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On Matrix Transformations Of Sequnce Spaces Defined In An Incomplete Space

by

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On Matrix Transformations Of Sequnce Spaces Defined In An Incomplete Space

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SUMMARY

The purpose of this note is to characterize the matrices which transform some sequence spaces into the same or another sequence space in an incomplete space and to give a new method to prove the necessity of the norm condition $f(A) < \infty$ for these type of transformations.

1. INTRODUCTION

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} $(n, k = 1,2,\ldots)$ and v, w be two subsets of the space s of complex sequences. We say that the matrix A defines a matrix transformation from v into w and denote it by writing $A \in (v, w)$, if for every sequence $x=(x_k) \in v$ the sequence $Ax=(A_n(x)) \in w$,

where $A_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k$.

It is known that most of the Toeplitz theory on transformations of sequence spaces, i.e., characterizetions of the matrices $A \in (v, w)$ seem to have been solved for the case in which x_k s are complex numbers. (See, for example, [1], [2], [4] [5], [6], [7]). However it can easily be shown that many of the important results are still valid in any complete seminormed complex linear space X.

In this paper we are going to deal with the Toeplitz theory on the transformations of sequence spaces defined in an incomplete seminormed complex linear space $X = X(p)$ with the seminorm p and zero θ. The main difficulty for these type of characterizetions is to prove the necessity of the norm condition which easily comes out from the Banach-Steinhaus type theorem when X is complete. Since the Banach-Steinhaus type theorem is not valid in an incomplete space, one has to find some other methods. First, Maddox modified the original argument used by Toeplitz, [3]. This modification consists of a construction of a special sequence which gives a contradiction, [6]. Sometimes this procedure is a hard and even a painstaking job. Therefore, we are going to establish a lemma which will save us to construct such special sequences for each of the transformations. (See Lemma 3.3).

2. NOTATIONS

As far as we know, the first paper on the matrix transformations of the sequence spaces defined in an incomplete space is due to Maddox [3]. In that paper, he has chosen the sequence spaces L_∞, $\mathcal{C}$ and **C**, the space of bounded sequences, Cauchy sequences and convergent sequences, respestively, and characterized the matrix transformations between any two of these spaces. Now we shall add the sequence spaces

$$\Gamma = \{ x=(x_k) : \Sigma\, x_k \text{ converges and } x_k \in X\},$$

$$L_r = \{x = (x_k) : \Sigma\, [p(x_k)]^r < \infty \ (1 < r < \infty) \text{ and } x_k \in X\}$$

and

$$L_s = \{x = (x_k) : \Sigma\, [p(x_k)]^s < \infty \ (0 < s \leqq 1) \text{ and } x_k \in X\}$$

to the spaces mentioned above to extend the range.

Throughout the paper, S will denote the space of all sequences defined in $X = X(p)$. V and W will be any subspaces of S. When $X = C$, the set of complex numbers, we are going to use the usual notations l_∞, c, γ, 1_r, 1_s for the corresponding spaces to L_∞, $\mathcal{C}$,

C, Γ, L_r, L_s, respectively. Of course, the space of Cauchy sequences is equal to the space of convergent sequences in the case of $X = C$.

Φ will denote the space of finite sequences of complex numbers, i. e., sequences which have only a finite number of non-zero coordinates and R denotes the set of row-finite infinite matrices, i. e., whose rows are in Φ.

By N we denote the set of natural numbers.

3. LEMMAS

Now, we are going to give some lemmas which will be used frequently throughout the paper.

Lemma 3.1. If X is incomplete, then L_∞, $\mathcal{C}$, **C**, Γ, L_r and L_s are also incomplete.

To fix the idea, we shall prove the incompleteness of **C** under the given seminorm. The others can be shown in a similar way.

