the distance between two lines it will leave also unchanged the dual angle between two lines. Therefore the corresponding trans formation in D^3 will leave the inner product

$$<\vec{A},\ \vec{B}> \ = A\ B^T \tag{2.11}$$

invariant. It is the action of an orthogonal matrix with dual coefficients. When the center of the dual unit sphere must remain fixed, the transformation grup in D^3, which is the image of the euclidean motions in R^3, does not contain any translations.

Theorem (2.3): The euclidean motions in R^3 are represented in D^3 by the dual orthogonal martices $X = (x_{ij})$, $XX^T = I$, x_{ij} dual numbers.

The Lie algebra $L(O_D3)$ of the group of 3x3 orthogonal dual matrices is the algebra of skew-symmetric 3x3 dual matrices.

This is seen by differentiaton of $XX^T = I$. Therefore we can easily entend all known formulas about real spherical motions. But it is necessary to pay attention to the zero divisors [1].

The two coordinate systems $\{O; \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and $\{O'; \vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ are right-handed orthogonal coordinate systems which represent the *moving space H* and the *fixed space H'*, respectively. Let us express the displacements (H / H') of H with respect to H' in a third orthonormal right-handed system (relative system) $\{N; \vec{r}_1, \vec{r}_2, \vec{r}_3\}$. Then the corresponding dual orthonormal coordinate axes are

$$\vec{E}_1 = \vec{e}_i + \varepsilon \vec{e}_i^*, \ \vec{E}' = \vec{e}'_i + \varepsilon \vec{e}_i^{*\prime}; \ \vec{R}_i = \vec{r}_i + \varepsilon \vec{r}^*_i . \ (i=1,2,3) \quad (2.12)$$

where

$$\vec{e}^*_i = \vec{OM} \Lambda \ \vec{e}_i, \ \vec{e}^{*\prime}_i = \vec{O'M} \Lambda \ \vec{e}'_i; \ \vec{r}^*_i = \vec{NM} \Lambda \ \vec{r}_i; (i=1,2,3) \quad (2.13)$$

and M is a fixed origin point in the space. The the correspondence dual 1- forms are

$$\Omega_i = w_i + \varepsilon w^*_i, \ \Omega'_i = w'_i + \varepsilon w'^*_i, \ (i=1,2,3). \quad (2.14)$$

Hence we can write the following formulas for the dual spherical motions which are equivalent to the real spherical motions [7]:

a) The displacements with respect to H are

$$dR = \Omega R \quad (2.15)$$

where

$$\Omega = \begin{bmatrix} O & \Omega_3 & -\Omega_2 \\ -\Omega_3 & O & \Omega_1 \\ \Omega_2 & -\Omega_1 & O \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \vec{R}_1 \\ \vec{R}_2 \\ \vec{R}_3 \end{bmatrix}. \quad (2.16)$$

b) The displacements with respect to H' are

$$d'R = \Omega' R \quad (2.17)$$

where

$$\Omega' = \begin{bmatrix} 0 & \Omega'_3 & -\Omega'_2 \\ -\Omega'_3 & 0 & \Omega'_1 \\ \Omega'_2 & -\Omega'_1 & 0 \end{bmatrix} . \tag{2.18}$$

The real and dual parts of (2.15) or (2. 17) correspond to the *pure rotation* and the *pure translation* of the motion H / H', respectively.

c) *Velocity and Acceleration in Spatial Motions*:

Velocity: Consider a point X of unit dual sphere such that its coordinates with respect to the relative system are

$$X_i = x_i + \varepsilon x_i^* \qquad (i = 1, 2. 3)$$

Then

$$\sum_{i=1}^{3} X_i^2 = 1 \tag{2.19}$$

and

$$\vec{X} = X^T R \tag{2.20}$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} .$$

A line $\vec{X}$ in space corresponds to this dual point X. According to (2.15), (2.17) and (2.20) the displacements of $\vec{X}$ with respect to H and H' are

$$d\vec{X} = (dX^T + X^T \Omega) R$$

and

$$d'X = (dX^T + X^T \Omega') R . \tag{2.22}$$

Therefore if $\vec{X}$ is fixed in H or H' then $d\vec{X} = 0$ or $d'\vec{X} = 0$ and we may write

$$dX^T = X^T \Omega^T \ , \tag{2.23}$$

$$d'X^T = X^T \Omega'^T . \tag{2.24}$$

Now, suppose that $\vec{X}$ is fixed in H and let us calculate its velocity $d_f \vec{X}$ with respect to H′ . Then we substitute (2.23) in (2.22) and obtain

$$d_f \vec{X} = X^T \ (\Omega' - \Omega) \ R \ . \tag{2.25}$$

If we define a new dual vector whose components in the relative system are

$$\Psi_i = \Omega'_i - \Omega_i \tag{2.26}$$

then (2.26) reduces to

$$d_f \vec{X} = \vec{\Psi} \wedge \vec{X} \tag{2.27}$$

where

$$\vec{\Psi} = \vec{\psi} + \varepsilon \vec{\psi}^* \tag{2.28}$$

is the dual rotation Pfaffian vector. The real part $\vec{\psi}$ and the dual part $\vec{\psi}^*$ *of* $\vec{\Psi}$ *correspond to the rotatation motions and the translation motions.* In order to leave out the pure translation motions we wil suppose that

$$\vec{\psi} \neq 0$$

Acceleration :

From (2.27) it follows that the acceleration of X is given by

$$\vec{J} = d^2_f \vec{X} = \Psi \wedge (\vec{\Psi} \wedge \vec{X}) + \dot{\vec{\Psi}} \wedge \vec{X}$$

$$= - \Psi^2 \vec{X} + < \vec{\Psi}, \vec{X} > \vec{\Psi} + \dot{\vec{\Psi}} \wedge \vec{X} \tag{2.29}$$

where $\dot{\vec{\Psi}} = d\vec{\Psi}$ is the instantaneous dual angular acceleration vector.

In the equation (2.29) we see that the components of the acceleration $\vec{J}$ of $\vec{X}$ are homogeneous linear functions of the coordinates x_i $(i = 1,2,3)$ of $\vec{X}$. Equations (2. 27) and (2.29) can be written in the matrix form:

$$d_f X = MX \tag{2.27}$$

$$J = (M^2 + \dot{M})\, X \tag{2.29}$$

where M, $\dot{M}$ and M^2 are the matrices

$$M = \begin{bmatrix} 0 & -\Psi_3 & \Psi_2 \\ \Psi_3 & 0 & -\Psi_1 \\ -\Psi_2 & \Psi_1 & 0 \end{bmatrix} ; \quad \dot{M} = dM = \begin{bmatrix} 0 & -\dot{\Psi}_3 & \dot{\Psi}_2 \\ \dot{\Psi}_3 & 0 & -\dot{\Psi}_1 \\ -\dot{\Psi}_2 & \dot{\Psi}_1 & 0 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} -\Psi^2 + \Psi^2_1 & \Psi_1\Psi_2 & \Psi_1\Psi_3 \\ \Psi_1\Psi_2 & -\Psi^2 + \Psi^2_2 & \Psi_2\Psi_3 \\ \Psi_1\Psi_3 & \Psi_2\Psi_3 & -\Psi^2+\Psi^2_3 \end{bmatrix}, \quad M^3 = -\Psi^2 M. \tag{2.31}$$

If we calculate the determinant D of the coefficients of (2.29)′, we obtain

$$D = -\Psi^2 \dot{\Psi}^2 \sin^2 \nabla \tag{2.32}$$

where

$$\nabla = \alpha + \varepsilon\alpha^*$$

is the dual angle between the vectors $\vec{\Psi}$ and $\dot{\vec{\Psi}}$. If both vectors $\vec{\Psi}$ and $\vec{\Psi}$ correspond to the same line of space, this line has no acceleration, in this special case $D = 0$. After the discussion of the general case $D \neq 0$ we shall return to this special case.

III.a) *Bresse Line Congruence* (C_2)

In this section, we show that the spatial equivalent of the Bresse circle of planar kinematics is a line congruence which we call Bresse line congruence.

We know that the acceleration of a point in spherical motion is the sum of three orthogonal components: (i) a component normal to the sphere, (ii) a component tangent to the path (or tangential acceleration), and (iii) a component normal to the path but lying in the plane tangent to the sphere (geodesic normal acceleration).

Definition (3.1): On the unit dual sphere the locus of points having zero tangential acceleration is the dual spherical equivalent of the Bresse circle of planar kinematics. Following Garnier's notation we denote this curve as C_2. The spatial equivalent of C_2 is the spatial equivalent of the Bresse circle of planar kinematics. This is a locus of lines each of which corresponds to a point of C_2. We will denote this locus by (C_2).