Proof. Since X is incomplete seminormed complex linear space, there exists a sequence $(x_n) = (x_1, x_2, \ldots)$ which is Cauchy but not convergent. Now, let us define

$$\begin{aligned} y_1 &= (x_1, \theta, \theta, \ldots) \\ y_2 &= (x_2, \theta, \theta, \ldots) \\ &\ldots\ldots\ldots\ldots \\ y_n &= (x_n, \theta, \theta, \ldots) \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Then

$$\bar{p}\,(y_n - y_m) = \sup p(x_n - x_m) \to 0,$$

since (x_n) is Cauchy in X, so (y_n) is Cauchy in **C**.

Now, suppose that

$$y_n \xrightarrow{\bar{p}} t \in \mathbf{C}, \text{ say,}$$

where $t = (t_1, t_2, \ldots)$. But $(y_n - t) = (x_n - t_1, -t_2, -t_3, \ldots)$ and

$$\bar{p}(y_n - t) = \sup_n p(x_n - t_1) \nrightarrow 0,$$

since (x_n) does not converge to any element of X. So

$$\bar{p}\ (y_n - t) \nrightarrow 0 \qquad \text{as } n \to \infty.$$

This is a contradiction.

Lemma 3.2. If the sequence $(\sum_{k=1}^{\infty} a_{nk}\ x_k)_{n \in N}$ converges for every $(x_k) \in V$, where V is a space which has the unit vector $e^{(k)} = (\theta, \theta, \ldots, \theta, u, \theta, \ldots)$ with $u \in X$ $(p(u) > 0)$ in k^{th} place and θ otherwise, e. g., L_∞, $\mathcal{C}$, $\mathbf{C}$, Γ, L_r, L_s, then

$$(a_{nk})_{n \in N} \in c \qquad (\forall\ k).$$

Proof. Let $(\sum_{k=1}^{\infty} a_{nk}\ x_k)_{n \in N} \in \mathbf{C}$ for each $(x_k) \in V$. Then taking $x_k = e^{(k)}$, we get

$$(a_{nk} \cdot u)_{n \in N} \in \mathbf{C} \qquad (\forall\ k),$$

which implies that

$$(a_{nk} \cdot u)_{n \in N} \in \mathcal{C} \qquad (\forall\ k),$$

i. e.,

$$p(u)\ |\ a_{nk} - a_{mk}\ | \to 0 \qquad \text{as } n, m \to \infty$$

Thus we have that

$$|a_{nk} - a_{mk}| \to 0 \qquad \text{as } n, m \to \infty$$

and therefore

$$(a_{nk})_{n \in N} \in c \qquad (\forall\ k).$$

REMARK. Let w is one of the sequence spaces l_∞, c, γ, 1_r and 1_s and W be the corresponding sequence spaces L_∞, $\mathcal{C}$, $\mathbf{C}$, Γ, L_r and L_s, then it is easy to check that $(x_k) \in w$ if and only if $(yx_k) \in W$ for each fixed vector $y \in X$ with $p(y) > 0$ where $x_k \in C$ Then we can give the following lemma:

Lemma 3.3. Let each of v, w be one of the sequence spaces l_∞, c, γ, 1_r and 1_s, and V, W be the corresponding sequence spaces

L_∞, $\mathcal{C}$, **C**, Γ, L_r and L_s Then if a norm-condition $f(A) < \infty$ is necessary for $A \in (v, w)$, it is also necessary for $A \in (V, W)$.

Proof. Let $A \in (V, W)$. But suppose that $f(A) = \infty$. Then there exists a sequence $(x_k) \in v$ such that $(\Sigma a_{nk} x_k) \notin w$. Now choose $y \in X$ with $p(y) > 0$. Hence $(yx_k) \in V$.

If $\Sigma a_{nk} x_k$ diverges for some n, then $\Sigma a_{nk} y x_k$ diverges for some n which is a contradiction.

If $\Sigma a_{nk} x_k$ converges for all n then we need $(y A_n(x)) \in W$. This implies that $(A_n(x)) \in w$, which is also a contradiction, whence $f(A) < \infty$. This completes the poof.