According to this definition, at first we must derive the equation of C_2 then the equation and the geometry of (C_2) can be obtain by the Study mapping. From (2. 27) and (2.29) the set of all points X having zero tangentical acceleration on the unit dual sphere satisfy the equation

$$<\vec{J}\ ,\ d_f\vec{X}> \ = 0.$$

If we calculate this equation we have

$$<(\dot{\vec{\Psi}} \wedge \vec{X})\ ,\ (\vec{\Psi} \wedge \vec{X})> \ = 0$$

or since $\vec{X}^2 = 1$

$$<\vec{\Psi}\ .\ \vec{X}>\ .\ <\dot{\vec{\Psi}}\ ,\ \vec{X}> \ = \ <\vec{\Psi}\ .\ \dot{\vec{\Psi}}>\ . \qquad (3.1)$$

(3.1.) is the dual spherical equivalent of the Bresse circle. In order to obtain the spatial equivalent of this circle we choose a relative system $\{N;\ \tilde{\vec{R}}_1\ ,\ \tilde{\vec{R}}_2\ ,\ \tilde{\vec{R}}_3\}$ such that

$$\vec{\Psi} = \Psi\,\tilde{\vec{R}}_3\ ,\ \ \Psi_1 = \Psi_2 = \vec{0} \ \text{ and } \ \Psi_3 = \Psi\ . \qquad (3.2)$$

Since $\Psi_1 = \Psi_2 = O$ (2.26) gives us that

$$\Omega_1 = \Omega'_1 \ , \ \Omega_2 = \Omega'_2 \ . \tag{3.3.}$$

Then we may write that

$$\left.\begin{array}{l} \tilde{\vec{R}}_1 = \vec{R}_1 \cos\theta + \vec{R}_2 \sin\theta \\ \tilde{\vec{R}}_2 = -\vec{R}_1 \sin\theta + \vec{R}_2 \cos\theta \\ \tilde{\vec{R}}_3 = R_3 \end{array}\right\} \tag{3.4}$$

and for the 1- forms

$$\left.\begin{array}{l} \tilde{\Omega}_1 = \Omega_1 \cos\theta + \Omega_2 \sin\theta \\ \tilde{\Omega}_2 = -\Omega_1 \sin\theta + \Omega_2 \cos\theta \\ \tilde{\Omega}_3 = \Omega_3 + d\theta \end{array}\right\} \tag{3.5}$$

In order to determine the new relative system in a unique way we choose θ such that

$$\tilde{\Omega}_1 = 0 \tag{3.6}$$

Thus, in the new relative system instead of (2.16) and (2.18), respectively, we have:

$$\Omega = \begin{bmatrix} 0 & \tilde{\Omega}_3 & -\tilde{\Omega}_2 \\ -\tilde{\Omega}_3 & 0 & 0 \\ \tilde{\Omega}_2 & 0 & 0 \end{bmatrix} , \quad \tilde{\Omega}' = \begin{bmatrix} 0 & \tilde{\Omega}'_3 & -\tilde{\Omega}_2 \\ \tilde{\Omega}'_3 & 0 & 0 \\ \tilde{\Omega}_2 & 0 & 0 \end{bmatrix} .$$

Therefore, instead of (2.15) and (2.17), respectively, we have

$$d\tilde{R} = \tilde{\Omega}\,\tilde{R} \quad \text{and} \quad d'\tilde{R} = \tilde{\Omega}'\,\tilde{R} \ . \tag{3.8}$$

In this new system if we differentiate the equation

$$\vec{\Psi} = \Psi_3 \; \vec{R}_3$$

we have

$$\dot{\vec{\Psi}} = \dot{\Psi}_3 \vec{R}_3 + \Psi_3 \dot{\vec{R}}_3$$

or from the Eq. (3.8) since

$$\dot{\vec{R}} = \dot{\tilde{\vec{R}}}_3 \rightarrow d\vec{R}_3 = \vec{R}_1 \tilde{\Omega}_2$$

$$\dot{\vec{\Psi}} = \Psi_3 \Omega_2 \tilde{\vec{R}}_1 + \dot{\Psi}_3 \tilde{\vec{R}}_3 \; . \tag{3.9}$$

Hence (3.1) reduces to

$$\Psi_3 X_3 \; (\Psi_3 \tilde{\Omega}_2 X_1 + \dot{\Psi}_3 X_3) = \dot{\Psi}_3 X_3$$

or since $\Psi_3 \neq 0$

$$\Psi_3 \tilde{\Omega}_2 X_1 X_3 + \dot{\Psi}_3 \psi_3{}^2 = \dot{\Psi}_3 \; . \tag{3.10}$$

If we calculate the real and the dual parts of (3.10) we have

$$\psi_3 \; (1 - x^2{}_3) - \tilde{\omega}_2 \psi_3 x_1 x_3 = 0$$

$$\psi^*{}_3 (1 - x^2{}_3) + (\tilde{\omega}_2 \psi^*{}_3 + \tilde{\omega}^*{}_2 \psi_3) x_1 x_3 + 2 \dot{\psi}^*{}_3 x_3 x^*{}_3 + (x_1 x^*{}_3 + x^*{}_1 x_3) \omega_2 \psi_3 = 0 \; . \tag{3.11}$$

Hence the Plückerian coordinates of the lines $\vec{X} \in (C_2)$ satisfy the equations (3.11) and the equations

$$\left. \begin{aligned} x^2{}_1 + x^2{}_2 + x^2{}_3 &= 1 \\ x_1 x^*{}_1 + x_2 x^*{}_2 + x_3 x^*{}_3 &= 0 \end{aligned} \right\} . \tag{3.12}$$

Since the equations (3.11) are second degree polynomails of the Plückerian coordinates of the lines $\vec{X}$, (C_2) is a *quadratic congruence.* This congruence is the spatial equivalent of the Bresse circle and is called the Bresse congruence. Each equation of (3.11) with the equations (3.12) represent a quadratic line complexes whose common lines form the congruence (C_2). Hence we have proved the folowing theorem:

Theorem (3.1): In one parameter spatial motion H/H', consider all the lines $\vec{X}$ of H, such that at the time t each of them has the zero tangential acceleration. Then these lines $\vec{X}$ form a quadratic congruence which is intersection of two quadratic complexes. This congruence is the spatial equvalent of the Bresse circle of planer kinematics.

b) *Bresse Line Congruence* (C_2), *Quadratic Complex* (Q), *Tangential Complex.*

Now, consider a fixed line $\vec{X}$ of the moving space H. During the motion H/H' the line $\vec{X}$ generate an orbit surface in the fixed space H'. The displacement of $\vec{X}$ with respect to H' can be determined from (2.27). On the other hand in the new relative system (2.27) reduces to the equation

$$d_f \vec{X} = \Psi_3 (\vec{R}_3 \Lambda \vec{X}) = \Psi_3 (\tilde{\vec{R}}_3 X_1 - \tilde{\vec{R}}_2 X_2) . \quad (3.13)$$

On the fixed unit dual sphere the unit dual vector $\vec{X}$ draws a curve whose dual arc lenght is

$$d\Phi = d\varphi + \varepsilon d\varphi^* .$$

For this arc lenght we have

$$d\Phi^2 = (d_f \vec{X})^2 = (X^2_1 + X^2_2) \Psi^2_3 = (1 - X^2_3) \Psi^2_3 . \quad 3.14)$$

And calculating the real and dual parts of (3.14) we obtain

$$\left.\begin{aligned} d\varphi^2 &= (1 - x^2_3)\ \psi^2_3 \\ d\varphi d\varphi^* &= (1 - x^2_3)\ \psi_3\, \psi^*_3 - x_3 x_3^*\ \psi^2_3 \end{aligned}\right\} . \tag{3.15}$$

Then th drall of the orbit surface of $\vec{X}$, in H' is

$$\frac{1}{d} = \frac{d\varphi\ d\varphi^*}{d\varphi^2} = \frac{(1-x^2_3)\ \psi_3\psi^*_3 - x_3\ x^*_3\ \psi^2_3}{(1 - x^2_3)\ \psi^2_3}$$

or denoting the pitch of H/H' (instantaneous helical motion) by

$$k = \frac{\psi^*_3}{\psi_3}$$

we have

$$\frac{1}{d} = k - \frac{x_3\ x^*_3}{1 - x^2_3} . \tag{3.16}$$

Now, during the motion $H\ /\ H'$ consider the lines $\vec{X} \in H$ which generates a ruled surface whose drall is zero. From the equations (3. 12) and (3. 16) we have

$$(k - \frac{1}{d})\ (x^2_1 + x^2_2) - x_3\ x^*_3 = 0 \tag{3.17}$$

which represents a quadratic line complex Q. Hence we have the following theorem.