Lemma 3.4. In (X, p), $A \in (V, \Gamma)$ if and only if $G \in (V, \mathbf{C})$

where the matrix $G=(g_{nk})$ is given by $g_{nk} = \sum_{j=1}^{n} a_{jk}$.

For the case in which $X = C$, this lemma is due to Vermes [7]. Since the proof is quite similar we omit to give it here.

Lemma 3.5. Let $0 < q < \infty$. Then, in (X, p), the necessary and sufficient condition for $\sum_{k=1}^{\infty} a_k x_k$ to be convergent

whenever $\sum_{k=1}^{\infty} [p(x_k)]^q < \infty$ is that $a=(a_k) \in \Phi$.

Proof. Since $\sum_{k=1}^{\infty} a_k x_k$ reduces to a finite sum, sufficiency is trivial.

For the necessity, suppose that $a \notin \Phi$. Then there is a sequence of positive integers $m_1 < m_2 < \dots$ such that $|a_{m_k}| > 0$. Also there is a sequence $y \in \mathcal{C}$ - **C**, whence there exist positive integers $n_1 < n_2 < \dots$ such that

$$p(y_{n_k} - y_{n_{k-1}}) < |a_{m_k}| / k^{1+1/q} \quad (k = 2, 3, \dots).$$

Let us define

$$x_{m_1} = y_{n_1}/a_{m_1}\ ,\ x_{m_k} = (y_{n_k} - y_{n_{k-1}})/a_{m_k}\quad (k \geqq 2),$$

and $x_n = \theta$ $(n \neq m_k)$ so that

$$\sum_{k=1}^{\infty} [p(x_k)]^q \leqq [p(y_{n_1})/|a_{m_1}|]^q + \sum_{k\geqq 2} 1/k^{1+q} < \infty,$$

since $q > 0$. But

$$\sum_{k=1}^{m_i} a_k\, x_k = a_{m_1} x_{m_1} + a_{m_2} x_{m_2} + \dots + a_{m_i} x_{m_i} = y_{n_i}.$$

Since $\sum_{k=1}^{\infty} a_k\, x_k$ converges, we have that the Cauchy sequence y has a convergent subsequence (y_{n_i}). Hence y converges, contrary to $y \in \mathcal{C}-\mathbf{C}$.

REMARK. The above lemma is still true if L_q is replaced by $L(q)$ where

$$L(q) = \{x = (x_k) : x_k \in X,\ \sum_{k=1}^{\infty} [p(x_k)]^{q_k} < \infty,\ q_k > 0\ (\forall\ k)\},$$

It is enough to take

$$p(y_{n_k} - y_{n_{k-1}}) < |a_{m_k}| \,/\, k^{2/q_k} \qquad (k = 2, 3, \dots)$$

in the above proof.

Lemma 3. 6. Let $W \supset L_q$ $(0 < q < \infty)$. Then the dual space of W is Φ, i. e.,

$$(W)^+ = \Phi.$$

Proof.. We have already shown is Lemma 3.5. that

$$(L_q)^+ = \Phi \qquad (0 < q < \infty).$$

Now, $W \supset L_q$ implies that $(W)^+ \subset (L_q)^+ = \Phi$, i.e.,

(1) $\quad (W)^+ \subset \Phi.$

Let $a = (a_k) \in \Phi$, i.e., $a = (a_1, a_2, \dots, a_{n_0}, 0, 0, \dots)$ and take any $(x_k) \in W$. Then

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{k=1}^{no} a_k x_k$$

exists, since it is a finite sum. Therefore

(2) $\Phi \subset (W)^+$.

So from (1) and (2), we get

$$(W)^+ = \Phi.$$

Corollary.

$(L_\infty)^+ = (\mathscr{C})^+ = (\mathbf{C})^+ = (L_q)^+ = \Phi$, where $0 < q < \infty$.

Lemma 3.7. In (X, p), the necessary and sufficient condition for $\sum_{k=1}^{\infty} a_k x_k$ to be convergent whenever $\sum_{k=1}^{\infty} x_k$ converges is that $\triangle a \in \Phi$, where $\triangle a = a_k - a_{k+1}$.

Proof. Sufficiency. $\triangle a \in \Phi$ implies that $a = (a_k)$ is ultimately constant. Then, obviously, $\sum_{k=1}^{\infty} a_k x_k$ is convergent whenever $\sum_{k=1}^{\infty} x_k$ converges.

Necessity. Let $(s_k) = (\sum_{i=1}^{k} x_k)$ be convergent. Then the series

$$\Sigma x_k = s_1 + (s_2 - s_1) + (s_3 - s_2) + \dots$$

converges and so

$$\sum_{k=1}^{n} a_k x_k = s_n a_n + \sum_{k=1}^{n-1} s_k \triangle a_k$$

tends to a limit as $n \to \infty$. Hence we can write

$$\sum_{k=1}^{n} b_{nk} s_k \rightarrow \text{a limit.}$$

So we get

$$B = (b_{nk}) \in (\mathbf{C}, \mathbf{C}),$$

i. e.,

$$(\lim_{n\rightarrow\infty} b_{nk}) = (\Delta a_k) \in \Phi.$$

This completes the proof.

4. MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES DEFINED IN AN INCOMPLETE SPACE

In this chapter, we are going to characterize the matrices which transform the sequence space V into L_∞, $\mathcal{C}$, $\mathbf{C}$ and Γ. The transformations between the spaces L_∞, $\mathcal{C}$ and $\mathbf{C}$ are due to Maddox [3]. Now we shall give the rest.

4. 1. Transformations of the form (V, L_∞).

Theorem 4. 1.1. In (X, p), $A \in (\Gamma, L_\infty)$ if and only if

(3) $\Delta A \in R$, i. e., $\Delta a_{nk} = 0$ for $k > k_o(n)$,

(4) $\sup_n \sum_{k=1}^{\infty} |\Delta a_{nk}| < \infty$,

and

(5) $\sup_n |a_{n1}| < \infty$.

Since the proof of this theorem is quite similar to the one in the case in which X is complete, we omit to giving it.

Theorem 4.1.2. Let $1 < r < \infty$. Then, in (X, p), $A \in (L_r, L_\infty)$ if and only if

(6) $A \in R$, i. e., $a_{nk} = 0$ for $k > k_o(n)$,

(7) $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}|^{r'} < \infty$, where $1/r + 1/r' = 1$.

Proof. Sufficiency. Suppose that the conditions hold. Then using the Hölder's inequality, we get

$$p\left(\sum_{k=1}^{\infty} a_{nk}\, x_k\right) = p\left(\sum_{k=1}^{k_o(n)} a_{nk}\, x_k\right)$$

$$\leq \sum_{k=1}^{k_o(n)} |a_{nk}|\; p(x_k)$$

$$\leq \left[\sum_{k=1}^{k_o(n)} |a_{nk}|^{r'}\right]^{1/r'} \left[\sum_{k=1}^{k_o(n)} [p(x_k)]^{r}\right]^{1/r}$$

$$< \infty,$$

whenever $(x_k) \in L_r$.

Necessity. According to Lemma 3.6. $A \in R$ and by Lemma 3.3. (7) is necessary.

Theorem 4.1.3. Let $0 < s \leq 1$. Then in (X, p), $A \in (L_s, L_\infty)$ if and only if

(6) $A \in R$, i. e., $a_{nk} = 0$ for $k > k_o(n)$,

(8) $M = \sup_{n,k} |a_{nk}| < \infty$.