Theorem (3.2): At the instant t, consider all of the lines $\vec{X} \in H$ having zero tangential acceleration and that each of these lines generates a ruled surface with the same drall. Then these lines $\vec{X}$ are the common lines of (C_2) and Q.

As a special case, if the lines $\vec{X} \in H$, at the time t, generate the developable ruled surfaces then $\frac{1}{d} = 0$ and (3. 17) reduces

$$k(x^2{}_1 + x^2{}_2) - x_3\, x^*{}_3 = 0 \tag{3.18}$$

which represents Q_0 quadratic complex. Q_0 is a special case of Q. The Q_0 quadratic complex and the tangential complex of tangent lines, at the time t, of the curves which are the orbits of the ∞^3 points of H are the same. Hence we have the following theorem:

Theorem (3.3): At the instant t, in the motion consider the lines $\vec{X} \in H$ having zero tangential acceleration and that each of these lines $\vec{X}$ generates a developable ruled surface. Then these lines are the common lines of (C_2) and the tangential complex.

IV. The Spatial Equivalent of The Inflection Circle.

In this section we derive that the spatial equivalent of the inflection circle of planar kinematics is a line congruence which we call "*inflection line congruence*".

Definition (4.1): On the unit dual sphere the locus of points having zero geodesic normal acceleration is the spherical equivalent of the inflection circle of the planar kinematics. Following Garnier's notation we denote this spherical curve as C_3. In R^3 the correspondence of C_3 is the spatial equivalent of the inflection circle of planar kinematics and is denoted by (C_3).

According this definition the equations and the geometry of (C_3) can be obtained. The normal vector to the path of a point $\vec{X}$ on the unit dual sphere but lying in the plane tangent to the sphere is

$$\vec{X} \wedge d_f\vec{X} .$$

According to the definition (4.1) from Equation (2.27) the locus of points having zero geodesic normal acceleration satisfy the equation

$$(\vec{J}\,,\ \vec{X}\,,\ d_f\vec{X}) = 0, \tag{4.1}$$

where the left hand side is a 3 x 3 determinant. If we replace (2.27) in (4.1) we have

$$< J\ ,\ \vec{X} \wedge d_f\vec{X} > = 0$$

or

$$<\vec{\Psi'} \wedge (\vec{\Psi'} \wedge \vec{X}) + \dot{\vec{\Psi'}} \wedge \vec{X},\ \vec{X} \wedge (\vec{\Psi'} \wedge \vec{X})> = 0,$$

$$<\vec{\Psi'}\,,\ \vec{X}> [\vec{\Psi'} - <\vec{\Psi'}\,,\ \vec{X}>^2] = <\vec{\Psi'} \wedge \dot{\vec{\Psi'}}\,,\ \vec{X}>. \tag{4.2}$$

Using the new relative system (4.2) *reduces to*

$$\Psi'_3\, X_3\, (X^2{}_1 + X^2{}_2) + \tilde{\Omega}_2\, X_2 = 0 \tag{4.3}$$

which is the dual spherical equivalent of the inflection circle of planar kinematics. As we see from (4.3) this spherical equivalent of the inflection circle is a dual spherical curve of third degree.

If we calculate the real and dual parts of (4.3) we have

$$\left.\begin{aligned} \psi_3\, x^*{}_3\, (x^2{}_1 + x^2{}_2) + \tilde{\omega}_2\, x_2 &= 0 \\ \psi_3\, x^*{}_3 + \psi^*{}_3\, x_3 + \tilde{\omega}_3 x_2 + \tilde{\omega}_2{}^*\, x_2 &= (\psi^*{}_3\, x_3 + 3\psi_3\, x^*{}_3) \end{aligned}\right\}. \tag{4.4.}$$

Since the Plückerian coordinates x_i, $x^*{}_i$ satisfy (3.12) and (4.4) the lines $\vec{X}$ having zero geodesic normal acceleration form a line congruence (C_3). Since the equations are third degree equations this line congruence consist of all the common lines of two cubic line complexes. We call (C_3) the inflection congruence. Hence we may express the following theorem:

Theorem (4.1): At the instant t all of the lines $\vec{X} \in H$ having zero geodesic normal acceleration form the inflection congruence (C_3) as a congruence consisting of the common lines of two cubic line complexes. (C_3) is the spatial equivament *of the inflection circle of* planar kinematics.

b) Inflection Congruence (C_3), *Quadratic Complex Q and Tangential Complex.*

If we repeat the discussion in the paragraph (b) of the section III for the (C_3) instead of (C_2) we have the following theorem which is analogous to the Theorem (3.2).