Proof. Sufficiency. If the conditions hold and $x \in L_s$, then we can easily get

$$p\left(\sum_{k=1}^{\infty} a_{nk}\, x_k\right) = p\left(\sum_{k=1}^{k_o(n)} a_{nk}\, x_k\right)$$

$$\leq \sum_{k=1}^{k_o(n)} |a_{nk}|\; p(x_k)$$

$$\leq M.\left(\sum_{k=1}^{\infty} [p(x_k)]^{s}\right)^{1/s}$$

$$< \infty.$$

Necessity. According to Lemma 3.5. we have $A \in R$, and Lemma 3.3. gives the necessity of (8).

4.2. Transformations of the form (V, $\mathcal{C}$)

Theorem 4.2.1. In (X, p), $A \in (\Gamma, \mathcal{C})$ if and only if

(3) $\Delta A \in R$, i. e., $\Delta a_{nk} = 0$ for $k > k_o(n)$,

(4) $\sup_n \sum_{k=1}^{\infty} |\Delta a_{nk}| < \infty$,

and

(9) $\lim_{n \to \infty} a_{nk} = \alpha_k$ exists for each fixed k.

Proof. Sufficiency. We may notice that these conditions are sufficient for $A \in (\Gamma, \mathbf{C})$ (See, Theorem 4.3.1.), and we also have the inclusion

(10) $(\Gamma, \mathbf{C}) \subset (\Gamma, \mathcal{C}) \subset (\Gamma, L_\infty)$.

So, the conditions are sufficient for $A \in (\Gamma, \mathcal{C})$, too.

Necessity. Since the conditions (3) and (4) are necessary for $A \in (\Gamma, L_\infty)$, (10) implies that they are also necessary for $A \in (\Gamma, \mathcal{C})$. Finally, Lemma 3.2. gives the necessity of (9).

Theorem 4. 2. 2. Let $1 < r < \infty$. Then, in (X, p), $A \in (L_r, \mathcal{C})$ if and only if

(6) $A \in R$, i. e., $a_{nk} = 0$ for $k > k_o(n)$,

(7) $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}|^{r'} < \infty \quad (1/r + 1/r' = 1)$.

and

(9) $\lim_{n \to \infty} a_{nk} = \alpha_k$ exists for each fixed k.

Proof. Sufficiency. Let $(x_k) \in L_r$ and the conditions hold. Then choose and fix an $m_o \geqq 1$ such that

$$\Big(\sum_{k=m_o+1}^{\infty} [p(x_k)]^r\Big)^{1/r} \leqq \varepsilon/4M^{1/r'},$$

where ε is a given positive number. So we write

$$p(A_n(x) - A_m(x)) \leqq \sum_{k=1}^{m_o} |a_{nk} - a_{mk}|\, p(x_k) +$$

$$\sum_{k=m_o+1}^{\infty} |a_{nk}|\, p(x_k) + \sum_{k=m_o+1}^{\infty} |a_{mk}|\, p(x_k)$$

$$\leqq \Big(\sum_{k=1}^{m_o} |a_{nk} - a_{mk}|^{r'}\Big)^{1/r'} \Big(\sum_{k=1}^{m_o} [p(x_k)]^r\Big)^{1/r} + \varepsilon/2.$$

Now, (9) implies that $(a_{nk})_{n\in N}$ is Cauchy for each k, i. e., there is an N_o such that

$$|a_{nk} - a_{mk}| < \varepsilon/2\,T\,m_o^{1/r'} \quad (n, m \geqq N_o),$$

where

$$T = \Big(\sum_{k=1}^{m_o} [p(x_k)]^r\Big)^{1/r}.$$

Therefore we get

$$p(A_n(x) - A_m(x)) \leqq \varepsilon, \text{ i. e., } (A_n(x)) \in \mathcal{C}.$$

Necessity. Let $(A_n(x)) \in \mathcal{C}$ whenever $(x_k) \in L_r$. Then, by Lemma 3. 6. we have $A \in R$ and according to Lemma 3.3. (7) is necessary. Finally, Lemma 3.2. gives the necessity of (9),

Theorem 4.2.3. In (X, p), $A \in (L_1, \mathcal{C})$ if and only if

(6) $A \in R$, i. e., $a_{nk} = 0$ for $k > k_o(n)$,

(8) $M = \sup_{n,k} |a_{nk}| < \infty$,

and

(9) $\lim_{n\to\infty} a_{nk} = \alpha_k$ exists for each fixed k.

Proof. Sufficiency. Let $(x_k) \in L_1$. Then

$$p(A_n(x) - A_m(x)) = \sum_{k=1}^{m_o} |a_{nk} - a_{mk}|\, p(x_k) + 2M \sum_{k=m_o+1}^{\infty} p(x_k)$$

where $m_o \geqq 1$. Now, choose m_o sufficiently large such that for a given $\varepsilon > 0$, $\sum_{k=m_o+1}^{\infty} p(x_k) < \varepsilon / 4M$. Then

$$p(A_n(x) - A_m(x)) \leqq \sum_{k=1}^{m_o} |a_{nk} - a_{mk}| \cdot p(x_k) + \varepsilon / 2.$$

and using (9) we get

$$|a_{nk} - a_{mk}| < \varepsilon / 2H \quad \text{for each } k, \ (n, m \geqq N_o),$$

where $H = \sum_{k=1}^{m_o} p(x_k) < \infty$. Therefore we get

$$p(A_n(x) - A_m(x)) \leqq \varepsilon \quad \text{for } n, m \geqq N_o.$$

Necessity. According to Lemma 3.6. we have $A \in R$; Lemma 3.3. and Lemma 3.2. give the necessity of (8) and (9), respectively.

Now, in the light of this theorem and the fact that $L_s \subset L_1$ $(0 < s \leqq 1)$, we can give the following

Theorem 4. 2. 4. Let $0 < s \leqq 1$. Then, in (X, p), $A \in (L_s, \mathcal{C})$ if and only if $A \in (L_1, \mathcal{C})$.

4.3. Transformations of the form (V,C)

Theorem 4.3.1. In (X, p), $A \in (\Gamma, \mathbf{C})$ if and only if

(3) $\Delta A \in R$, i. e., $\Delta a_{nk} = 0$ for $k > k_o(n)$,

(4) $\sup_n \sum_{k=1}^{\infty} |\Delta a_{nk}| < \infty$,

and

(9) $\lim_{n \to \infty} a_{nk} = \alpha_k$ exists for each fixed k.

Proof. Sufficiency part is trivial. Then, by Lemma 3.7. we have $\Delta A \in R$; Lemma 3.3. and Lemma 3.2. give the necessity of (4) and (9), respectively.

Theorem 4. 3. 2. Let $1 < r < \infty$. Then, in (X, p), $A \in (L_r, \mathbf{C})$ if and only if

(6) $A \in R$, i. e., $a_{nk} = 0$ for $k > k_o(n)$,

(7) $M = \sup_n \sum_{k=1}^{\infty} |a_{nk}|^{r'} < \infty$, $(1/r + 1/r' = 1)$,

(9) $\lim_{n \to \infty} a_{nk} = \alpha_k$ exists for each fixed k,

and

(12) $\alpha = (\alpha_k) \in \Phi$.

Proof. Sufficiency. Let $(x_k) \in L_r$. Then obviously, $A \in R$ implies that $\sum_{k=1}^{\infty} a_{nk} x_k$ exists for each n and for each $(x_k) \in L_r$ and $\alpha \in \Phi$ implies that $\sum_{k=1}^{\infty} \alpha_k x_k$ exists for each $(x_k) \in L_r$, finite sums in fact. Now we are going to show that $(\sum_{k=1}^{\infty} a_{nk} x_k)_{n \in N}$ converges to $\sum_{k=1}^{\infty} \alpha_k x_k$. For a given $\varepsilon > 0$, let us choose and fix an $m_o > 0$ such that

$$(\sum_{k=m_o+1}^{\infty} [p(x_k)]^r)^{1/r} \leqq \varepsilon / 4M^{1/r'}.$$

We may note that we also have that $\sum_{k=1}^{\infty} |\alpha_k|^{r'} \leqq M$. So we write