Theorem (4.2): At the instant t, consider the lines $\vec{X} \in H$ having zero geodeşic normal acceleration and that each of these lines generates a ruled surface with the same drall. Then these lines $\vec{X}$ are the common lines of (C_3) and Q.

The analogous theorem to the Theorem (3.3) is as foollows:

Theorem (4.3): At the instant t, in the motion H/H′ consider all of the lines $\vec{X} \in H$ having zero geodesic normal accelerationand that each of these lines generates a developable orbit surface. Then these lines $\vec{X}$ are the common lines of (C_3) and the tangential complex.

c) The Common Lines of (C_2) *and* (C_3) *(Acceleration Axes).*

Let denote the mowing and fixed unit dual spheres by K and K′, respectively. K and K′ correspond to the spaces H and H′, respectively.Then the dual spherical motion K/K′ corresponds to the spatial motions H/H′. At every instant t, in the motion K/K′ there are some points of K having neither a tangential acceleration nor a geodesic normal acceleration: their acceleration $\vec{J}$ is purely normal to the sphere and they are the acceleration centers. These points are the intersections of the curves C_2 and C_3. These points are three points and located as the vertics of a spherical triangle on K [7,8].

The corresponding lines of these points, in general, are the three skew lines [7]. Thus the common lines of (C_2) and (C_3), in general, are the three skew acceleration axes of the motion H′ / H at that instant t. If the motion H /H'is, as a special case, a spherical motion then these three common lines intersect each other at the center of the spheres [7], These three lines discussed as the acceleration axes [7, 8].

V. *Special Cases.*

a) *The case of* $D = 0$:

In this case $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are linearly dependent. There are two cases: (i) $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are coincident, and (ii) $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are parallel.

(i) If the lines $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are coincident with a line l, then the accelerations of all the points of l are zero. Therefore the line l is a common line of (C_2) and (C_3). The geometry of this line l has been discussed [7, p: 34].

(ii) If the lines $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are parallel then the points of these parallel lines the acceleration $\vec{J}$ is zero. Thus these two lines are also the common lines in (C_2) and (C_3). The set of all the lines intersecting the both lines $\vec{\Psi}$ and $\dot{\vec{\Psi}}$, form a hyperbolic congruence [7]. Then the lines $\vec{\Psi}$ and $\dot{\vec{\Psi}}$ are the *principal directions* of this congruence [9].

b) *The case of* $<\vec{\Psi}, \dot{\vec{\Psi}}> = 0$:

In this case from the equation (3.1) we have

$$<\vec{\Psi}, \vec{X}> = 0 \qquad \text{or} \qquad <\dot{\vec{\Psi}}, \vec{X}> = 0.$$

Therefore, in this case, (C_2) consist of all lines of two linear line complexes whose axes intersects each other. Thus (C_2), in this special case, is linear. In order to obtain the equations of (C_2) it is enough to calculate (3.10) and (3.11) for this case.

c) *Spherical Motion*:

If the motion H / H′ has a fixed point, then the motion is called a pure rotation or a spherical motion. In this case in order to have the equations of (C_2) and (C_3) it is enogh to replace $\vec{\psi}^* = 0$ and $\dot{\vec{\psi}}^* = 0$ in their equations obtained in the sections III and IV. Since the pitch of H / H″ is $k = \frac{\psi^*}{\psi}$, in this case k becomes zero and therefore H / H′ reduces to a spherical motion. In a spherical motion (C_2) and (C_3) has three common lines (acceleration axes of the motion) form a pencil of straight lines whose vertex is the center of the spheres [8].

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ÖZET

Düzlemsel Kinematiğin iyi bilinen Bresse çemberi ve Dönüm çemberi birim dual küre üzerinde hesaplandı. Bu iki çemberin uzaydaki karşılıkları Study dönüşümünü kullanarak elde edildi. Böylece iki doğru kongrüansı tanımlandı. Hareketin herhangi bir t anında bu kongrüansların üç ortak doğrusu vardır. Bu üç doğru iki kongruansa karşılık gelen bir parametreli uzay hareketnin ani ivme eksenleridirler. Genel halde ve bazı özel hallerde bu kongruansların özelikleri incelendi.

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