$$p(\sum_{k=1}^{\infty} a_{nk} x_k - \sum_{k=1}^{\infty} \alpha_k x_k) \leqq p(\sum_{k=1}^{m_o} (a_{nk} - \alpha_k) x_k) + \varepsilon/2.$$

where $m_o \geqq 1$. Letting $n \to \infty$, we get

$$p(\sum_{k=1}^{\infty} a_{nk} x_k - \sum_{k=1}^{\infty} \alpha_k x_k) < \varepsilon \quad \forall\ n \geqq N_o.$$

Necessity. According to Lemma 3.6. $A \in R$; by Lemma 3.3. we get that (7) is necessary, Lemma 3.2. gives the necessity of (9). Finally, since (12) is necessary for $(L_1, \mathbf{C})$, it is also necessary for $(L_r, \mathbf{C})$.

Theorem 4. 3. 3. Let $0 < s \leq 1$. Then in (X, p). $A \in (L_s, \mathbf{C})$ if and only if

(6) $A \in R$, i. e., $a_{nk} = 0$ for $k > k_o(n)$.

(8) $M = \sup_{n,k} |a_{nk}| < \infty$,

(9) $\lim_{n \to \infty} a_{nk} = \alpha_k$ exists for each fixed k

and

(12) $\alpha = (\alpha_k) \in \Phi$.

Proof. Sufficiency. It can easily be shown that the conditions above are sufficient for $A \in (L_1, \mathbf{C})$. Since $(L_s, \mathbf{C}) \supset (L_1, \mathbf{C})$, it is easy to say that the conditions are sufficient for $A \in (L_s, \mathbf{C})$, too.

Necessity. According to Lemma 3.7., Lemma 3.3. and Lemma 3.2. we get the necessityof (6), (8) and (9), respectively. Now, we shall prove that condition (12) is necessary for $A \in (L_1, \mathbf{C})$. A similar proof can be given for $A \in (L_s, \mathbf{C})$. Since $A \in R$, we have that $a_{nk} = 0$ for $k > k_o(n)$. Consider two cases : (i) $k_o(n)$ bounded. Then $a_{nk} = 0$ $(k > \max k_o(n))$, whence $\alpha = (\alpha_k) \in \Phi$. (ii) Suppose that $k_o(n)$ is unbounded, but $\alpha \notin \Phi$. We are then assuming that $(A_n(x)) \in \mathbf{C}$ whenever $(x_k) \in L_1$ and since $A \in R \cap (l_1, c)$, it follows that the sequence

(13) $\left(\sum_{k=1}^{k_o(n)} \alpha_k x_k \right)_{n \in N}$

converges whenever $(x_k) \in L_1$. Using the proof of Lemma 3.5. with α in place of a we constract a sequence x in terms of the Cauchy sequence y, then extract from (13) a subsequence of y which will converge, since (13) converges. This contradits the fact that $y \in (\mathcal{C} - \mathbf{C})$ and so completes the proof of the theorem.

4.4. Transformations of the form (V, Γ)

By Lemma 3.4. we have reduced the problem of finding the matrices A ∈ (V, Γ) to the problem of finding the matrices

G ∈ (V, **C**), where the matrix $G = (g_{nk})$ is given by $g_{nk} = \sum_{j=1}^{n} a_{jk}$.

Since we have already characterized the transformations of the form (V, **C**) in paragraph 4.3., we can directly get the matrices A ∈ (V, Γ) just writing the matrix G instead of the matrix A in those transformations.

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ÖZET

Bu makalede, tam olmayan bir uzay üzerinde tanımlanmış dizi uzaylarını kendileri veya bir diğer dizi uzayının içine dönüştüren matrisler karakterize edilmekte ve bu tip dönüşümler için $f(A) < \infty$ norm-şartının gerekliliğini ispat etmek için yeni bir metot verilmektedir.

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